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UNIQUENESS THEOREMS FOR MINIMAL SURFACES

BY

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Introduction

There are three basic uniqueness theorems in the theory of compact minimal surfaces in $\mathbf{R}^3$. The first theorem is due to Rado [19] and states that if γ is a Jordan curve which admits a one-to-one orthogonal or central projection on a convex plane curve, then γ bounds a unique minimal disk which is a graph over the plane. The second theorem due to Nitsche ([14] and also [5]) states that a smooth Jordan curve with total curvature less or equal to 4π bounds a unique minimal disk and the disk is immersed. A third theorem which follows from convergence properties for minimal disks is that any C^2 Jordan curve sufficiently C^2 close to a fixed plane C^2 Jordan curve bounds a unique minimal disk which is a graph over the plane (see [4] and [24]).

In this paper we generalize these three theorems to show that the curves above bound unique compact minimal surfaces. In the case of a Jordan curve with total curvature less than or equal to 4π, we need to make the additional assumption that the curve lies on the boundary of a convex set. It is still not known if this additional assumption is needed. The tools used in the proofs are the maximum principle for minimal surfaces, the geometric Dehn's lemma in [11], Nitsche's uniqueness theorem [14] and the manifold representation of smooth immersed minimal disks in $\mathbf{R}^3$ [24]. Except for Theorem 2, the results in this paper also appear in the author's book "Lectures on Plateau's Problem" [9] published by I.M.P.A. A generalization of Theorem 3 using a completely different approach also appears in [12].

1. Generalization of a theorem by Rado

The following well-known geometric inequality will be one of our basic tools for proving uniqueness theorems.

MAXIMUM PRINCIPLE. *Suppose M_1 and M_2 are graphs of two C^2 functions f_1, $f_2 \colon D \to \mathbf{R}$ where*

(1) *D is the unit disk in $\mathbf{R}^2$,*
(2) *M_1 has zero mean curvature,*
(3) *M_1 and M_2 are tangent to the xy plane at the origin,*
(4) *$f_1(p) \le f_2(p)$ for all $p \in D$.*

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Then the mean curvature of M_2 at the origin in $\mathbf{R}^3$ is nonpositive with respect to the normal vector $(0, 0, -1)$. *Furthermore, if M_2 also has zero mean curvature, then $M_1 = M_2$.*

Actually the last statement in the above maximum principle is what is usually referred to as the maximum principle for minimal surfaces. It geometrically states that an open minimal surface M_1 in $\mathbf{R}^3$ can not locally lie on one side of an open minimal surface M_2 in $\mathbf{R}^3$ at a point of intersection. This last statement continues to hold if M_1 and M_2 have branch points.

The following theorem was proved by Rado [19] in the case the topology of the minimal surface was restricted to be the disk. Note that convex will always be used in the weak sense.

THEOREM 1. *Let γ be a Jordan curve in $\mathbf{R}^3$ with a one-to-one orthogonal projection onto a convex plane Jordan curve. Then γ is the boundary curve of a unique compact minimal surface and this surface is a graph over the plane.*

Proof. Suppose γ has a one-to-one orthogonal projection onto a convex Jordan curve γ' in the xy plane $\mathbf{R}^2$. The curve γ' is the boundary curve of a disk D in $\mathbf{R}^2$. Geometrically the curve γ lies on the boundary of the solid cylinder $C = \{(x, y, z) \mid z \in \mathbf{R} \text{ and } (x, y) \in D\}$. Since C is a convex set, the maximum principle for harmonic functions implies that the interior of any compact minimal surface $f\colon M \to \mathbf{R}^3$ with boundary curve γ lies in the interior of the solid cylinder C.

We will now show that $f(M)$ is a graph above the xy plane. If $f(M)$ is not the graph of a continuous function, then by elementary differential topology there is a point $q \in \operatorname{int}(D)$ with $\pi^{-1}(q) \cap f(M)$, consisting of at least two points where $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$ is orthogonal projection onto the xy plane. Let

$$p_1, p_2 \in \pi^{-1}(p) \cap f(M)$$

be two points with the z coordinate of p_2 greater than that of p_1. Then the surfaces $f(M)$ and $f(M) + (p_2 - p_1)$ intersect in the point p_2. Thus there is a nontrivial vertical translation $(0, 0, t)$ so that the intersection of $(f + (0, 0, t))(M)$ and $f(M)$ is nonempty.

Let

$$T = \max\{t \in \mathbf{R} \mid (f + (0, 0, t))(M) \cap f(M) \neq \emptyset\}.$$

Note that $T > 0$ and exists by the compactness of $f(M)$. Now let p be an element in the intersection $(f + (0, 0, T)(M)) \cap f(M)$. Since $(f + (0, 0, t))(\partial M)$ is disjoint from γ for all $t > 0$, p must correspond to two points $p_3, p_4 \in \operatorname{int}(M)$ with $f(p_3) = (f + (0, 0, T))(p_4) = p$. By our choice of T, the immersed surfaces $f(M)$ and $(f + (0, 0, T))(M)$ must locally lie on one side of each other near the point p. (Otherwise the surface $(f + (0, 0, T + \varepsilon))(M)$ would intersect $f(M)$ for some small $\varepsilon > 0$ which contradicts the definition of T.)

The maximum principle now implies that the surfaces

$$f(M) \quad \text{and} \quad (f + (0, 0, T))(M)$$

have the same images on open neighborhoods U_3 of p_3 and U_4 of p_4. Unique continuation properties for analytic maps imply that the intersection of these surfaces is also an open subset of $f(M)$. On the other hand, the intersection of the surfaces is also a closed subset of $f(M)$ in the interior of the surface. Since $f(M)$ is connected, the intersection must be all of $f(M)$. However, this contradicts the fact that $f(\partial M)$ is disjoint from $(f + (0, 0, T))(M)$. This contradiction proves the surface $f(M)$ is a graph and hence M is a disk D'.

If $f_1, f_2'' D' \to \mathbf{R}^3$ are two distinct minimal surfaces which are different graphs, then there is a nontrivial vertical translation $(0, 0, t)$ so that

$$(f_1 + (0, 0, t))(D') \cap f_2(D')$$

is nonempty. The argument used above gives a contradiction and this proves the theorem.

Remark. As the Gauss map for an open minimal surface is an open mapping, the projection map onto the xy plane of the unique minimal surface given in the above theorem has nondegenerate Jacobian on the interior of the minimal surface (see also [18]). Theorem 1 holds in the greater generality where the minimal surface with boundary curve γ is allowed to have branch points. If a Jordan curve γ in $\mathbf{R}^3$ has a central projection onto a convex plane curve, then a similar proof shows γ is the boundary curve of a unique branched minimal surface.

THEOREM 2. *If γ is a Jordan curve in $\mathbf{R}^3$ which has a monotonic orthogonal or central projection onto a convex plane Jordan curve, then γ is the boundary curve of a unique compact minimal surface and the interior is a graph over the plane.*

Proof. We will give a proof of the theorem in the case that the Jordan curve γ projects orthogonally and monotonically onto a convex plane Jordan curve $\bar{\gamma} \subset \mathbf{R}^2$. The proof of the second case of the theorem is similar and will be left to the reader.

Let $\pi\colon \mathbf{R}^3 \to \mathbf{R}^2$ be the orthogonal projection and $C = \pi^{-1}(\bar{\gamma})$ be the cylinder over $\bar{\gamma}$. The Jordan curve γ is the uniform limit of Jordan curves $\{\alpha_i \mid i \in N\}$ which project in a one-to-one way onto $\bar{\gamma}$ and which lie on the part of C above the Jordan curve γ. Similarly, γ is the uniform limit of Jordan curves $\{\beta_i \mid i \in N\}$ which project in a one-to-one way onto $\bar{\gamma}$ and which lie on the part of C below γ. By the previous theorem the curves α_i and β_i bound unique minimal surfaces which are disks D_{α_i} and D_{β_i}, respectively. Also these minimal disks are graphs over the disk Δ bounded by $\bar{\gamma}$.

Now pick three points p_1, p_2, p_3 on γ and points r_{i1}, r_{i2}, r_{i3} on α_i and points s_{i1}, s_{i2}, s_{i3} on β_i which converge to p_1, p_2, p_3, respectively. After picking the three points on α_i and three points x_1, x_2, x_3 on the boundary of the disk D, there is a unique conformal harmonic map $f_{\alpha_i}: D \to \mathbf{R}^3$ of finite area which parametrizes α_i and has image D_{α_i} with $f(x_k) = r_{ik}$ (see [8]). The family $\{f_{\alpha_i} | i \in N\}$ is equicontinuous (see for example [11]). Therefore a subsequence converges to a disk $f_\alpha: D \to \mathbf{R}^3$ with boundary γ. By lower semicontinuity of area, Area $(f_\alpha) \leq \inf (f_{\alpha_i})$. In fact, since f_{α_i} are disks of least area with respect to their boundary curves and the area of the annular regions of C bounded by α_i and γ have area going to zero as i gets large, the map f_α is a disk of least area with boundary curve γ. Similarly, the three point condition for minimal disks shows that the disks D_{β_i} give rise to a disk $f_\beta: D \to \mathbf{R}^3$ of least area with respect to the boundary curve γ.

Assertion 1. The interiors of the least area disks $D_\alpha = f_\alpha(D)$ and $D_\beta = f_\beta(D)$ are graphs over the interior of the disk Δ in $\mathbf{R}^3$ with boundary $\bar{\gamma}$.

Proof. The maps f_{α_i} converge smoothly to f_α on compact subsets contained in the interior of D (see [11]). Since the Gauss map G: interior $(D_\alpha) \to S^2$ is conformal and the images of the Gauss map for $f_{\alpha_i}(D)$ lie in the upper hemisphere of S^2, the image of G also lies in the upper hemisphere of S^2. Thus the projection of the interior of D_α onto the place $\mathbf{R}^2$ is a submersion. The standard monodromy argument shows that in this case the interior of D_α is a graph over the interior of the disk Δ bounded by $\bar{\gamma}$. The same argument shows that the interior of D_β is also a graph over the interior of Δ. This proves assertion 1.

Assertion 2. If $D_\alpha = D_\beta$, then γ is the boundary curve of a unique compact minimal surface.

Proof. Consider another compact minimal surface M with boundary γ. By convexity of the region $\pi^{-1}(\Delta)$ the interior of M is contained in the interior of $\pi^{-1}(\Delta)$. Since the disks D_{α_i} converge to D_α from above and the disks D_{α_i} converge to D_α from below, either some disk D_{α_i} or D_{β_i} intersects M.

Suppose $D_{\alpha_i} \cap M$ is nonempty. By definition of α_i, $\alpha_i + (0, 0, t)$ is disjoint from the boundary curve α of the surface M for all nonnegative t. As in the proof of Theorem 1, $D_{\alpha_i} + (0, 0, t)$ intersects M in its interior for a largest value T which by the maximum principle is impossible. Hence $D_{\alpha_i} \cap M$ is empty and a similar argument shows $D_{\beta_i} \cap M$ is empty. Since either $D_{\alpha_i} \cap M$ or $D_{\beta_i} \cap M$ is nonempty, we have a contradiction which proves Assertion 2.

We now show by using the least area property of D_α and D_β that $D_\alpha = D_\beta$. Since the disks D_{α_i} are disjoint and above D_{β_i} for all i, it is not difficult to prove that the limiting disk D_α lies "above" D_β and these disks intersect only along their common boundary curve γ. As the disks D_{α_i} and D_{β_i} are disks of least area with common boundary curve γ on the boundary of a convex set, the disjointness property of D_α and D_β also follows from Dehn's lemma in [11]. In fact we are now going to apply an approximation argument in the proof of Dehn's lemma in [11] to show that $D_\alpha = D_\beta$.

If D_α is different from D_β, then there is a negative T so that the translated disk $D_1 = (D_\alpha + (0, 0, T))$ and the disk $D_2 = D_\beta$ intersect transversely along some arc κ': $[0, 1] \to \mathbf{R}^3$ which lies in the interior of D_2 (see Lemma 2 in [11]). By compactness of the embedded disks D_1 and D_2 we may assume the intersection of a small ball B^3, centered at $\kappa'(\frac{1}{2})$, with each of the disks are subdisks E and F, respectively. We may also assume that E and F intersect transversely along an arc κ: $[0, 1] \to B^3$ with $\kappa(0), \kappa(1) \in \partial B^3$ and that intersection of E and F on ∂B^3 consist of smooth Jordan curves r and s, respectively.

Let r_1, r_2 and s_1, s_2 be the subarcs of r and s, respectively, which join one point of the intersection of r and s to the other point of intersection. The arc κ divides E and F into closed subdisks E_1, E_2 and F_1, F_2, respectively, with r_1, r_2; s_1, s_2 being arcs of part of the boundary of the respective subdisks.

Let $A_{ij} = \text{Area}\ (E_i \cup F_j)$, B_{ij} be the area of a solution to Plateau's problem for the Jordan curve $r_i s_j^{-1}$ and $\varepsilon = \inf \{(A_{ij} - B_{ij}) \mid 1 \le i, j \le 2\}$. Since the disks E and F intersect transversely along κ, ε must be positive (see Lemma 6 of [11] for a rigorous proof).

As the curves β_i converge uniformly to γ, we may assume that i is large enough so that the annulae of least area bounded by β_i and γ are each less than min $(\varepsilon/4, \text{Area}\ (E))$ and the area of these annular regions converges to zero as i gets large. The assumption that the area is less than Area (E) is to guarantee the existence of least area annuli with boundaries being the translated curves $\beta_i + (0, 0, T)$ with r. This follows from Morrey's condition for an annulus of least area to exist by checking the required area inequality directly.

Since the curves $\gamma + (0, 0, T)$ and r bound the unique annular solution $D_1 \sim$ interior (E) to Plateau's problem, the smooth convergence argument used in the proof of Theorem 5 in [11] shows we may assume, after picking a subsequence, that there are annular solutions F_i: $\Omega \to \mathbf{R}^3$ to Plateau's problem for $\beta_i + (0, 0, T)$ which converge uniformly to the original unique embedded solution to Plateau's problem for the curve $\gamma + (0, 0, T)$ and r. Since the convergence of F_i is uniform in C^∞ norm near r, we may assume for large i, $F_i(\Omega)$ is embedded near the smooth curve r, transverse to D_1 near ∂B^3, and $F_i(\Omega^0)$ is disjoint from B^3. Dehn's lemma for planar domains in [11] shows that F_i are embedded for large i.

Fix i large enough so that F_i has the above properties. Consider the continuous piecewise differential map f_i: $D \to C = \pi^{-1}(\Delta)$ which is the embedding obtained by gluing the embedded annulus $F_i(\Omega)$ to the embedded disk E along the common boundary curve r. After a small C^2 perturbation of f_i outside of B^3 there is a new embedding f'_i: $D \to C$ with

$$\text{Area}\ (f'_i) \le \text{Area}\ (f_i) + \varepsilon/2,$$

$f'_i(\partial D) = \beta_i + (0, 0, T)$, f'_i is transverse to D_1 in interior $(C) - B^3$. As f'_i is transverse to D_1 and these disks only intersect in their interiors, the intersection of $f'_i(D)$ and D_1 consists of a finite collection of Jordan curves. Let K be the Jordan curve in this intersection which contains the arc $\kappa([0, 1]) = E \cap F$. The

Jordan curve K bounds a subdisk Δ_i' on $f_i'(D)$ and a subdisk Δ_i on D_1. As D_1 is a disk of least area, a cut and paste argument shows Area $(\Delta_i) \leq$ Area (Δ_i'). Hence by forming the disk D_i' by gluing together $(f_i'(D) - \Delta_i') \bigcup_K \Delta_i$, we have a piecewise smooth disk with boundary $\beta_i + (0, 0, T)$ and

$$\text{Area } (D_i') \leq \text{Area } (f_i) + \varepsilon/2.$$

By our choice of ε, we can decrease the area of D_i' in B^3 by at least ε. This shows that there is a solution g_i: $D \to \mathbf{R}^3$ to Plateau's problem for β_i with area less than Area $(f_i) - \varepsilon/2$. Since in our case

$$\lim_{i \to \infty} (\text{Area } (g_i)) = \text{Area } (D_2),$$

we have a contradiction to the previous inequality and the fact that

$$\lim_{i \to \infty} (\text{Area } (f_i)) = \text{Area } (D_2).$$

This contradiction shows $D_\alpha = D_\beta$ as was to be proved. The theorem now follows from Assertion 2.

2. Positive envelopes

In the previous section we used the fact that the interior of a compact minimal surface in $\mathbf{R}^3$ is always contained in the interior of the convex hull of its boundary. It is well known that this "convexity property" continues to hold in $\mathbf{R}^3$ in the following case. If the boundary of a minimal surface is contained in a star-shaped sphere of positive mean curvature, then the interior of the minimal surface is contained in the interior of the ball bounded by the sphere (this is well known). The following lemma generalizes this convexity property of surfaces of positive mean curvature.

LEMMA 1. *Let B be a compact region of $\mathbf{R}^3$ with C^2-boundary having positive mean curvature. Suppose f: $M \to \mathbf{R}^3$ is a continuous conformal harmonic map of a compact Riemann surface with boundary such that $f(M)$ is contained in B and $f \mid \partial M$ gives a monotonic parametrization of a collection $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of C^2 Jordan curves on the boundary of β. Then*

(1) f (interior M) $\subset$ interior (B),
(2) *f is an immersion in a neighborhood of ∂M and is transversal at ∂B.*

Proof. Part (1) follows immediately from the inequality in Lemma 1. We refer the reader to Section 3 of [8] for many of the definitions and computational details used in the proof of part (2). Also we refer the reader to [11] and to page 366 in [15] where a proof is given in the case the curve lies on the boundary of a convex set. The reader should note that f is a map of class C^2 on all of M by the boundary regularity theorem in [6].

Let δ be the distance function from ∂B. For each point $x \in B$ which is close to ∂B, choose an orthonormal frame e_1, e_2, e_3 so that e_3 is tangential to the unique geodesic which realizes the distance from x to ∂B. Then

$$\delta_{33} = e_3 e_3(\delta) - (\nabla_{e_3} e_3)\, \delta = 0.$$

Further computations show δ_{13}, δ_{31}, δ_{23} and δ_{32} are also zero. By the first variation formula for area (in the unit normal direction), one immediately has $\delta_{11}(x) + \delta_{22}(x) < 0$ when x is close to ∂B. In particular $\Delta\delta = \delta_{11} + \delta_{22} + \delta_{33} < 0$.

Now consider the function $g = -\delta \circ f$. As f is conformal, we may choose an orthonormal frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ at x so that

$$f_*(\partial/\partial x) = c\tilde{e}_1, \qquad f_*(\partial/\partial y) = c\tilde{e}_2.$$

Then by harmonicity of the coordinate functions f_j,

$$\begin{aligned}\Delta g &= -\sum_{i=1}^{2} (\delta \circ f)_{ii} = -\sum_{k,l,i} \delta_{kl} f_i^k f_i^l \\ &= -c^2[H(\delta)(\tilde{e}_1, \tilde{e}_1) + H(\delta)(\tilde{e}_2, \tilde{e}_2)]\end{aligned}$$

where $H(\delta)$ is the Hessian of δ at x.

Now consider a point $x \in \partial M$ where the map $f\colon M \to B$ is not transverse to ∂B. Since the Gauss map of f is well defined on M (see [8]), even at a branch point, there is a well defined tangent plane at $f(x)$. It follows from the local ramification properties of branch point ([16] or [18]) that if x is a branch point, then the tangent plane at $f(x)$ is the tangent plane to ∂B. Hence the lemma will be proved once we have shown the tangent plane at $f(x)$ is transverse to the boundary of B.

If the tangent plane to ∂B at $f(x)$ is not transverse, then the previous calculations show that

$$\begin{aligned}\Delta g(x) &= -c^2[H(\delta)(\tilde{e}_1, \tilde{e}_1) + H(\delta)(\tilde{e}_2, \tilde{e}_2)] \\ &= -c^2[H(\delta)(e_1, e_1) + H(\delta)(e_2, e_2)] \geq 0.\end{aligned}$$

Since $(H(\delta)(e_1, e_1) + H(\delta)(e_2, e_2)) = \delta_{11} + \delta_{22}$ is strictly negative near ∂B and the map f is of class C^2, then $\Delta g \geq 0$ for y close to x.

We now claim that $\partial g(x)/\partial n \neq 0$ for $x \in \partial M$ where n is the inward normal on the boundary of M. This will prove that $f(M)$ is transversal to ∂B along $f(\partial M)$. Indeed it shows by the chain rule that for the normal N to ∂B,

$$\left(N, \frac{\partial f}{\partial n}\right) = \left(\nabla_\delta^+, \frac{\partial f}{\partial n}\right) = \frac{\partial g}{\partial n} \neq 0.$$

This will show that f is an immersion near M and transversal to ∂B.

Let D be a small disk in M so that $\partial D \cap \partial M = \{x\}$ and $f(D)$ lies in a neighborhood of ∂B so that $\Delta g \geq 0$. The Hopf maximum principle on page 68 of [17] now shows that $\partial g(x)/\partial n \neq 0$. This completes the proof of Lemma 1.

DEFINITION. A Jordan curve γ in $\mathbf{R}^3$ has a positive envelope of surfaces $\{M_t \mid t \in [0, 1]\}$ if:

(1) M_t is a continuous family of embedded compact connected C^2-surfaces in $\mathbf{R}^3$ which have positive mean curvature with respect to the inward normal on the compact regions of $\mathbf{R}^3$ that they bound.

(2) $\gamma \subset M_0$ and γ is disjoint from M_t for $t > 0$.

(3) The convex hull of γ is contained in the interior of the compact region of $\mathbf{R}^3$ bounded by M_1.

Example. The simplest example of a positive envelope of surfaces is to take a star shaped sphere S_0 of positive mean curvature and then consider the natural family of positive mean curvature spheres which arise from S_0 by taking outward radial expansions from the star point (see the figure below).

PROPOSITION 1. *Suppose a Jordan curve γ in $\mathbf{R}^3$ has a positive envelope of surfaces $\{M_t \mid t \in [0, 1]\}$. Then every compact minimal surface S with boundary curve γ is contained in the compact region C of $\mathbf{R}^3$ bounded by the surface M_0. Furthermore, the interior of S is contained in the interior of C.*

Proof. Separation theorems in algebraic topology imply that the curve γ is contained in the interior of the compact region of $\mathbf{R}^3$ bounded by any surface M_t for $t > 0$. Suppose now that there is a minimal surface S with boundary curve γ whose interior is not contained in the interior of the compact region bounded by M_0.

Consider $\sigma = \max \{t \in [0, 1] \mid M_t \cap S \neq \phi\}$. By compactness of M and the unit interval, σ exists. Since the convex hull of γ is contained in the interior of the compact region bounded by the surface M_1, we know that S is contained in

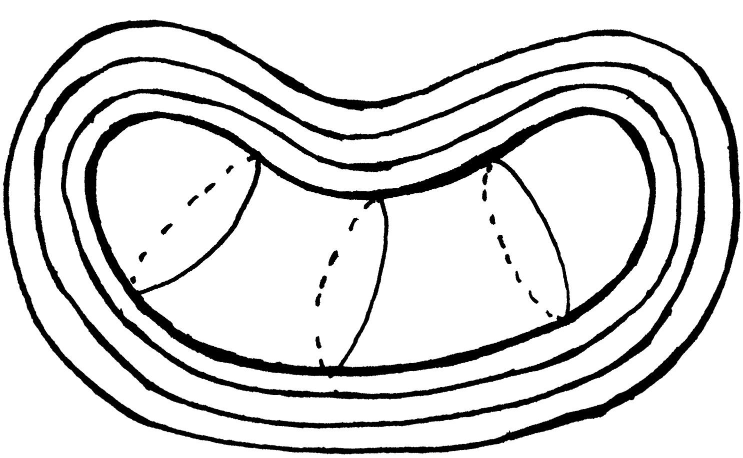

FIG. 1

the interior of the compact region bounded by M_1 and hence $\sigma \neq 1$. Lemma 2 implies the proposition if $\sigma = 0$. Thus we may assume that σ is contained in the interior of $[0, 1]$.

Since the family of surfaces M_t is continuous and disjoint from γ for $t > 0$, the surface S is contained in the compact region of $\mathbf{R}^3$ bounded by M_σ and S intersects M_σ from the "inside" of M_σ at an interior point $p \in S$. Since S lies inside the compact region bounded by M_σ, and with respect to the inward normal to M_σ at p and M_σ has positive mean curvature, the maximum principle gives a contradiction and proves the proposition.

In [24], F. Tomi and T. Tromba proved that the space $\mathscr{D}$ of immersed minimal disks with C^2-boundary curves and without branch points on the boundary curves forms a C^2 Banach manifold. Furthermore they proved the natural projection map $P: \mathscr{D} \to \mathscr{C}$ to the manifold of C^2 Jordan curves in $\mathbf{R}^3$ is a C^2 Fredholm map of index zero. By the Sard-Smale theorem [21] almost every $\gamma \in \mathscr{C}$ is a regular value for P.

In [24], Tomi and Tromba use these facts about the projection map P to deduce the existence of an embedded minimal disk for a C^2-Jordan curve on the boundary of a convex set in $\mathbf{R}^3$. Using their technique we will show that one can achieve the existence of more than one embedded disk if the curve is the boundary curve of more than one minimal surface.

THEOREM 3. *Suppose α_0 is a C^2-Jordan curve in $\mathbf{R}^3$ which is contained in a positive envelope of spheres $\{S_t \mid t \in [0, 1]\}$. If α_0 is a regular value for $P: \mathscr{D} \to \mathscr{C}$ and α_0 is the boundary curve of more than one compact minimal surface, then α_0 is the boundary curve of at least three embedded minimal disks, one of which is the solution to Plateau's problem.*

Proof. By Proposition 1 we know that every minimal surface with boundary curve α_0 is contained in the ball B bounded by the sphere S_0. Dehn's lemma in [11] shows that any solution to Plateau's problem in $\mathbf{R}^3$ will be an embedded minimal disk in B. It now only remains to prove that α_0 is the boundary curve of at least three embedded minimal disks.

Let $p \in S_0 - \alpha_0$ be a point on the sphere S_0 with positive Gaussian curvature and let M be an arbitrary compact minimal surface with boundary curve α_0. Since M and the disks $P^{-1}(\alpha_0)$ form a compact subset of the ball β bounded by the sphere S_0 which only intersects S_0 along the curve α_0, the point p lies a positive distance ε from M and the disks $P^{-1}(\alpha_0)$.

Since p is a point of positive Gaussian curvature on S_0, there is a C^2-plane Jordan curve α_{-1} on S_0 close to p which bounds a plane disk D with all the points of D of distance less than ε from p. The curve α_{-1} on S_0 is a regular value for $P: \mathscr{D} \to \mathscr{C}$ by Tromba's transversality theorem in [25] and α_{-1} is the boundary value of a unique branched minimal disk which is embedded and disjoint from M and the disks $P^{-1}(\alpha_0)$.

By the Jordan curve theorem α_0 disconnects S_0 into two disks D_1 and D_2. Suppose the curve α_{-1} is contained in D_1 and let p_2 be a point in int (D_2). As in the previous paragraph, the point p_2 is a positive distance from M and the disks $P^{-1}(\alpha_0)$.

Choose a small C^2 Jordan curve α_1 around p_2 which has total curvature less than 4π and such that the convex hull of α_1 is disjoint from M and $P^{-1}(\alpha_0)$. Such a curve α_1 can be obtained as the intersection curve of a sufficiently small sphere centered at p_2 with the sphere S_0.

The uniqueness theorem of Nitsche states that if α_1 is analytic, then α_1 spans a unique branched minimal disk or equivalently $P^{-1}(\alpha_1)$ contains one point. The argument in [15] shows Nitsche's theorem continues to hold if the curve is of class C^4. In fact it can be shown that Nitsche's theorem holds in the case of C^2 curves. Since this fact is not explicitly written down, the reader may assume that all of our surfaces and curves are of class C^4. It follows from the transversality theorem in [25] that α_1 is a regular value for $P: \mathscr{D} \to \mathscr{C}$.

The curves α_{-1} and α_0 are the boundary curves of a C^2-annulus A_{-1} in S_0. By the conformal classification of annular domains [2] there is a conformal C^2 diffeomorphism $h: A_{-1} \to A'$ where A' is an annulus in the plane with concentric circles for boundary.

Hence the annular region A_{-1} has a C^2 foliation $\{\alpha_t \mid t \in [-1, 0]\}$ by circles. Similarly the annular region bounded by the Jordan curves α_0 and α_1 has a C^2 foliation by circles $\{\alpha_t \mid t \in [0, 1]\}$. By being careful one can show these two C^2 foliations give rise to a C^2 foliation of the annular region A bounded by the curves α_{-1} and α_1. This foliation induces a natural C^2 embedding $\alpha: [-1, 1] \to \mathscr{C}$ such that $\alpha(-1) = \alpha_{-1}$, $\alpha(0) = \alpha_0$, $\alpha(1) = \alpha_1$ and the variation vector field at the circle α_t is everywhere nonzero.

However, the curve $\alpha: [-1, 1] \to \mathscr{C}$ might not be transverse to the map $P: \mathscr{D} \to \mathscr{C}$. As $\alpha(-1)$, $\alpha(0)$ and $\alpha(1)$ are regular values and the derivative of α is never zero, α is transverse to P at the points $\{-1, 0, 1\}$. The Smale-Sard theorem [21] shows there is small C^2-perturbation α' of α that is transverse to P and which agrees with α at the points $\{-1, 0, 1\}$.

In fact the compactness of the sets M and $P^{-1}(\alpha_0)$ together with the fact that the variational vector field of α_t is everywhere nonzero can be used to prove the following: there is a C^2-perturbation α' of α such that:

(1) α' agrees with α in a neighborhood of $\{-1, 0, 1\}$.

(2) $\bar{S} = (\bigcup_{t \in [-1,1]} \alpha_t') \cup (S_0 - A)$ is a C^2-sphere with positive mean curvature.

(3) The interior of M and the interiors of the disks in $P^{-1}(\alpha)$ are contained in the interior of the ball bounded by the sphere $\bar{S}$.

(4) The curves α_t' give rise to a foliation of the annular region on $\bar{S}$ bounded by the curves $\alpha_{-1} = \alpha'(-1)$ and $\alpha_1 = \alpha'(1)$.

(5) α' is transverse to P.

Since $\alpha'[-1, 1] \subset \mathscr{C}$ is an embedded submanifold and α' is transverse to the

C^2 map $P: \mathscr{D} \to \mathscr{C}$ which is Fredholm of index zero (see [25]), the implicit function theorem states that $\mathscr{M} = p^{-1}(\alpha'[-1, 1])$ is a one-dimensional submanifold of $\mathscr{D}$ and has boundary $P^{-1}(\alpha_{-1})$ and $P^{-1}(\alpha_1)$. Here $P^{-1}(\alpha_{-1})$ and $P^{-1}(\alpha_1)$ are the unique minimal disks bounded by the curves α_{-1} and α_1. Lemma 1 or Lemma 2 easily implies that the subset $\bar{\mathscr{M}} \subset \mathscr{M}$ consisting of disks in $\mathscr{M}$ which are contained in the ball $\bar{B}$ bounded by the sphere $\bar{S}$ is the submanifold of $\mathscr{M}$ which consists of all connected components of $\mathscr{M}$ which contain a minimal disk in $\bar{B}$. (Actually $\bar{\mathscr{M}} = \mathscr{M}$.)

Since a limit of minimal disks in $\bar{B}$ bounding a C^2-Jordan curve γ on $\bar{S}$ is another minimal disk in $\bar{B}$ without branch points on the boundary, it follows that $\bar{\mathscr{M}}$ is a one-dimensional manifold. Our choice of the curves α_{-1} and α_1 shows that the boundary of $\bar{\mathscr{M}}$ consists of the unique minimal disks bounded by the curves α_{-1} and α_1. It follows from [24] that the manifold $\bar{\mathscr{M}}$ contains exactly one component which is an interval $T \subset \bar{\mathscr{M}}$ which joins the minimal disk bounded by α_{-1} to the minimal disk bounded by α_1 and the disks in T are embedded. The following geometric picture describes our situation.

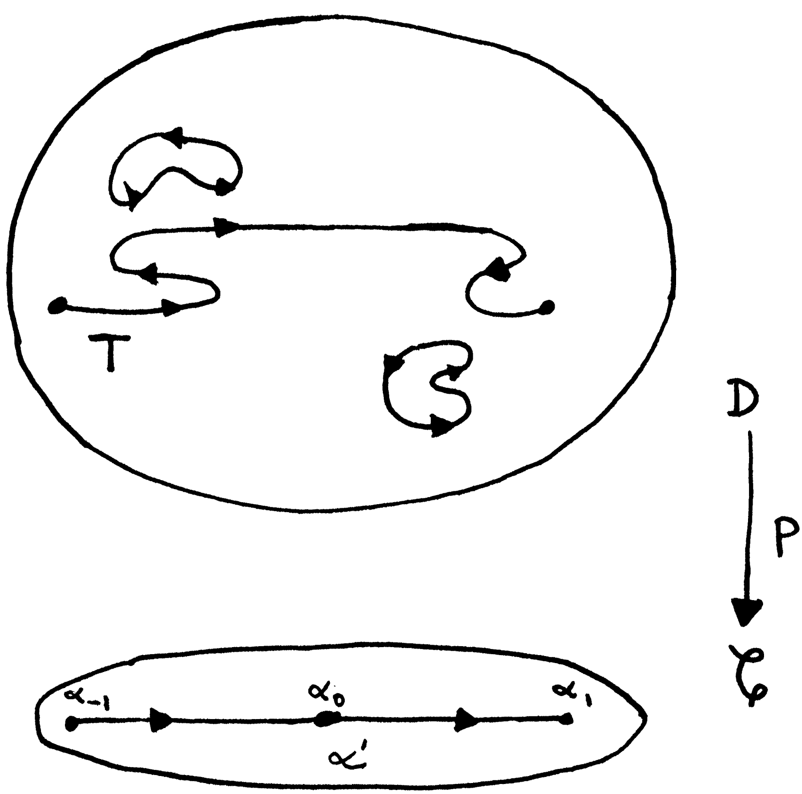

FIG. 2

Since an odd number of disks in T have boundary curve α_0 (see Figure 2), and all these disks are embedded, we may assume that there is a unique disk $\beta \in T$ with boundary curve α_0 (or else there would already by three minimal disks). We will now show that the minimal surface M is the minimal disk β.

First define

$$\sigma_1 = \min \{\sigma \in T = [0, 1] \mid \sigma \cap M \neq \phi\},$$

$$\sigma_2 = \max \{\sigma \in T = [0, 1] \mid \sigma \cap M \neq \phi\}.$$

Here the point $0 \in T$ corresponds to the unique minimal disk with boundary curve α_{-1} and $1 \in T$ corresponds to the unique minimal disk with boundary curve α_1 and β corresponds to $1/2 \in T = [0, 1]$. By compactness of M and the interval the numbers σ_1 and σ_2 exist.

Since the disks corresponding to the numbers $0, 1 \in T$ are disjoint from M, it follows that σ_1 and σ_2 lie in the interior of T. Since β is the unique minimal disk in T with boundary curve γ, if σ_1 (and likewise σ_2) is different from 1/2, then any point p of intersection of σ_1 and M must be a point in int $(\sigma_1) \cap$ int (M). At this point p the surface σ locally lies on one side of the surface M. Hence the maximum principle implies the surface σ_1 and M must agree on an open set. However, it now follows that M and σ_1 have the same image and hence the same boundary curve α_0. This shows σ_1 and similarly σ_2 are the same minimal disk $\beta \in T$. In turn this implies by continuity that $M = \beta$, as was to be proved.

Since M is arbitrary, we have shown that if α_0 bounds more than one minimal surface, then there are at least three embedded minimal disks in T with boundary curve α_0. This completes the proof of the theorem.

Remark. Using Dehn's lemma, the author and S. T. Yau have shown that if a continuous Jordan curve is contained in a positive envelope of spheres and is the boundary curve of more than one minimal surface in $\mathbf{R}^3$, then γ is the boundary curve of two embedded stable minimal disks. It follows from the arguments in the proof of Theorem 3 that a generic rectifiable curve which is the boundary curve of more than one minimal surface is the boundary curve of at least three embedded minimal disks.

3. Two more uniqueness theorems

The first uniqueness theorem in this section is related to a theorem of Nitsche [14] which states in its proper generalization that a C^2 Jordan curve in $\mathbf{R}^3$ with total curvature less than or equal to 4π is the boundary curve of a unique branched minimal disk which in fact does not have branch points. We will first give a generalization of this result.

DEFINITION. The total interior or I curvature of a unit speed C^2-Jordan curve $\gamma\colon [a, b] \to M$ where M is a surface in $\mathbf{R}^3$ with unit normal vector field N

is defined to be

$$I = \int_a^b I(t)\,dt \quad \text{where } I(t) = \begin{cases} \|\gamma''(t)\| & \text{if } \gamma''(t)\cdot N \geq 0, \\ \|\gamma''(t) - (\gamma''(t)\cdot N)N\| & \text{if } \gamma''(t)\cdot N \leq 0. \end{cases}$$

LEMMA 2. *Suppose γ is a C^2-Jordan curve contained in a positive envelope of spheres $\{S_t \mid t \in [0, 1]\}$ and the total I curvature of γ is less than or equal to 4π with respect to the inward normal on the sphere S_0. Then γ is the boundary curve of a unique branched minimal disk which in fact does not have branch points.*

Proof. By Proposition 1 in the previous section, a minimal immersion $f\colon D \to \mathbf{R}^3$ with boundary curve γ will be contained in the ball bounded by the sphere S_0. The Gauss-Bonnet formula and the fact that the Gauss curvature of a minimal surface is nonpositive give the formula

$$1 + \frac{1}{2\pi}\int_D |K|\,dA = \frac{1}{2\pi}\int_\gamma \kappa_g\,d\gamma.$$

On the other hand, the geodesic curvature $\kappa_g(t) \leq I(t)$ by our definition of $I(t)$ and the fact that $f(D)$ is contained inside the ball bounded by S_0. Hence if the total I curvature of γ is less than or equal to 4π, then

$$1 + \frac{1}{2\pi}\int_D |K|\,dA \leq 2.$$

By Lemma 2 in the previous section, a branched minimal immersion $g\colon D \to \mathbf{R}^3$ with boundary curve γ does not have branch points on the boundary. The existence of a branch point in the interior of D for the map g contributes at least 1 to the left-hand side of the above inequality (see p. 321 of [14] for this calculation) which would imply $\iint_D |K|\,dA = 0$. This in turn would imply that D is embedded in a plane and the lemma is known in this case.

Hence every branched minimal immersion $f\colon D \to \mathbf{R}^3$ with boundary curve γ in fact does not have branch points. Another calculation in [14] shows that we may in fact assume that $\iint_D |K|\,dA < 2\pi$. By the stability theorem of L. Barbosa and M. do Carmo the map $f\colon D \to \mathbf{R}^3$ is stable. Hence if there are two minimal immersions f_1 and f_2 of the disk having boundary curve γ, then the two minimal disks are strict local minima for area in the variational sense. If the immersions f_1 and f_2 were local minima in the C^0 sense, then the theorem of Shiffman [20] would imply the existence of an unstable minimal disk with boundary curve γ.

In [14] Nitsche shows that the minimal immersions f_1 and f_2 will be local minima in the C^0 sense if the curve γ is analytic. However, the results of F. Tomi [23] and the argument in [5] show that γ would bound an unstable minimal disk in the case γ is of class C^2. Since the I curvature of γ is less than or equal to 4π, the above discussion shows γ can only be the boundary curve of a strictly stable minimal disk. This contradiction proves the lemma.

THEOREM 4. *Suppose γ is a C^2-Jordan curve with a positive envelope of spheres $\{S_t \mid t \in [0, 1]\}$. If the total I curvature of γ is less than or equal to 4π with respect to the inward normal on the sphere S_0, then γ is the boundary curve of a unique compact branched minimal surface in $\mathbf{R}^3$. Furthermore, this surface is an embedded disk.*

Proof. That γ bounds an embedded minimal disk follows from Dehn's lemma. If γ bounds more than one minimal surface and γ is a regular value for $P: \mathscr{D} \to \mathscr{C}$, then Theorem 2 in the previous section shows γ is the boundary curve of at least three embedded minimal disks. The calculations in the previous lemma show that in fact every branched minimal disk bounding γ is free of branch points and is stable. Hence by Tromba's stability theorem γ is a regular value for $P: \mathscr{D} \to \mathscr{C}$. Since γ now is the boundary curve of more than one minimal disk, the previous lemma gives a contradiction which proves the theorem.

Recently a number of authors ([4] and [25]) proved that if γ is a C^2-Jordan curve in a plane P in $\mathbf{R}^3$, then there is a C^2-neighborhood N of γ in the space $\mathscr{C}$ of C^2 Jordan curves such that every element in N bounds a unique minimal disk. An elementary proof of this fact using the idea of proof in Theorem 1 is as follows. The proof of Theorem 1 implies the result for a given γ' if every minimal disk bounding γ' is a graph over the plane P. If N does not exist, then there is a sequence γ_i which converges in a C^2-manner to γ and which bounds minimal disks D_i which are not graphs. By uniqueness of the minimal disk D with boundary γ and the usual C^2 convergence theorems, D_i converges to D in the C^2-norm. (Note we must also use the three point condition.) Hence eventually the orthogonal projection of D_i on the plane is a submersion with the projection of γ_i one-to-one. The usual monodromy theorem shows D_i is actually a graph. This contradiction shows some N must exist.

We now give a generalization of this uniqueness theorem which only uses the simplest geometric principles and which shows uniqueness of minimal surfaces of varying topological type as well. Since the method of proof is quite explicit, one can actually compute an ε from the geometry of a C^2-plane curve γ in the unit disk so that any ε C^2-perturbation of γ bounds a unique compact minimal surface which is a graph over the plane.

THEOREM 5. *Suppose γ is a C^2-plane Jordan curve. Then there is an $\varepsilon > 0$ so that any Jordan curve in $\mathbf{R}^3$ which is ε close to γ in the C^2-norm is the boundary curve of a unique compact minimal surface. Furthermore, this minimal surface is a graph over the plane.*

Proof. Let $\gamma\colon [0, 1] \to \mathbf{R}^3$ be a C^2-Jordan curve contained in the xy plane P. What we will show is that if $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ is a C^2 diffeomorphism which is sufficiently close to the identity map in the C^2 topology, then $f(\gamma)$ is the boundary curve of a unique minimal surface which is a graph over the xy plane. It will be sufficient to prove this since a neighborhood of $\mathrm{id}_{\mathbf{R}^3}$ in the space of C^2-diffeomorphisms generates a neighborhood of γ in $\mathscr{C}$ by composition with γ.

The main step in the proof of this theorem will be to construct what is essentially a positive envelope of spheres for the Jordan curve γ. We will now carry out this construction.

Suppose that γ is a C^2 curve in the xy plane which is contained in the interior of the unit disk D_1. Let $F = \{\gamma_t \mid t \in [0, 1]\}$ be a C^2-foliation by circles of the annular region A in the xy plane bounded by the Jordan curves $\gamma_0 = \gamma$ and let γ_1 be the boundary circle of the disk D_2 of radius 2. Then there exists an $\varepsilon > 0$ that depends on this foliation such that for any point p with distance less than 10ε from the circle γ_t there is a unique geodesic arc which minimizes the distance from p to γ_t. Such an ε exists by the following argument. For each t there exists a largest $\varepsilon(t)$ such that for any point p with distance less than $10\varepsilon(t)$ from γ_t there exists a unique shortest geodesic which joins p to γ_t. Since $\varepsilon(t)\colon [0, 1] \to \mathbf{R}$ is continuous and positive and $[0, 1]$ is compact, the function $\varepsilon(t)$ has a positive minimum which is our ε. Note that $\varepsilon < 1/10$.

Now consider the associated foliation $F = \{\alpha_t(u) = \gamma_t(u) + \varepsilon N(\gamma_t(u))\colon S^1 \to \mathbf{R}^2\}$ where $N(\gamma_t(u))$ is the inward unit normal of the curve γ_t at $\gamma_t(u)$ with respect to the disk it bounds. The reader should note that it follows from the focal properties of geodesics (see p. 34 of [13]) that 9ε is less than the minimum radius of curvature of the curves γ_t. Similarly, 8ε is less than the minimum radius of curvature of the curves α_t. Thus the following embedded tori have positive mean curvature with respect to the inward normal on the compact regions that they bound. In fact they have positive mean curvature greater than one.

For each $t \in [0, 1]$ we define $T_t^+ : S^1 \times S^1 \to \mathbf{R}^3$ and $T_t^- : S^1 \times S^1 \to \mathbf{R}^3$ by

$$\begin{aligned}
T_t^+(\theta_1, \theta_2) &= \alpha_t(\theta_1) + (0, 0, \varepsilon) + \sqrt{2\varepsilon} \\
&\qquad \times [-\cos(\theta_2) N(\gamma_t(\theta_1)) + (0, 0, \sin(\theta_2))] \\
T_t^-(\theta_1, \theta_2) &= \alpha_t(\theta_1) - (0, 0, \varepsilon) + \sqrt{2\varepsilon} \\
&\qquad \times [-\cos(\theta_2) N(\gamma_t(\theta_1)) + (0, 0, \sin(\theta_2))].
\end{aligned}$$

Note that the curve γ_t lies on the tori T_t^+ and T_t^-.

Now consider the piecewise smooth spheres S_t defined as follows. Let D_t^+ be the disk in the plane $P_+ = \{(x, y, (\sqrt{2\varepsilon} - \varepsilon)) \mid x, y \in \mathbf{R}\}$ bounded by the curve $\alpha_t + (0, 0, \sqrt{2\varepsilon} - \varepsilon)$. Let D_t^- be the disk in the plane

$$P = \{(x, y, (\varepsilon - \sqrt{2\varepsilon})) \mid x, y \in R\}$$

bounded by the curve $\alpha_t + (0, 0, \varepsilon - \sqrt{2\varepsilon})$. Let

$$A_t^+ = \{T_t^-(\theta_1, \theta_2) \mid \pi/4 \le \theta_2 \le \pi/2\}$$

and

$$A_t^- = \{T^+(\theta_1, \theta_2) \mid -\pi/2 \le \theta_2 \le -\pi/4\}.$$

Now $S_t = D_t^+ \cup D_t^- \cup A_t^+ \cup A_t^-$ defines a piecewise smooth sphere with positive mean curvature greater than one on the annuluar regions A_t^+ and A_t^- and the associated tangent planes to A_t^+ and A_t^- along γ_t have an angle of $\pi/4$ radians with the vertical vector $(0, 0, 1)$.

Let $G = \{f\colon \mathbf{R}^3 \to \mathbf{R}^3 \mid f$ is a diffeomorphism of class C^2 which is the identity outside of the ball B of radius two$\}$. Define a metric on G by

$$d(f_1, f_2) = \max_{x \in B} \left| \sum_{i=0}^{2} \|(f_1 - f_2)^i(x)\|_i \right|$$

where $f^i(x)$ denote the ith derivative of f and $\| \quad \|_i$ is sup norm for the ith derivative of a map $g\colon \mathbf{R}^3 \to \mathbf{R}^3$. Let N be the neighborhood of $\mathrm{id}_{\mathbf{R}^3} \in G$ such that $f \in N$ if and only if

(1) $\|f(x) - x\| < (\sqrt{2\varepsilon} - \varepsilon)/4$ for all $x \in B$.
(2) $(Df_x(v)/\|Df_x(v)\|, v) > \cos(\pi/4)$ for all $x \in B$ and $v \in \mathbf{R}^3$ with $\|v\| = 1$.
(3) $f(\operatorname{int}(A_t^+) \cup \operatorname{int}(A_t^-))$ has positive mean curvature.

It is not too difficult to check that N contains an open neighborhood of $\mathrm{id}_{\mathbf{R}^3}$ in G.

Now suppose M is a minimal surface with boundary curve $f \quad \gamma$ where $f \in N$. We will first show that M is contained in the ball $\tilde{B}$ bounded by the sphere $f(S_0)$. If M is not contained in $\tilde{B}$, then $\sigma = \max\{t \in [0, 1] \mid M \cap f(S_t) \neq \phi\}$ is greater than 0. Condition (1) above for f implies that the convex hull of $f \quad \gamma$ is contained in the region between the surfaces $f(P_+)$ and $f(P_-)$ where P_+ and P_- were defined previously in the proof. This shows that the surface M intersects the sphere $f(S_\sigma)$ in the region $f(A_\sigma^+ \cup A_\sigma^-)$. This fact and condition (1) show that M is contained inside the interior of the ball bounded by the sphere S_1 and hence σ is less than 1.

By condition (2) on f and the fact that the interior angles between A_σ^+ and A_σ^- are $\pi/2$ radians, the interior angles along the surfaces $f(A_\sigma^+)$ and $f(A_\sigma)$ are less than π. This fact together with Lemma 1 in Section 2, condition (3), and the discussion in the previous paragraph imply that if $\sigma = 0$, i.e., $M \subset \tilde{B}$, then $\operatorname{int}(M) \subset \operatorname{int}(\tilde{B})$.

On the other hand, if $\sigma \in (0, 1)$ and $p \in M \cap f(S_\sigma)$, then because the interior angles along the surfaces $f(A_\sigma^+)$ and $f(A_\sigma^-)$ are less than π, we have

$$p \in f(\operatorname{int}(A_\sigma^+) \cup \operatorname{int}(A_\sigma^-)).$$

Condition (3) together with Lemma 1 in Section 2 and the fact that σ occurs in the interior of $[0, 1]$ now give an immediate contradiction (see also the Proof of

Proposition 1 in Section 2 for a similar argument). Thus we have shown that the interior of M is contained in the interior of the ball $\tilde{B}$ bounded by the sphere $f(S_0)$.

The argument given in the Proof of Theorem 1 in Section 1 will show that M is a graph and unique if we can prove that the nontrivial vertical translates of $f \circ \gamma$ are disjoint from M. By condition (2) on f and our choices of the annular surfaces A_σ^+ and A_σ^-, the orthogonal projection of $f(A_\sigma^+ \cup D^+)$ and of $f(A_\sigma^- \cup D^-)$ on the xy plane are submersions. By condition (1) and (2) on f and our original choice of ε, we can be sure that $f(\gamma)$ projects in a one-to-one way onto a Jordan curve in the xy plane. The standard monodromy argument shows that $f(A_\sigma^+ \cup D^+)$ and $f(A_\sigma^- \cup D^-)$ are graphs over the xy plane. This fact together with the fact that M is contained in the ball bounded by $f(S_0)$ implies that the nontrivial vertical translates of $f \quad \gamma$ are disjoint from M. As remarked above, this proves the theorem.

COROLLARY. *Let γ be a C^2-Jordan curve in the xy plane which bounds a disk D. If $f: \gamma \to \mathbf{R}$ is a C^2 function that is sufficiently close to zero in the C^2 norm, then f is the boundary values of a function $F: D \to \mathbf{R}$ which satisfies the Euler-Lagrange equations for the area integral. Furthermore, the graph of F is the unique compact minimal surface with boundary curve $(\gamma(t), f(\gamma(t)))$.*

In Theorem 5 we used the maximum principle for minimal surfaces. This principle fails in codimension greater than one. Thus the proof does not prove that small C^2-perturbations of a plane C^2-Jordan curve in $\mathbf{R}^n$ bounds a unique minimal surface which is a graph over this plane. The author conjectures that this generalization of Theorem 5 is true.

Theorem 1 and Theorem 5 have codimension-one generalizations in higher dimensions. Theorem 1 in higher dimensions can be stated as follows: Suppose a sphere $S^n \subset \mathbf{R}^{n+1}$ has a one-to-one orthogonal or central convex projection onto a hyperplane. Then any immersed minimal submanifold with boundary S^n is unique, and it is a graph over this codimension-one subspace. Theorem 5 generalizes as follows: If $f: S^n \to \mathbf{R}^{n+1} \subset \mathbf{R}^{n+2}$ is a C^2 embedded sphere, then there is a C^2-neighborhood N of S in the space of immersions of S^n in $\mathbf{R}^{n+2}$ such that each $g \in N$ is the boundary of a unique minimal submanifolds. Furthermore, this minimal submanifold is a graph over $\mathbf{R}^{n+1}$. Existence of the minimal submanifold for the above two theorems follows from the work of Jenkins and Serrin [7] and the part of the proof of Theorem 5 which shows that the perturbed sphere lies on boundary of region with positive mean curvature.

The proof of Theorem 1 generalizes directly in the codimension-one situation. However, to prove Theorem 5, one needs to apply the Schoenflies theorem to create a foliation of part of $\mathbf{R}^{n+1}$ by spheres. As the Schoenflies theorem is not known to hold in dimension four, the argument does not generalize so well in that dimension. For this reason, we can only claim Theorem 5 in dimensions n different from three.

With respect to Theorem 4, the reader should note that there are Jordan curves γ_n in $\mathbf{R}^3$ which are contained in positive envelopes of spheres, γ_n has total curvature greater than $4\pi + n$ and the total interior curvature I is less than $2\pi + 1/n$. Such γ_n can be constructed by taking close parallel interval parts of two helixes on the cylinder $x^2 + y^2 = 1$ and connecting the end points on the cylinder. If the curvature of the helix is sufficiently small, the required positive envelope can be constructed.

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