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ON DIFFERENTIATION OF MULTIPLE TRIGONOMETRIC SERIES

BY

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1. In this paper we obtain some multidimensional analogues to the theorems of Riemann and Lebesgue on the differentiation of formally integrated one dimensional trigonometric series. To develop these results we define symmetric derivatives for functions of several variables by expanding weighted spherical means of the functions into power series of even or odd terms. We use surface harmonics for the weights. We show that for each surface harmonic a weighted symmetric derivative can be defined, and that for each weighted symmetric derivative a different theorem of "Riemann type" can be constructed.

In p dimensions, $p \geq 2$, we let

$$x = (x_1, \ldots, x_p) \in E_p \quad \text{and} \quad n = (n_1, \ldots, n_p) \in \mathbf{Z}_p.$$

We let $\Sigma = \{x \in E_p\colon |x| = 1\}$ and $x' = |x|^{-1}x$. We write $ds(\eta)$ to denote the surface element in $(p-1)$-dimensional surface integrals. Let ν be a non-negative integer and let $S_\nu(x)$ be a harmonic polynomial homogeneous of degree ν. For $\xi \in \Sigma$, let $\Omega(\xi) = S_\nu(\xi)$. Ω is called a *surface harmonic* of degree ν.

Suppose a function $F(x)$ is defined in a neighborhood of $x_0 \in E_p$ and is integrable over the surface of each sphere $|x - x_0| = t$, for t small. Let k be an integer of the form $k = \nu + 2r$, where r is a non-negative integer. We make the following definition.

DEFINITION. F has at x_0 a kth Ω-derivative with value s_k if

$$(1.1) \quad (2\pi)^{-p/2}\int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta) = a_\nu t^\nu + a_{\nu+2}t^{\nu+2} + \cdots + a_k t^k + o(t^k),$$

as $t \to 0$, where

$$a_k = \frac{2^{-p/2-k+1}s_k}{((k-\nu)/2)!\,\Gamma((k+\nu+p)/2)}.$$

This definition may be thought of as an analogue to the definitions (1.2) and (1.3) of [10, volume 2], p. 59, depending on whether ν is even or odd.

When $p = 2$, $\nu = 0$, $k = 2r$, and $\Omega(\xi) \equiv 1$, our formula (1.1) gives the rth *generalized Laplacian*, which is developed in [9]. When $p = 2$, $\nu = 1$, and

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$\Omega(x) = x_1 + x_2$, our formula is used in [6]. When $p = 2$, $v = k = 2$, and $\Omega(x) = x_1 x_2$, the definition is used in [7].

For general p, v, and k, and for smooth enough $F(x)$, the following result gives the value of the Ω-derivative.

THEOREM 1. *Let $S_v(x)$ be a harmonic polynomial homogeneous of degree v. Let $\Omega(\xi) = S_v(\xi)$ for $\xi \in \Sigma$. Let $k = v + 2r$ where r is a non-negative integer. Suppose $F(x)$ and all partial derivatives of F of order $\leq k + 1$ are defined and continuous in a neighborhood of $x_0 \in E_p$. Then F has at x_0 a kth Ω-derivative with value $s_k = S_v(\text{grad})\Delta^r F(x_0)$.*

Now let

$$T: \sum_{n \in \mathbf{Z}_p} c_n e^{in \cdot x} \tag{1.2}$$

be a trigonometric series in p variables. We will say T is Bochner-Riesz-β summable at x_0 to sum s if

$$\lim_{R \to \infty} \sum_{|n| < R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\beta c_n e^{in \cdot x_0} = s.$$

The following result is a multidimensional analogue to the one dimensional results on differentiation of formally integrated trigonometric series. (See [10, vol. 1, p. 319, Theorem 2.4]; [10, vol. 1, p. 322, Theorem 2.18]; [10, vol. 2, p. 66, Theorem 2.1].)

THEOREM 2. *Let $S_v(x)$, $\Omega(\xi)$, k and r be as in Theorem 1. Let the series (1.2) be Bochner-Riesz-β summable at x_0 to finite sum s_0, for some $\beta < k + (p - 3)/2$. Suppose the coefficients of (1.2) satisfy $c_n = 0$ if $S_v(n) = 0$ and*

$$\sum_{n \in \mathbf{Z}_p} |n|^\alpha |c_n|^2 |S_v(n)|^{-2} < \infty \tag{1.3}$$

for some number $\alpha > p - 1 - 4r$. Let

$$F(x) = \sum_{S_v(n) \neq 0} \frac{c_n}{i^v S_v(n) |n|^{2r}} e^{\text{in} \cdot x}. \tag{1.4}$$

Then $F(x)$ has at x_0 a kth Ω-derivative equal to s_0.

Remark. The hypothesis in Theorem 2 requiring $c_n = 0$ when $S_v(n) = 0$ is necessary for the definition of $F(x)$. Given a specific $S_v(x)$, it is often possible to delete this hypothesis by altering the definition of $F(x)$ so that the series in (1.4) has different terms when $S_v(n) = 0$. For example, when $p = 2$, $v = 2$, and $S_v(x) = x_1 x_2$ we can define

$$F(x) = c_0 x_1 x_2 - \sum_{n_1 n_2 \neq 0} \frac{c_n}{n_1 n_2} e^{in \cdot x} + x_1 \sum_{n_1 = 0}' \frac{c_n}{in_2} e^{in \cdot x} + x_2 \sum_{n_2 = 0}' \frac{c_n}{in_1} e^{in \cdot x}.$$

Unfortunately, this author has not been able to find a general formula to give the terms of $F(x)$, when $S_v(n) = 0$, for arbitrary $S_v(x)$.

2. We begin by establishing some lemmas that will be used in the proofs of theorems. After these lemmas have been established, the proofs of the theorems will closely follow the proofs in [6] and [9]. In what follows we let $S_\nu(x)$ be a harmonic polynomial homogeneous of degree ν and $\Omega(\xi)$ be the restriction of $S_\nu(x)$ to the unit sphere. We denote the Bessel's function of order ν by $J_\nu(t)$.

LEMMA 1. *Let j be a non-negative integer. Let $F(x)$ and all partial derivatives of F of order less than or equal to j exist and be integrable in a neighborhood of $x_0 \in E_p$. If $j \geq \nu$ and $j - \nu$ is even, then*

$$(2.1) \qquad \int_{\eta \in \Sigma} (\eta \cdot \operatorname{grad})^j F(x_0)\Omega(\eta)\, ds(\eta)$$
$$= \frac{2^{1-j}\pi^{p/2} j!}{((j-\nu)/2)!\,\Gamma((j+\nu+p)/2)} S_\nu(\operatorname{grad})\Delta^{(j-\nu)/2}F(x_0).$$

If $j < \nu$ or if $j - \nu$ is odd, then

$$\int_{\eta \in \Sigma} (\eta \cdot \operatorname{grad})^j F(x_0)\Omega(\eta)\, ds(\eta) = 0.$$

Note. Here $\operatorname{grad} = (-i\,\partial/\partial x_1, \ldots, -i\,\partial/\partial x_p)$ where the $-i$ is chosen so that the Fourier transform $\hat{F}(y) = \int F(x) \exp(-ixy)\, dx$ will satisfy

$$(\operatorname{grad} F)^\wedge(y) = y \cdot \hat{F}(y).$$

LEMMA 2. *Let $n \in \mathbf{Z}_p$. Then, if $n \neq 0$,*

$$(2.2) \quad (2\pi)^{-p/2} \int_{\eta \in \Sigma} \exp(in \cdot t\eta)\Omega(\eta)\, ds(\eta)$$
$$= \Omega(n')i^\nu(|n|t)^{-(p-2)/2} J_{\nu+(p-2)/2}(|n|t).$$

Proof. Lemmas 1 and 2 are consequences of the Funk-Hecke Theorem, see [4, p. 181], (where p has a different meaning): *If Ω is a surface harmonic of degree ν, $\xi \in \Sigma$ and $f(h)$ is continuous for $-1 \leq h \leq 1$* then

$$(2.3) \qquad \int_{\eta \in \Sigma} f(\xi \cdot \eta)\Omega(\eta)\, ds(\eta) = c(p, \nu, f)\Omega(\xi),$$

where

$$c(p, \nu, f) = \frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu + (p-1)/2)} \int_{-1}^{1} f^{(\nu)}(h)(1-h^2)^{\nu+(p-3)/2}\, dh.$$

To prove Lemma 2 we fix $n \in \mathbf{Z}_p$ and $t > 0$. We set

$$f(h) = (2\pi)^{-p/2} \exp(i|n|th)$$

and let $\xi = |n|^{-1}n$. Applying (2.3) we get

$$\int_{\eta \in \Sigma} (2\pi)^{-p/2} \exp(i|n|t|n|^{-1}n \cdot \eta)\Omega(\eta)\, ds(\eta)$$

$$= \Omega(n')\frac{2^{-v+1}\pi^{(p-1)/2}}{\Gamma(v + (p-1)/2)}\int_{-1}^{1} (2\pi)^{-p/2}(i|n|t)^{v} \exp(i|n|th)(1-h^2)^{v+(p-3)/2}\, dh$$

Thus

$$(2\pi)^{-p/2}\int_{\eta \in \Sigma} \exp(in \cdot t\eta)\Omega(\eta)\, ds(\eta)$$

$$= \Omega(n')\frac{2^{-v+1-p/2}\pi^{-1/2}(i|n|t)^{v}}{\Gamma(v+(p-1)/2)}\int_{-1}^{1} \exp(i|n|th)(1-h^2)^{v+(p-3)/2}\, dh.$$

By formula 7, p. 81 of [2],

$$\int_{-1}^{1} \exp(i|n|th)(1-h^2)^{v+(p-3)/2}\, dh = \frac{\Gamma(v+(p-1)/2)}{\pi^{-1/2}(\frac{1}{2}|n|t)^{v+(p-2)/2}} J_{v+(p-2)/2}(|n|t),$$

so

$$(2\pi)^{-p/2}\int_{\eta \in \Sigma} \exp(in \cdot t\eta)\Omega(\eta)\, ds(\eta)$$

$$= \Omega(n')\frac{2^{-v+1-p/2}\pi^{-1/2}(i|n|t)^{v}}{\Gamma(v+(p-1)/2)} \frac{\Gamma(v+(p-1)/2)}{\pi^{-1/2}(\frac{1}{2}|n|t)^{v+(p-2)/2}} J_{v+(p-2)/2}(|n|t)$$

$$= \Omega(n')i^{v}(|n|t)^{-(p-2)/2}J_{v+(p-2)/2}(|n|t).$$

This completes the proof of Lemma 2.

For the proof of Lemma 1 we apply formula (2.3) again, this time with $f_j(h) = h^j$ and $\xi = |y|^{-1}y$, where $y \in E_p$. We have

$$\int_{\eta \in \Sigma} (|y|^{-1}y \cdot \eta)^j\Omega(\eta)\, ds(\eta) = c(p, v, f_j)\Omega(y').$$

Thus

$$\int_{\eta \in \Sigma} (y \cdot \eta)^j\Omega(\eta)\, ds(\eta) = c(p, v, f_j)|y|^{j-v}S_v(y), \tag{2.4}$$

where

$$c(p, v, f_j) = \frac{2^{-v+1}\pi^{(p-1)/2}}{\Gamma(v+(p-1)/2)}\int_{-1}^{1} f_j^{(v)}(h)(1-h^2)^{v+(p-3)/2}\, dh. \tag{2.5}$$

If $j < v$, then $f_j^{(v)}(h) \equiv 0$, so $c(p, v, f_j) = 0$. Also, if $j - v$ is odd, then the integral on the right side of (2.5) vanishes, so again $c(p, v, f_j) = 0$.

If $j \geq \nu$ and $j - \nu$ is even, then

$$c(p, \nu, f_j) = \frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu + (p-1)/2)} \frac{j!}{(j-\nu)!} \int_{-1}^{1} h^{j-\nu}(1-h^2)^{\nu+(p-3)/2}\, dh.$$

Calculating this last integral by reduction formulae, we get

$$\begin{aligned} c(p, \nu, f_j) &= \frac{2^{-\nu+1}\pi^{(p-1)/2}}{\Gamma(\nu + (p-1)/2)} \frac{j!}{(j-\nu)!} \cdot \frac{2^{\nu-j}(j-\nu)!\,\pi^{1/2}\Gamma(\nu+(p-1)/2)}{((j-\nu)/2)!\,\Gamma((j+\nu+p)/2)} \\ &= \frac{2^{1-j}\pi^{p/2}j!}{((j-\nu)/2)!\,\Gamma((j+\nu+p)/2)}. \end{aligned}$$

We now complete the proof of Lemma 1. We may assume, by changing $F(x)$ outside a neighborhood of x_0 if necessary, that $F(x)$ has compact support. Let

$$\Phi(x) = \int_{\eta \in \Sigma} (\eta \cdot \mathrm{grad})^j F(x)\Omega(\eta)\, ds(\eta)$$

and let

$$\Psi(x) = c(p, \nu, f_j) S_\nu(\mathrm{grad})\Delta^{(j-\nu)/2} F(x).$$

Then

$$\Phi^\wedge(y) = \int_{\eta \in \Sigma} (\eta \cdot y)^j \hat{F}(y)\Omega(\eta)\, ds(\eta) = \hat{F}(y) \int_{\eta \in \Sigma} (\eta \cdot y)^j \Omega(\eta)\, ds(\eta),$$

and

$$\Psi^\wedge(y) = c(p, \nu, f_j) S_\nu(y) |y|^{(j-\nu)} \hat{F}(y).$$

Hence, by (2.4) $\Phi^\wedge(y) = \Psi^\wedge(y)$. Therefore $\Phi(x) = \Psi(x)$, and the proof of Lemma 1 is complete.

Now let $T\colon \sum_{n \in \mathbf{Z}_p} c_n \exp(in \cdot x)$ be a multiple trigonometric series. Let

$$S_R = S_R(x) = \sum_{|n|<R} c_n e^{in \cdot x}$$

and, for $\beta > 0$,

$$S_R^\beta = S_R^\beta(x) = \frac{1}{\Gamma(\beta)} \int_0^R S_u(x)(R-u)^{\beta-1}\, du. \tag{2.6}$$

Hardy [3] showed that a series is Bochner-Riesz-β summable to 0 at x_0 if and only if

$$\sum_{|n|<R} c_n e^{in \cdot x_0}\left(1 - \frac{|n|}{R}\right)^\beta \to 0$$

as $R \to \infty$. Hence if the series T is Bochner-Riesz-β summable to 0 at x_0, then

$$S_R^\beta(x_0) = o(R^\beta) \tag{2.7}$$

as $R \to \infty$.

Let $I^\alpha(f)$ represent the fractional integral of a function f, see [1]. Note that (2.6) means that for fixed x, $S_R^\beta(x) = I^\beta(f(R))$, where $f(R) = S_R(x)$.

LEMMA 3. *Suppose the series* (1.2) *is Bochner-Riesz-*$(m+1)$ *summable to* 0 *at* $x_0 = 0$, *and suppose that the coefficients of* (1.2) *satisfy the hypothesis of Theorem* 2. *Then*

$$S_R^{(j)} = S_R^{(j)}(0) = o(R^{k+(p-1)/2})$$

as $R \to \infty$, *for* $j = 0, 1, \ldots, m+1$.

Proof. The proof is similar to the corresponding proof in [6] and is omitted.

3. *Proof of Theorem* 1. We use Taylor's Formula:

$$F(x_0 + t\eta) = \sum_{j=0}^{k} \frac{1}{j!}(t\eta \cdot \mathrm{grad})^j F(x_0) + \frac{1}{(k+1)!}(t\eta \cdot \mathrm{grad})^{k+1} F(x_0 + h\eta),$$

for some $0 < h < t$. Then, using Lemma 1,

$$\begin{aligned}
&(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta) \\
&\quad = \sum_{j=v}^{k} \frac{t^j}{j!}(2\pi)^{-p/2} \int_{\eta \in \Sigma} (\eta \cdot \mathrm{grad})^j F(x_0)\Omega(\eta)\, ds(\eta) \\
&\qquad + \frac{t^{k+1}}{(k+1)!} \int_{\eta \in \Sigma} (\eta \cdot \mathrm{grad})^{k+1} F(x_0 + h\eta)\Omega(\eta)\, ds(\eta) \\
&\quad = \sum_{\substack{j=v,\\ j-v \text{ even}}}^{k} \frac{t^j}{j!}(2\pi)^{-p/2} \frac{2^{1-j}\pi^{p/2} j!}{((j-v)/2)!\,\Gamma((j+v+p)/2)} S_v(\mathrm{grad})\Delta^{(j-v)/2} F(x_0) + R \\
&\quad = \sum_{\substack{j=v,\\ j-v \text{ even}}}^{k} \frac{2^{1-j-p/2}}{((j-v)/2)!\,\Gamma((j+v+p)/2)} S_v(\mathrm{grad})\Delta^{(j-v)/2} F(x_0 w) t^j + R.
\end{aligned}$$

Here

$$R = \frac{t^{k+1}}{(k+1)!} \int_{\eta \in \Sigma} (\eta \cdot \mathrm{grad})^{k+1} F(x_0 + h\eta)\Omega(\eta)\, ds(\eta) = o(t^k)$$

as $t \to 0$.

Proof of Theorem 2. We may assume without loss of generality that $x_0 = 0$ and $s_0 = 0$. Write $\beta = m + \alpha$, where m is an integer and $0 \le \alpha < 1$. We begin with the case when $\beta = m$ and $\alpha = 0$. The condition on the coefficients (1.3) assures us that the series defining $F(x)$ converges spherically on each sphere $|x| = t$ and that we can integrate the series termwise on each sphere, see [5].

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta)$$
$$= \sum_{n \in \mathbf{Z}_p} \frac{c_n}{i^\nu S_\nu(n)\,|n|^{2r}} (2\pi)^{-p/2} \int_{\eta \in \Sigma} e^{in \cdot t\eta}\Omega(\eta)\, ds(\eta)$$
$$= \sum_n \frac{c_n}{i^\nu S_\nu(n)\,|n|^{2r}} \cdot \Omega(n') i^\nu (|n|\,t)^{-(p-2)/2} J_{\nu+(p-2)/2}(|n|\,t)$$

by Lemma 2. Thus

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta)$$
$$= \sum_n c_n |n|^{-2r-\nu} (|n|\,t)^{-(p-2)/2} J_{\nu+(p-2)/2}(|n|\,t)$$
$$= t^k \sum c_n (|n|\,t)^{-k-(p-2)/2} J_{\nu+(p-2)/2}(|n|\,t)$$
$$= t^k \sum c_n \gamma(|n|\,t),$$

where

$$\gamma(z) = z^{-k-(p-2)/2} J_{\nu+(p-2)/2}(z).$$

We write the last sum as an integral, using the notation of (2.6), and integrate by parts m times:

$$\sum_{|n|<R} c_n \gamma(|n|\,t) = S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut)\, du$$
$$= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2} \gamma(ut)\, du$$
$$= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \cdots + (-1)^m S_R^m \frac{d^m}{dR^m} \gamma(Rt) \tag{3.1}$$
$$+ (-1)^{m+1} \int_0^R S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du.$$

By Lemma 3, $S_R^{(j)} = o(R^{k+(p-1)/2})$ as $R \to \infty$, for $j = 0, \ldots, m$. Also, repeatedly using equation 51, p. 11 of [2], and the fact that $J_\nu(z) = O(z^{-1/2})$ as $z \to \infty$, we get

$$\gamma^{(j)}(z) = O(z^{-k-(p-2)/2-1/2}) = O(z^{-k-p/2+1/2}) \tag{3.2}$$

as $z \to \infty$, for $j = 1, 2, \ldots$. Hence as $R \to \infty$ all of the integrated terms of (3.1) tend to 0. Therefore

$$\lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|\,t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du,$$

and

$$(3.3)\quad (2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0+t\eta)\Omega(\eta)\,ds(\eta) = t^k(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du.$$

We now make use of the series expansion for $J_{\nu+(p-2)/2}(z)$, see [2], p. 4. Let $\mu = \nu + (p-2)/2$. We have

$$\begin{aligned}\gamma(z) &= z^{-2r-\mu}J_\mu(z)\\ &= z^{-2r-\mu}\{c_\mu z^\mu + c_{\mu+2}z^{\mu+2} + \cdots + c_{\mu+2r-2}z^{\mu+2r-2} + c_{\mu+2r}z^{\mu+2r} + \cdots\}.\end{aligned}$$

If $r = 0$, let $P(z) \equiv 0$. If $r \neq 0$, let

$$P(z) = c_\mu z^\mu + c_{\mu+2}z^{\mu+2} + \cdots + c_{\mu+2r-2}z^{\mu+2r-2}.$$

Let

$$(3.4)\qquad \lambda(z) = \gamma(z) - z^{-2r-\mu}P(z).$$

Then $\lambda(z)$ is an entire function. Returning to (3.3),

$$\begin{aligned}&(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0+t\eta)\Omega(\eta)\,ds(\eta)\\ &= t^k(-1)^{m+1}\int_0^\infty S_u^m\frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du\\ &= t^k(-1)^{m+1}\int_0^\infty S_u^m\frac{d^{m+1}}{du^{m+1}}\{(ut)^{-2r-\mu}P(ut)+\lambda(ut)\}\,dt\\ &= t^k(-1)^{m+1}\int_0^\infty S_u^m\frac{d^{m+1}}{du^{m+1}}\left\{\sum_{j=0}^{r-1}c_{\mu+2j}(ut)^{-2r+2j}+\lambda(ut)\right\}du\\ &= \sum_{j=0}^{r-1}t^{-2r+2j+k}(-1)^m c'_{\mu+2j}\int_0^\infty S_u^m u^{-2r+2j-m-1}\,du + t^k(-1)^{m+1}\\ &\qquad\int_0^\infty S_u^m\frac{d^{m+1}}{du^{m+1}}\lambda(ut)\,du.\end{aligned}$$

Recall $S_u^m = o(u^m)$ as $u\to\infty$. Also, since $c_0 = 0$, $S_u^m = 0$ for $0 \le u < 1$. Hence,

$$\int_0^\infty S_u^m u^{-2r+2j-m-1}\,du = \int_{1/2}^\infty o(u^m)\cdot u^{-2r+2j-m-1}\,du = O(1)$$

for $j = 0, \ldots, r-1$. Thus,

$$\begin{aligned}&(2\pi)^{-p/2}\int_{\eta\in\Sigma} F(x_0+t\eta)\Omega(\eta)\,ds(\eta)\\ &\qquad = \sum_{j=0}^{r-1}a_{2j}t^{\nu+2j} + 0\cdot t^k + t^k(-1)^{m+1}\int_0^\infty S_u^m\frac{d^{m+1}}{du^{m+1}}\lambda(ut)\,du.\end{aligned}$$

To complete the proof of Theorem 2, we must show

$$(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du \to 0 \quad \text{as} \quad t \to 0.$$

Using (3.4),

$$\begin{aligned}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du &= \int_0^{1/t} S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du + \int_{1/t}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du \\ &= \int_0^{1/t} S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du + \int_{1/t}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du \\ &\quad - \int_{1/t}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-2r-\mu} P(ut)\}\, du \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Since $\lambda(z)$ is entire, when $|ut| \leq 1$,

$$\left| \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \right| \leq C t^{m+1}.$$

Thus,

$$I_1 = \int_0^{1/t} S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du = \int_0^{1/t} o(u^m) C \cdot t^{m+1}\, du = o(1) \quad \text{as } t \to 0.$$

We use (3.2) to evaluate I_2.

$$\begin{aligned} I_2 &= \int_{1/t}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)\, du = \int_{1/t}^\infty o(u^m) t^{m+1} O((ut)^{-k-p/2+1/2})\, du \\ &= t^{m-k-p/2+3/2} \int_{1/t}^\infty o(u^{m-k-p/2+1/2})\, du \\ &= o(1). \end{aligned}$$

(Note we needed $m < k + (p-3)/2$ for this last integral to converge.) To evaluate I_3, we note

$$\begin{aligned}\frac{d^{m+1}}{du^{m+1}} \{(ut)^{-2r-\mu} P(ut)\} &= t^{m+1} \sum_{j=0}^{r-1} c'_{\mu+2j} (ut)^{2j-2r-m-1} \\ &= t^{m+1} O(ut)^{-m-3}.\end{aligned}$$

Thus

$$I_3 = \int_{1/t}^\infty o(u^m) t^{m+1} O(ut)^{-m-3}\, du = o(1).$$

This completes the proof of Theorem 2 in the case when $\beta = m$ is an integer.

We now consider the general case $\beta = m + \alpha$, $0 < \alpha < 1$. We begin as before, but this time integrate by parts the last integral on the right of (3.1) one more time. We get, after showing the integrated terms tend to zero,

$$(2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta) = t^k(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut)\, du.$$

Letting $P(z)$ and $\lambda(z)$ be defined as before, we obtain

$$(3.5) \quad (2\pi)^{-p/2} \int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta)$$
$$= \sum_{j=0}^{r-1} a_{2j} t^{\nu + 2j} + 0 \cdot t^k + t^k(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du.$$

Recall that for any $\delta \geq 0$, $S_u^\delta = I^\delta(\mathrm{S}_u)$, where I^δ denotes the fractional integral of order δ. Hence

$$\begin{aligned} S_u^{m+1} &= I^{m+1}(S_u) \\ &= I^{1-\alpha} I^{m+\alpha} S_u \\ &= I^{1-\alpha} S_u^{m+\alpha} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{(1-\alpha)-1} S_z^{m+\alpha}\, dz \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha}\, dz. \end{aligned}$$

Thus the last integral on the right of (3.5) becomes

$$(3.6) \quad (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du$$
$$\begin{aligned} &= (-1)^m \int_0^\infty \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha}\, dz \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du \\ &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \left\{ \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du \right\} dz \\ &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R)\, dz, \end{aligned}$$

where

$$H(z, t, R) = \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du.$$

We obtain the following estimates on $H(z, t, R)$. (For details, which we omit here, see [6, p. 445–448], where the notation is slightly different).

$$H(z, t, R) = \begin{cases} O(t^{m+1})(1/t - z)^{-\alpha} & \text{for } 0 < tz < 1 \\ t^{-2}O(z)^{-m-\alpha-3} + t^{m+1+\alpha}O(tz)^{-k-(p-1)/2} & \text{for } tz > 1. \end{cases}$$

Returning to (3.6)

$$\begin{aligned}
(-1)^m \int_0^\infty & S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du \\
&= \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R)\, dz \\
&\quad + \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R)\, dz \\
&= \int_0^{1/t} o(z)^{m+\alpha} O(t^{m+1}) \left(\frac{1}{t} - z\right)^{-\alpha} dz \\
&\quad + \lim_{R\to\infty} \int_{1/t}^R o(z)^{m+\alpha} \{t^{-2} O(z^{-m-\alpha-3}) \\
&\quad + t^{m+1+\alpha} O(tz)^{-k-(p-1)/2}\}\, dz \\
&= o\left(\frac{1}{t}\right)^{m+\alpha} O(t^{m+1}) \int_0^{1/t} \left(\frac{1}{t} - z\right)^{-\alpha} dz + t^{-2} \int_{1/t}^\infty o(z^{-3})\, dz \\
&\quad + t^{m+1+\alpha-k-(p-1)/2} \int_{1/t}^\infty o(z^{m+\alpha-k-(p-1)/2})\, dz \\
&= o(1) + o(1) + o(1).
\end{aligned}$$

(Note that the hypothesis $\beta = m + \alpha < k + (p-3)/2$ is necessary for the last integral to converge.) This completes the proof of Theorem 2.

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