COMPLEMENT THEOREMS BEYOND THE TRIVIAL RANGE[1]

BY

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1. Introduction

By a well known theorem of Chapman [2], if X and Y are Z-sets in the Hilbert cube Q, then X and Y have the same shape (abbreviated Sh $(X) =$ Sh (Y)) if and only if $Q - X$ is homeomorphic with $Q - Y$. In recent years there has been a great deal of interest in finite-dimensional analogues of this result, the principal aim being to find conditions on compacta X, $Y \subset E^n$ such that Sh $(X) =$ Sh (Y) if and only if $E^n - X \cong E^n - Y$. Thus far all results along this line have required either that the dimensions or fundamental dimensions of X and Y lie at most in the trivial $([n/2] - 1)$ range with respect to n [3], [7], [8], [11], [17], [20] or that Sh (X) and Sh (Y) have particularly nice representatives, such as spheres, manifolds, or finite complexes [4], [11], [12], [13], [15], [21]. It is our purpose to present here a theorem in (fundamental) codimension four. We are able to go beneath the trivial range in ambient dimension by assuming appropriate connectivity conditions on the embedded compacta; these conditions allow us to replace general position arguments which suffice in the trivial range by ones using engulfing. Our main result is as follows.

THEOREM A. *Let X and Y be r-shape connected continua in E^n of fundamental dimension at most k and satisfying ILC, where*

$$n \geq \max (2k + 2 - r, k + 3, 5).$$

Then Sh $(X) =$ Sh (Y) implies $E^n - X \cong E^n - Y$. The converse holds if $n \geq k + 4$.

Nowak [14] has shown that if X is a finite dimensional approximatively 1-connected compactum and $\check{H}^i(X) = 0$ for $i > k$, then $Fd(X) \leq k$. This fact along with Theorem A yields the following.

THEOREM B. *Let X and Y be r-shape connected continua in E^n satisfying ILC and such that $\check{H}^i(X) = 0 = \check{H}^i(Y)$ for $i > k$, where*

$$n \geq \max (2k + 2 - r, k + 3, 5)$$

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and $r \geq 1$. *Then* $\mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y)$ *implies* $E^n - X \cong E^n - Y$. *The converse holds if* $n \geq k + 4$.

In Section 5 we show that it is possible to weaken the hypotheses of Theorem A slightly, replacing connectivity in dimension r with pointed r-movability.

THEOREM C. *Let* X *and* Y *be* $(r-1)$-*shape connected, pointed* r-*movable continua in* E^n *of fundamental dimension at most* k *and satisfying ILC, where* $n \geq \max\,(2k + 2 - r,\ k + 3,\ 5)$. *Then* $\mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y)$ *implies* $E^n - X \cong E^n - Y$. *The converse holds if* $n \geq k + 4$ *and* X *and* Y *are* 1-*shape connected.*

Remark. It follows from [6, Theorem 4] that an $(r-1)$-shape connected continuum X is pointed r-movable if and only if X has the shape of some locally $(r-1)$-connected continuum.

The codimension 4 hypothesis of Theorem C is not needed in case X and Y have polyhedral shape. We obtain the following which generalizes results in [11] and [12].

THEOREM D. *Let* X *and* Y *be continua in* E^n *satisfying ILC and each having the shape of an* r-*connected finite complex of dimension at most* k, *where* $n \geq \max\,(2k + 1 - r, k + 3, 5)$. *Then* $\mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y)$ *implies* $E^n - X \cong E^n - Y$. *The converse holds if* $r \geq 1$.

Theorem A is an immediate consequence of Theorem 3 and Theorem 5, using Theorems 1 and 2 to see that Theorem 3 applies. These results are obtained in Sections 2, 3, and 4. In Section 5 we consider movable continua, and modify the work of Section 3 to obtain Theorem C. Section 6 consists of some brief remarks concerning Borsuk's theory of positions [1].

We use either the fundamental sequence approach or the ANR-sequence approach to shape theory, as convenience dictates. Our notations are those commonly used in shape theory as found, for example, in [1] or [5].

2. The inessential loops condition and neighborhoods of compacta

If X is a compactum in the manifold M, then X is said to satisfy the *inessential loops condition, ILC*, if for each neighborhood U of X in M there exists a neighborhood V of X in U such that each loop in $V - X$ which is null-homotopic in V is also nullhomotopic in $U - X$. The *fundamental dimension of* the compactum X, $Fd(X)$, is

$$\min\,\{\dim\,Y: \mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y),\ Y \text{ a compactum}\}.$$

For our purpose the following relationship between these notions, which is stated as Theorem 4.1 of [22], will be of primary importance.

THEOREM 1. *Suppose* X *is a compactum in the interior of a PL* n-*manifold* M^n, $n \geq 5$, *and* $Fd(X) \leq k \leq n - 3$. *Then* X *satisfies ILC if and only if* X *has arbitrarily close compact PL manifold neighborhoods with* k-*dimensional spines.*

If X is a compactum we say that X is *approximatively k-connected, $k \geq 1$,* if the homotopy pro-group pro-$\pi_k(X, x)$ is trivial for all $x \in X$. It is well known that if $X \subset M \in$ ANR, then X is approximatively k-connected if and only if for each neighborhood U of X in M there exists a neighborhood V of X in U such that any map of the k-sphere S^k into V is nullhomotopic in U. If X is approximatively k-connected for $1 \leq k \leq r$, we say that X is *r-shape connected.* If $X \in$ ANR, and $r \geq 0$, we say that X is *r-connected* if X is connected and $\pi_k(X) = 0$ for $1 \leq k \leq r$.

Our first result concerns the problem of finding neighborhoods of a compactum X having connectivity matching the shape connectivity of X.

THEOREM 2. *Suppose $X \subset$ int M^n is an r-shape connected compactum in the PL n-manifold M having arbitrarily close compact PL manifold neighborhoods with k-dimensional spines, where $k + 3 \leq n$ and $n \geq 5$. Then X has arbitrarily close compact PL manifold neighborhoods having r-connected components with k-dimensional spines.*

Proof. Let U be a neighborhood of X in M^n. We shall find a compact PL manifold neighborhood V of X such that $V \subset U$, each component of V is r-connected, and V has a k-dimensional spine. We may assume $r \geq 1$.

First assume $r \geq k$ and, since the case $k = 0$ is trivial, $k \geq 1$. Let U be a neighborhood of X, and let U_1 be a neighborhood of X such that $U_1 \subset U$ and U_1 is a compact PL manifold having k-dimensional spine. Let V_1 be a neighborhood of X such that $V_1 \subset U_1$ and every connected r-dimensional polyhedron in V_1 is inessential in U_1. Let V_2 be a neighborhood of X such that $V_2 \subset V_1$ and V_2 is a compact PL manifold having k-dimensional spine. Then each component of V_2 is inessential in U_1 and each loop in $V_2 - X$ is nullhomotopic in $U_1 - X$. By the lemma of [16], there exists a neighborhood V of X such that $V \subset U$ and each component of V is a PL n-cell.

Now assume $r < k$. Fix j, $1 \leq j \leq r$, and assume inductively that there exists a compact PL manifold neighborhood V_j of X such that $V_j \subset U$, each component of V_j is $(j-1)$-connected, and V_j has k-dimensional spine. Let V' be a compact PL manifold neighborhood of X such that $V' \subset$ int V_j, V' has k-dimensional spine K, and each connected polyhedron in V' of dimension j is inessential in V_j. We assume henceforth that V_j is connected; otherwise we apply the following construction to each component of V_j.

Let $L = K \cup vK^{(j)} \subset vK$, where vK is the cone on K with vertex v and $K^{(j)}$ is the j-skeleton of K. By our assumptions, the inclusion of K into int V_j can be extended to a PL general position map $f: L \to$ int V_j. If $S(f)$ denotes the singular set of f, then

$$\dim S(f) \leq (j+1) + k - n \leq j - 2.$$

Since $j \leq r < k \leq n - 3$, dim $S(f) \leq n - 5$. Note further that since V_j is $(j-1)$-connected and L is j-connected, f is $(j-1)$-connected. Theorem 4.3 of

[19] then applies, and yields a k-dimensional polyhedron $P \subset$ int V_j such that $f(L) \subset P$ and the composition $L \to f(L) \subset P$ is a simple homotopy equivalence. Let $N \subset$ int V_j be a regular neighborhood of P. Then N contains a regular neighborhood N' of K. Let $h: V_j \to V_j$ be a PL homeomorphism such that $h(N') = V'$. Then $V_{j+1} = h(N) \subset U$ is a compact j-connected PL manifold neighborhood of X having k-dimensional spine. Continuing the induction we obtain $V_{r+1} = V$.

3. Continua in E^n having the same shape

We now state a lemma which will be used to maintain an inductive argument in the proof of Theorem 3. It is similar to Lemma 4.1 of [7] and Lemma 4 of [20].

LEMMA 1. *Let X and Y be continua in E^n such that X has arbitrarily close compact PL manifold neighborhoods with k-dimensional spines and Y has arbitrarily close open $(2k + 2 - n)$-connected neighborhoods, where $k \le n - 3$. Let $\mathbf{f} = \{f_i, X, Y\}$ and $\mathbf{f'} = \{f'_i, Y, X\}$ be fundamental sequences in E^n such that $\mathbf{f'f} \simeq \mathrm{id}_X$ in E^n. Let U_0 be an open $(2k + 2 - n)$-connected neighborhood of X, and $h: E^n \to E^n$ be a PL homeomorphism such that $Y \subset h(U_0)$ and such that there exists a neighborhood W_0 of Y with $h^{-1} | W_0 \simeq f'_i | W_0$ in U_0 for almost all i. Then for every neighborhood V_0 of Y, there exist an open $(2k + 2 - n)$-connected neighborhood V of Y lying in $V_0 \cap h(U_0)$, a PL homeomorphism $q: E^n \to E^n$, and a neighborhood U_1 of X such that*

(1) $q | E^n - U_0 = h | E^n - U_0$,
(2) $q(X) \subset V$, and
(3) $q | U_1 \simeq f_i | U_1$ in V for almost all i.

Proof. There exists neighborhoods $V \subset V_0 \cap h(U_0)$ of Y and $U_1 \subset U_0$ of X and a positive integer N such that if $i \ge N$, then $h^{-1} | V \simeq f'_i | V$ in U_0, $f_i | U_1 \simeq f_N | U_1$ in V, and $f'_i f_i | U_1 \simeq \mathrm{id}_{U_1}$ in U_0. By hypothesis we may assume V to be open and $(2k + 2 - n)$-connected and U_1 to be a compact PL manifold having k-dimensional spine L. Let $\hat{f}: L \to V$ be a PL map such that $\hat{f} \simeq f_N | L$ in V and such that $\hat{f}$ and $h | L$ are in general position.

Note that $h^{-1} \hat{f} \simeq f'_N \hat{f} \simeq f'_N f_N | L \simeq \mathrm{id}_L$ in U_0, and so $\hat{f} h^{-1} | h(L) \simeq \mathrm{id}_{h(L)}$ in $h(U_0)$. Let $G: h(L) \times I \to h(U_0)$ be a PL general position map such that $G(x, 0) = x$ and $G(x, 1) = \hat{f} h^{-1}(x)$ for all $x \in h(L)$. We may assume, by Zeeman's Piping Lemma (Lemma 48 in Chapter 7 of [23]), that there exists a polyhedron $J \subset h(L) \times I$ such that $S(G) \subset J$, $\dim J \le 2k + 2 - n \le n - 4$, and

$$h(L) \times I \searrow J \cup (h(L) \times \{1\}).$$

By our connectivity conditions, the pair $(h(U_0), V)$ is $(2k + 2 - n)$-connected, and so we may apply Stalling's Engulfing Theorem [18] to obtain a

homeomorphism

$$r: E^n \to E^n$$

such that r is fixed on $G(h(L) \times \{1\}) \cup (E^n - h(U_0))$ and $r(V) \supset G(J)$. Using the fact that $S(G) \subset J$, we see that

$$G(h(L) \times I) \searrow G(J \cup (h(L) \times \{1\})).$$

Since we may engulf along the track of this collapse, we may assume

$$r(V) \supset G(h(L) \times I).$$

Now, let

$$h_1: E^n \to E^n$$

be a homeomorphism such that h_1 is fixed on $L \cup (E^n - U_0)$ and

$$hh_1(U_1) \subset r(V),$$

and let $q = r^{-1}hh_1: E^n \to E^n$.

(1') If $x \in E^n - U_0$, $q(x) = r^{-1}hh_1(x) = h(x)$, so (1) is satified.
(2') $q(U_1) = r^{-1}(hh_1(U_1)) \subset r^{-1}(r(V)) = V$, so (2) is satisfied.
(3') Let $F: L \times I \to V$ be defined by

$$F(x, t) = r^{-1}G(h(x), t), \quad x \in L, t \in I.$$

Then, if $x \in L$,

$$F(x, 0) = r^{-1}G(h(x), 0) = r^{-1}(h(x)) = r^{-1}h(h_1(x)) = q(x)$$

and

$$F(x, 1) = r^{-1}G(h(x), 1) = G(h(x), 1) = \hat{f}h^{-1}(h(x)) = \hat{f}(x).$$

Thus $q|L \simeq \hat{f}$ in V, and so, if $i \geq N$, $q|L \simeq f_i|L$ in V. The fact that $q(U_i) \subset V$, as shown in (2'), along with the fact that L is a strong deformation retract of U_1, shows that if $i \geq N$, then $q|U_1 \simeq f_i|U_1$ in V, and so (3) holds.

We are now prepared to state and prove the main result of this section.

THEOREM 3. *Suppose X and Y are continua in E^n such that X and Y each have arbitrarily close compact PL manifold neighborhoods with k-dimensional spines and arbitrarily close open $(2k + 2 - n)$-connected neighborhoods, where $k \leq n - 3$. Then $\mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y)$ implies $E^n - X \cong E^n - Y$.*

Proof. Suppose that $\mathrm{Sh}\,(X) = \mathrm{Sh}\,(Y)$. Then an easy modification of the proof of Lemma 4.2 of [7] shows that $E^n - X \cong E^n - Y$, where our Lemma 1 is used in place of Lemma 4.1 of [7]. We need only note that in our case we may keep the induction going by replacing the sets $W_1, S_2, S_3, \ldots$, used in that proof by appropriately chosen $(2k + 2 - n)$-connected open sets.

We remark that because of linking phenomena we must assume X and Y connected in Theorem 3. This is the case even if X and Y have polyhedral shape. As an example, let X and Y each consist of the disjoint sum of two piecewise linear copies of S^3 in S^7, with linking occurring for X but not Y. Then $E^n - X \not\cong E^n - Y$.

4. Continua in E^n having homeomorphic complements

We begin this section with a result which generalizes Theorem 1' of [12]. (Our "niceness of embeddings" condition is ILC. However, for 1-shape connected compacta in E^n of fundamental dimension at most $n - 3$, ILC is equivalent to the formally stronger cellularity criterion [13] or the condition that the embedding be globally $1 - ALG$ (Definition in [15]).)

THEOREM 4. *Let X and Y be 1-shape connected continua in E^n of fundamental dimension at most k and satisfying ILC, where $k \leq n - 3$. Then*

$$E^{n+1} - X \cong E^{n+1} - Y$$

implies Sh $(X) =$ Sh (Y).

Proof. If $n \leq 2$ the statement is vacuous, while if $n = 4$ the result follows from Theorem 1 of [20]. In case $n = 3$, the result follows from standard techniques, see [22] for example. We may thus assume $n \geq 5$.

By Theorems 1 and 2 there exists a sequence $\{M_i\}_{i=1}^{\infty}$ of compact connected PL n-manifolds in E^n such that $X = \bigcap_{i=1}^{\infty} M_i$ and, for $j = 1, 2, \ldots$, $M_{j+1} \subset \text{int } M_j$, $\pi_1(M_j) = 0$, and M_j has k-dimensional spine K_j. Let

$$N_j = M_j \times [-1/j, 1/j] \subset E^n \times E^1 = E^{n+1}.$$

Note that $\pi_1(N_j) = \pi_1(N_j - K_j) = \pi_1(\partial N_j) = 0$.

Let $h: E^{n+1} - X \to E^{n+1} - Y$ be a homeomorphism. We may assume that h induces a homeomorphism of the quotient space E^{n+1}/X onto E^{n+1}/Y. For if this is not the case, then the end of $E^{n+1} - X$ corresponding to X is isomorphic to the end of E^{n+1}, as is the end of $E^{n+1} - Y$ corresponding to Y; from this we see that X and Y are cellular in E^{n+1}, hence both have trivial shape and the desired conclusion is obtained. If $j = 1, 2, \ldots$, let $N_j' = E^{n+1} - h(E^{n+1} - N_j)$. Then $Y = \bigcap_{i=1}^{\infty} N_i'$ and, for $j = 1, 2, \ldots$, N_j' is a compact topological manifold and $N_{j+1}' \subset \text{int } N_j'$. By Lemma 1 of [20], $\pi_2(N_j, N_j - X) = 0$, so $\pi_1(N_j - X) = 0$. Hence $\pi_1(N_j' - Y) = 0$. This, along with the fact that $\pi_1(N_j', N_j' - Y) = 0$, implies $\pi_1(N_j') = 0$.

If $j = 1, 2, \ldots$, let $p_j: N_j \to M_j \times \{1/j\}$ denote the natural projection, and define $f_j: N_j \to N_j'$ by $f_j(x) = h(p_j(x))$ for all $x \in N_j$. The neighborhoods $\{N_i\}_{i=1}^{\infty}$ of X form, along with the inclusion maps, an inverse sequence $\mathbf{X}$ of ANR's whose inverse limit is X. Similarly $\{N_i'\}_{i=1}^{\infty}$ determines an inverse sequence $\mathbf{Y}$ whose inverse limit is Y. We shall show that (1) $\mathbf{f} = (f_i, \text{id}): \mathbf{X} \to \mathbf{Y}$ is a (level preserving) system map and (2) if $j = 1, 2, \ldots$, then $f_j: N_j \to N_j'$ is a homotopy

equivalence. Our conclusion will follow, since it is readily seen that if $f_j': N_j' \to N_j$ is a homotopy inverse of f_j, then $\mathbf{f}' = (f_i', \text{id}): \mathbf{Y} \to \mathbf{X}$ is a system map inverse to $\mathbf{f}$, thereby showing that $\text{Sh}(X) = \text{Sh}(Y)$.

Now, if $j = 1, 2, \ldots,$ then $p_{j+1} \simeq p_j | N_{j+1}$ in $N_j - X$. It follows that $f_{j+1} \simeq f_j | N_{j+1}$ in $N_j' - Y$ (consequently in N_j'). Hence $\mathbf{f}$ is a system map.

It now remains to be shown that if $j = 1, 2, \ldots,$ then $f_j: N_j \to N_j'$ is a homotopy equivalence. Since N_j and N_j' are simply connected ANR's, it suffices to show that $(f_j)_*: H_q(N_j) \to H_q(N_j')$ is an isomorphism for $q = 2, \ldots$. To this end, let $N_j = N$, $N_j' = N'$, $f_j = f$, $p_j = p$, and let

$$\alpha: \partial N \hookrightarrow N, \quad \beta: \partial N \hookrightarrow E^{n+1} - \text{int } N, \quad \delta: \partial N' \hookrightarrow N' \text{ and } \gamma: \partial N' \hookrightarrow E^{n+1} - \text{int } N'$$

denote the inclusion maps. Considering the Mayer-Vietoris sequence for

$$(E^{n+1}; N, E^{n+1} - \text{int } N),$$

we find that the sequence

$$0 = H_{q+1}(E^{n+1}) \to H_q(\partial N) \overset{\phi}{\to} H_q(N) \oplus H_q(E^{n+1} - \text{int } N) \to H_q(E^{n+1}) = 0$$

is exact. Hence $\phi = (\alpha_*, -\beta_*)$ is an isomorphism. It follows that

$$H_q(\partial N) = \ker \alpha_* \oplus \ker \beta_*$$

and that

$$\alpha_* | \ker \beta_*: \ker \beta_* \to H_q(N) \quad \text{and} \quad \beta_* | \ker \alpha_*: \ker \alpha_* \to H_q(E^{n+1} - \text{int } N)$$

are isomorphisms. Similarly

$$H_q(N') = \ker \delta_* \oplus \ker \gamma_*,$$

and

$$\delta_* | \ker \gamma_*: \ker \gamma_* \to H_q(N') \quad \text{and} \quad \gamma_* | \ker \delta_*: \ker \delta_* \to H_q(E^{n+1} - \text{int } N')$$

are isomorphisms.

Note that $\alpha p \cong \text{id}_N$. Therefore $p_*: H_q(N) \to H_q(\partial N)$ is a monomorphism and $H_q(\partial N) = p_*(H_q(N)) \oplus \ker \alpha_*$. Note also that $p_*(H_q(N)) \subset \ker \beta_*$ since $p(N)$ is inessential in $E^{n+1} \times \{1/j\} \subset E^{n+1} - \text{int } N$. Therefore $p_*(H_q(N)) = \ker \beta_*$. Note finally that if $h': \partial N \to \partial N'$ is defined by $h'(x) = h(x)$ for all $x \in \partial N$, then $h_*' | \ker \beta_*$ carries $\ker \beta_*$ isomorphically onto $\ker \gamma_* \subset H_q(\partial N')$ since h' extends to a homeomorphism of $E^{n+1} - \text{int } N$ onto $E^{n+1} - \text{int } N'$.

From the above and the commutative diagram

$$
\begin{array}{ccc}
H_q(N) & \xrightarrow{\; p_* \;} & p_*(H_q(N)) = \ker \beta_* \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle h_*} \\
H_q(N') & \xleftarrow[\; \delta_* \;]{} & \ker \gamma_*
\end{array}
$$

we see that f_* is an isomorphism, thereby completing the proof.

THEOREM 5.   *Let X and Y be r-shape connected continua in E^n of fundamental dimension at most k and satisfying ILC, where*

$$n \geq \max\ (2k + 2 - r, k + 4, 5).$$

Then $E^n - X \cong E^n - Y$ implies Sh (X) = Sh (Y).

Proof.   If $r \leq 0$, the result follows from Theorem 1 of [20]. We assume, then, that $r \geq 1$. Since $k \leq n - 4$, $k - (2k + 2 - n) \geq 2$, and so Theorem 5 of [10] shows that X and Y may be embedded up to shape in E^{n-1}, say as X' and Y'. The proof of Theorem 5 of [10] shows that the hypothesis of Theorem 3 may be assumed to apply to X' and Y'. Hence

$$E^n - X' \cong E^n - X \cong E^n - Y \cong E^n - Y'.$$

By Theorem 4, Sh (X') = Sh (Y'), hence Sh (X) = Sh (Y).

5. Movable continua

Let X be a continuum in a *PL* n-manifold M^n. Then X is *pointed m-movable* if there exists a point $x \in X$ with the following property: for every neighborhood U of X there exists a neighborhood V of X in U such that if $\phi \colon (K, k) \to (V, x)$ is a map of a pointed complex of dimension $\leq m$ into V and W is any neighborhood of X, then ϕ is homotopic in U to a map into W (keeping the base point fixed). All continua considered in this section shall be pointed 1-movable. It follows from Theorem 7.1.3 of [5] that all shape equivalences may be regarded as equivalences in the pointed shape category. Since the work of this section leans heavily on the notion of homotopy progroups, we shall find this of use. We therefore assume henceforth that when a certain shape morphism is given, it is a pointed morphism; however, we suppress base points from our notations.

A shape morphism $\mathbf{f} \colon X \to Y$ is said to be *shape r-connected* if $\mathbf{f}$ induces an isomorphism $\mathbf{f}_\# \colon \text{pro-}\pi_k(X) \to \text{pro-}\pi_k(Y)$ for $1 \leq k < r$ and an epimorphism $\mathbf{f}_\# \colon \text{pro-}\pi_r(X) \to \text{pro-}\pi_r(Y)$. A map is *shape r-connected* if it generates a shape r-connected shape morphism.

LEMMA 2.   *Let M^n be a PL n-manifold and let $X \subset$ int M^n be an r-shape connected, pointed $(r + 1)$-movable continuum of fundamental dimension at most k, where $(r + 1) < k \leq n - 3$ and $n \geq 5$. If X satisfies ILC then X has arbitrarily close compact PL manifold neighborhoods V such that V has a k-dimensional spine and the inclusion of X into V is shape $(r + 1)$-connected.*

Proof.   Let U be an open set containing X. By Theorems 1 and 2 there exists a neighborhood U_1 of X in U such that U_1 is r-connected. Choose a neighborhood V_1 of X satisfying the movability condition with respect to U_1. By Theorems 1 and 2 we may assume that V_1 is a compact *PL* manifold with an r-connected, k-dimensional spine K. Choose a smaller neighborhood V_2 of X satisfying the movability condition with respect to V_1.

We will find a neighborhood V of X such that $V_1 \subset V \subset U_1$, V has a k-dimensional spine P, P is r-connected, and the inclusion induced map $\pi_{r+1}(V_2) \to \pi_{r+1}(V)$ is onto. That will finish the proof since the choice of V_2 shows that if W is any neighborhood of X in V_2 then the image of the inclusion induced map $\pi_{r+1}(W) \to \pi_{r+1}(V_1)$ equals the image of $\pi_{r+1}(V_2) \to \pi_{r+1}(V_1)$ and thus $\pi_{r+1}(W) \to \pi_{r+1}(V)$ is an epimorphism.

Now $\pi_i(V_1) = 0$ for $i \leq r$, so $\pi_{r+1}(V_1)$ is finitely generated. Let

$$\beta_i: (S^{r+1}, s) \to (V_1, x), \quad i = 1, \ldots, j,$$

denote representatives of a generating set. The choice of V_1 implies that β_i is homotopic in U_1 to $\gamma_i: (S^{r+1}, s) \to (V_2, x)$. Let $p: V_1 \to K$ denote the end of a strong deformation retraction and let L denote the complex obtained from K by attaching $(r + 2)$-cells along $[p\beta_i][p\gamma_i]^{-1}$, $i = 1, \ldots, j$. The inclusion of K into U_1 extends to a map $f: L \to U_1$. Exactly as in the proof of Theorem 2, we can apply Theorem 4.3 of [19] to find a k-dimensional polyhedron $P \subset U_1$ such that $L \to f(L) \subset P$ is a simple homotopy equivalence. Let V be a regular neighborhood of P. As in the proof of Theorem 2 there exists a homeomorphism h such that $V_1 \subset h(V) \subset U_1$. Then $h(V)$ is the neighborhood we want.

We are now prepared to state a variant of Lemma 1.

LEMMA 3.   *Let X and Y be continua in E^n each satisfying ILC and having the shape of an $(r - 1)$-shape connected, pointed r-movable continuum of dimension at most k, where $n \geq \max (2k + 2 - r, k + 3, 5)$. Let $\mathbf{f} = \{f_i, X, Y\}$ and $\mathbf{f}' = \{f_i', Y, X\}$ be mutually inverse fundamental sequences in E^n. Let U_0 be an open $(2k + 1 - n)$-connected neighborhood of X in E^n such that the inclusion of X into U_0 is shape $(2k + 2 - n)$-connected, and $h: E^n \to E^n$ be a PL homeomorphism such that $Y \subset h(U_0)$ and such that there exists a neighborhood W_0 of Y with $h^{-1} | W_0 \simeq f_i' | W_0$ in U_0 for almost all i. Then for every neighborhood V_0 of Y, there exist an open $(2k + 1 - n)$-connected neighborhood V of Y lying in $V_0 \cap h(U_0)$ such that the inclusion of Y into V is shape $(2k + 2 - n)$-connected, a PL homeomorphism $q: E^n \to E^n$, and a neighborhood U_1 of X such that*

(1) $q \,|\, E^n - U_0 = h \,|\, E^n - U_0$,
(2) $q(X) \subset V$, and
(3) $q \,|\, U_1 \simeq f_i \,|\, U_1$ in V for almost all i.

Proof.   By Theorem 1, X has arbitrarily close compact PL manifold neighborhoods with k-dimensional spines. The proof of Lemma 1 will thus apply here if we can find an arbitrarily small open neighborhood V of Y such that V is $(2k + 1 - n)$-connected, the inclusion of Y into V is shape $(2k + 2 - n)$-connected, and the pair $(h(U_0), V)$ is $(2k + 2 - n)$-connected.

By Lemma 2, there exist arbitrarily small neighborhoods V of Y such that V is $(r - 1)$-connected and the inclusion of Y into V is shape r-connected. We need to check that the pair $(h(U_0), V)$ is r-connected. Let

$$\mathbf{h}^{-1}: h(U_0) \to U_0, \quad \mathbf{j}: X \to U_0 \quad \text{and} \quad \mathbf{k}: Y \to h(U_0)$$

be shape morphisms generated by $h^{-1}: h(U_0) \to U_0$ and the inclusions $j: X \to U_0$, $k: Y \to h(U_0)$. Note that the hypothesis $h^{-1} | W_0 \simeq f_i' | W_0$ in U_0 for almost all i shows that $\mathbf{j}f' \simeq \mathbf{h}^{-1}\mathbf{k}$, and so $\mathbf{hj}f' \simeq \mathbf{k}$. By hypothesis $\mathbf{j}$ induces an epimorphism

$$\mathbf{j}_\# : \text{pro-}\pi_r(X) \to \text{pro-}\pi_r(U_0)$$

and so, since $\mathbf{f}'$ and $\mathbf{h}$ are shape equivalances, $\mathbf{k}$ induces an epimorphism

$$\mathbf{k}_\# : \text{pro-}\pi_r(Y) \to \text{pro-}\pi_r(h(U_0)).$$

It follows that in the exact sequence

$$\cdots \to \pi_r(V) \xrightarrow{\alpha} \pi_r(h(U_0)) \to \pi_r(h(U_0), V) \to 0,$$

α is an epimorphism, and so $\pi_r(h(U_0), V) = 0$.

We are now prepared to prove Theorem C.

Proof of Theorem C. Suppose first that Sh $(X) = $ Sh (Y). Then use Lemma 3 as Lemma 1 was used in the proof of Theorem 3 to show that $E^n - X \cong E^n - Y$. Next suppose that $E^n - X \cong E^n - Y$. Since $k \leq n - 4$, the case $k \leq 2$ follows from Theorem 1 of [20]. If $k \geq 3$, X and Y can be embedded up to shape in E^{n-1} by Corollary 2 of [9]. By the direction of the theorem proved above, $E^n - X \cong E^n - X'$ and $E^n - Y \cong E^n - Y'$. Thus Sh $(X) = $ Sh (Y) by Theorem 4.

The next theorem generalizes Theorem 1 of [12].

THEOREM 6. *Let K and L be simply connected subpolyhedra of E^n. If $E^{n+1} - K \cong E^{n+1} - L$ then K and L have the same homotopy type.*

Proof. Let $h: E^{n+1} - K \to E^{n+1} - L$ be a homeomorphism. As in the proof of Theorem 4, we may assume that h induces a homeomorphism of the quotient space E^{n+1}/K to E^{n+1}/L. Choose regular neighborhoods $M_1 \supset M_2 \supset M_3$ of K and $N_1 \supset N_2 \supset N_3 \supset N_4$ of L such that $h(\partial M_i) \subset N_i - \text{int } N_{i+1}$, for $i = 1, 2$, and 3. We may assume that M_i is of the form

$$M_i = M_i' \times [-1/i, \, 1/i] \subset E^n \times E^1 = E^{n+1}.$$

Define $f: h(\partial M_2) \to \partial N_2$ to be the map which pushes $h(\partial M_2)$ across the product structure $N_2 - \text{int } N_3 \cong \partial N_2 \times [0, 1]$ into ∂N_2. Define

$$f': \partial N_2 \to h(\partial M_2)$$

by pushing along the product structure $h(M_1 - \text{int } M_2) \cong h(\partial M_2) \times [0, 1]$. Then $f'f \simeq \text{id}$ in $h(M_1 - M_3)$ and so $f'f \simeq \text{id}$ in $h(\partial M_2)$. Similarly, $ff' \simeq \text{id}$ in ∂N_2. Thus $\bar{f}: \partial M_2 \to \partial N_2$ defined by $\bar{f}(x) = f(h(x))$ is a homotopy equivalence. Note that $\bar{f}$ extends to a homotopy equivalence of $E^{n+1} - \text{int } M_2$ to $E^{n+1} - \text{int } N_2$. Let $p: M_2 = M_2' \times [-1/2, 1/2] \to M_2' \times \{1/2\}$ denote the natural projection. Then $\bar{f}p: M_2 \to N_2$ is a homotopy equivalence exactly as in the proof of Theorem 4.

Proof of Theorem D. If Sh (X) = Sh (Y), then $E^n - X \cong E^n - Y$, by Theorem C. If $E^n - X \cong E^n - Y$, we first find polyhedra $K, L \subset E^{n-1}$ representing the shape classes of X and Y respectively by Corollary 4.2 of [19] and then apply Theorem 6.

6. Borsuk's notion of position

A complement theorem is a sort of weak unknotting theorem. Another weak type of unknotting, the notion of *position*, has been considered by Borsuk (Chapter XI of [1]). We recall a definition of Borsuk from [1]. Let M and N be spaces, $M \supset A_1 \supset A_2 \supset \cdots$ and $N \supset B_1 \supset B_2 \supset \cdots$. Then the sequences $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ are said to be *similar* if there exist homeomorphisms $h_i \colon M \to N$, $i = 1, 2, \ldots$, such that (1) $h_i(A_i) = B_i$, $i = 1, 2, \ldots$, and (2) if $1 \le i < j$, then

$$h_i | M - A_i = h_j | M - A_i \quad \text{and} \quad h_i | A_i \simeq h_j | A_i \quad \text{in } B_i.$$

A careful reading of the proof of Theorem 3 shows that under its hypothesis, if Sh (X) = Sh (Y), then there exist compact *PL* manifolds

$$M_1 \supset \text{int } M_1 \supset M_2 \supset \text{int } M_2 \supset M_3 \supset \cdots \quad \text{and}$$

$$N_1 \supset \text{int } N_1 \supset N_2 \supset \text{int } N_2 \supset N_3 \supset \cdots$$

such that $\{M_i\}_{i=1}^{\infty}$ and $\{N_i\}_{i=1}^{\infty}$ are similar, $X = \bigcap_{i=1}^{\infty} M_i$, and $Y = \bigcap_{i=1}^{\infty} N_i$. (Specifically, M_i may be taken to be cl (S_{2i}) in the proof of Lemma 4.2 of [7], noting that the homeomorphism q constructed in Lemma 1 may be constructed so that $q | U_0 \simeq h | U_0$ in $h(U_0)$.) The following result is a consequence of this observation and Theorem 8.6 on page 336 of [1].

THEOREM 7. *Let X and Y be r-shape connected continua in E^n of fundamental dimension at most k and satisfying ILC, where*

$$n \ge \max (2k + 2 - r, k + 3, 5).$$

Then Sh (X) = Sh (Y) *if and only if* Pos (E^n, X) = Pos (E^n, Y).
Similarly we have the following, corresponding to Theorem C.

THEOREM 8. *Let X and Y be $(r-1)$-shape connected, pointed r-movable continua in E^n of fundamental dimension at most k satisfying ILC where*

$$n \ge \max (2k + 2 - r, 5, k + 3).$$

Then Sh (X) = Sh (Y) *if and only if* Pos (E^n, X) = Pos (E^n, Y).

REFERENCES

1. K. BORSUK, *Theory of shape*, Mathematical Monographs, vol. 59, Polish Scientific Publishers, Warsaw, 1975.
2. T. A. CHAPMAN, *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund. Math., vol. 76 (1972), pp. 181–193.

3. ——, *Shapes of finite-dimensional compacta*, Fund. Math., vol. 76 (1972), pp. 261–276.

4. D. CORAM and P. DUVALL, JR., *Neighborhoods of sphere-like continua*, General Topology and Appl., vol. 6 (1976), pp. 191–198.

5. J. DYDAK and J. SEGAL, *Shape theory, an introduction*, Lecture Notes in Mathematics No. 688, Springer-Verlag, New York, 1978.

6. S. FERRY, *A stable converse to the Vietoris-Smale theorem with applications to shape theory*, Trans. Amer. Math. Soc., vol. 261 (1980), pp. 369–386.

7. R. GEOGHEGAN and R. R. SUMMERHILL, *Concerning the shapes of finite-dimensional compacta*, Trans. Amer. Math. Soc., vol. 178 (1973), pp. 281–292.

8. J. G. HOLLINGSWORTH and T. B. RUSHING, *Embeddings of shape classes of compacta in the trivial range*, Pacific, J. Math., vol. 60 (2) (1975), pp. 103–110.

9. L. S. HUSCH and I. IVANŠIĆ, *Embeddings and concordances of embeddings up to shape*, to appear.

10. I. IVANŠIĆ, *Embeddings compacta up to shape*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., vol. 25 (1977), pp. 471–475.

11. V.-T. LIEM, *Certain continua in S^n of the same shape have homeomorphic complements*, Trans. Amer. Math. Soc., vol. 218 (1976), pp. 207–217.

12. ——, *Certain continua in S^n with homeomorphic complements have the same shape*, Fund. Math., vol. 97 (1977), pp. 221–228.

13. D. R. MCMILLAN, JR., *A criterion for cellularity in a manifold*, Ann. of Math. (2), vol. 79 (1964), pp. 327–337.

14. S. NOWAK, *On the fundamental dimension of approximatively 1-connected compacta*, Fund. Math., vol. 89 (1975), pp. 61–79.

15. T. B. RUSHING, *The compacta X in S^n for which Sh (X) = Sh (S^k) is equivalent to $S^n - X \approx S^n - S^k$*, Fund. Math., vol. 97 (1977), pp. 1–8.

16. R. B. SHER, *Defining subsets of manifolds by cells*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., vol. 17 (1969), pp. 363–365.

17. L. C. SIEBENMANN, *Chapman's classification of shapes, a proof using collapsing*, Manuscripta Math., vol. 16 (1975), pp. 373–384.

18. J. R. STALLINGS, *The piecewise linear structure of Euclidean space*, Proc. Cambridge Philos. Soc., vol. 58 (1962), pp. 481–488.

19. ——, *The embedding of homotopy types into manifolds*, mimeographed notes, Princeton University, 1965.

20. G. A. VENEMA, *Embeddings of compacta with shape dimension in the trivial range*, Proc. Amer. Math. Soc., vol. 55 (1976), pp. 443–448.

21. ——, *Weak flatness for shape classes of sphere-like continua*, General Topology and Appl., vol. 7 (1977), pp. 309–319.

22. ——, *Neighborhoods of compacta in Euclidean space*, Fund. Math., to appear.

23. E. C. ZEEMAN, *Seminar on combinatorial topology*, Mimeographed Notes, Institut Hautes Etudes Sci., Paris, 1963.

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