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INEQUALITIES ON BERGMAN SPACES

BY

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I. Introduction

Let U denote the open unit disk. If $p > 0$, A^p denotes the Bergman space of functions f which are analytic in U and for which $|f|^p$ is integrable on U. If ϕ is a bounded measurable function on U let T_ϕ denote the operator

$$T_\phi f = P(\phi f), \qquad f \in A^2,$$

where P is the orthogonal projection from L^2 onto A^2 (the Bergman projection). These operators were considered in McDonald and Sundberg [2] where their compactness properties were studied. Let G be any measurable subset of U. Consider the following problems:

(A) What properties must G have in order that the operation $f \mapsto f|_G$ from A^p to $L^p(G)$ have closed range? i.e. If G has positive measure, when is

$$\iint_U |f|^p \, dm \le \text{const} \iint_G |f|^p \, dm, \qquad f \in A^p?$$

Here dm is two-dimensional Lebesgue measure.

(B) For which bounded analytic functions ϕ is the operator T_ϕ bounded below on the unit sphere of A^2?

(C) Define $H^p(w)$ to be the closure of the polynomials in the $L^p(w\, dm)$ norm, where w is a non-negative integrable function on U. What is $H^p(w)$?

(D) For which ϕ is T_ϕ invertible?

In this paper problems A and B are solved completely in the sense that necessary and sufficient conditions on G and ϕ are found. In problem A the condition on G describes how G "spreads out" at the boundary. The condition on ϕ in problem B describes the set where $|\phi|$ stays away from zero.

With regard to problem C, a sufficient condition on w is obtained that yields $A^p = H^p(w)$. Problem D is solved for non-negative functions ϕ.

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II. Statement of results

The main result is the following solution of Problem A.

MAIN THEOREM. *Let G be a measurable subset of U and $p > 0$. Then the following two conditions are equivalent.*

(1) *There is a constant $C > 0$ such that*

$$\iint_U |f|^p \, dm \leq C \iint_G |f|^p \, dm, \qquad f \in A^p.$$

(2) *There is a constant $\delta > 0$ such that $m(G \cap D) > \delta m(U \cap D)$ for all disks D whose centers lie on $|z| = 1$.*

The results on problems B, C, and D are now easy to obtain as corollaries.

COROLLARY 1. *Let ϕ be a bounded measurable function on U. Then there is a constant $\varepsilon > 0$ such that*

$$\iint_U |\phi f|^p \, dm \geq \varepsilon \iint_U |f|^p \, dm, \qquad f \in A^p,$$

if and only if there exists $r > 0$ such that the set $\{z \in U\colon |\phi(z)| > r\}$ satisfies condition (2).

This gives the promised solution of problem B since, when ϕ is analytic, $\|T_\phi f\|^2 = \iint |\phi f|^2 \, dm$. This should be compared with Proposition 22 in [2] which solves problem B for inner functions, and which can be shown to follow from Corollary 1.

COROLLARY 2. *Suppose w is a non-negative integrable function on U and suppose the following hold.*

(a) $\iint_{D \cap U} w \, dm \leq \text{const} \cdot m(D \cap U)$ *for all disks D with centers on $|z| = 1$.*
(b) *There exists $r > 0$ such that the set $\{z \in U\colon w(z) > r\}$ satisfies* (2).

Then $H^p(w) = A^p$. In fact (a) *implies $A^p \subseteq H^p(w)$ and* (b) *implies $H^p(w) \subseteq A^p$.*

The implication (a) $\Rightarrow A^p \subseteq H^p(w)$ is a special case of the main result in Hastings [1] and the other implication, (b) $\Rightarrow H^p(w) \subseteq A^p$, is a consequence of the "if" half of Corollary 1.

COROLLARY 3. *Let ϕ be a bounded positive measurable function on U. Then T_ϕ is invertible if and only if there exists $r > 0$ such that $\{z \in U\colon \phi(z) > r\}$ satisfies* (2).

Proof. If T_ϕ is invertible then there exists $\varepsilon > 0$ such that

$$\iint |\phi f|^2 \, dm \geq \|T_\phi f\|^2 \geq \varepsilon \|f\|^2. \tag{3}$$

If ϕ did not satisfy the condition given, then there is a sequence $r_n \to 0$ such that $\{z\colon \phi(z) > r_n\}$ does not satisfy (2). This means (1) is not satisfied so that a sequence $\{f_n\}$ in A^2 exists such that $\iint |f_n|^2 \, dm = 1$ and

$$\iint_{\phi > r_n} |f_n|^2 \, dm \to 0.$$

This violates (3) because (assuming $\phi \leq 1$)

$$0 < \varepsilon \leq \iint |\phi f_n|^2 \, dm \leq r_n \iint_{\phi \leq r_n} |f_n|^2 \, dm + \iint_{\phi > r_n} |f_n|^2 \, dm$$

and both term on the right tend to zero. Conversely, suppose $\{z\colon \phi(z) > r\}$ satisfies (2). Then it also satisfies (1) so

$$\iint \phi^2 |f|^2 \, dm \geq r^2 \iint_{\phi > r} |f|^2 \, dm \geq \frac{r^2}{C} \|f\|^2.$$

Without loss of generality we may suppose $\phi \leq 1$. Then

$$\begin{aligned} \|(I - T_\phi)f\|^2 &= \|T_{1-\phi} f\|^2 \\ &\leq \iint (1-\phi)^2 |f|^2 \, dm \\ &\leq \iint (1-\phi^2)|f|^2 \\ &\leq \left(1 - \frac{r^2}{C}\right)\|f\|^2. \end{aligned}$$

Thus $\|I - T_\phi\| < 1$ which implies T_ϕ is invertible. QED.

III. Geometry of disks

The proof of the main theorem is considerably simplified if condition (2) is replaced by either of two conditions equivalent to it. This section is devoted to establishing these equivalences.

Suppose we have a set G satisfying condition (2). Then we may regard $\delta > 0$ as given and unchanging throughout this argument. There exist constants $C > 0$ and $0 < \eta < 1$, depending only on δ, with the following properties. Associated with each disk $D = \{z\colon |z - b| < r\}$ with center b on $|z| = 1$, there is a disk which will be denoted

$$D(a, \eta) = \{z: |z - a| < \eta(1 - |a|)\}$$

whose center a lies on the radius from 0 to b such that

$$1 - |a| \le Cr \quad \text{and} \quad m(D \cap U - D(a, \eta)) < \tfrac{1}{2}\delta m(D \cap U).$$

(Note that the radius of $D(a, \eta)$ is $\eta(1 - |a|)$, not η, so that $D(a, \eta) \subseteq U$.) The constant C may be taken to be $4/\delta\pi$ and η may be $\sqrt{1 - \delta/(8C^2)}$. Then one condition equivalent to (2) is:

(2′) *There exist* $\delta_0 > 0$ *and* η *with* $0 < \eta < 1$ *such that*

$$m(G \cap D(a, \eta)) > \delta_0 m(D(a, \eta)) \quad \textit{for all } a \in U.$$

Proof. Let D and $D(a, \eta)$ be associated with one another as above. If G satisfies (2) then

$$m(G \cap D(a, \eta)) \ge m(G \cap D) - \frac{\delta}{2} m(D \cap U) > \frac{\delta}{2} m(D \cap U) > \frac{\delta\pi}{8C^2} m(D(a, \eta))$$

so G satisfies (2′) with $\delta_0 = \delta\pi/(8C^2)$.

Conversely if G satisfies (2′), let D be a disk centered on $|z| = 1$ that just contains $D(a, \eta)$. Then D has radius $(1 + \eta)(1 - |a|)$, so

$$\begin{aligned} m(G \cap D) &> m(G \cap D(a, \eta)) \\ &> \delta_0 m(D(a, \eta)) \\ &\ge \delta_0 \pi\eta^2(1 - |a|^2) \\ &\ge \frac{\delta_0 \eta^2}{(1 + \eta)^2} m(D \cap U). \end{aligned}$$

Therefore (2) holds with $\delta = \delta_0 \eta^2/(1 + \eta)^2$. QED.

In Lemmas 2 and 3 we will have to pass from the disks $D(a, \eta)$ to disks defined in terms of the pseudohyperbolic metric. For this we need the following. Define

$$\Delta(a, r) = \left\{z \in U: \frac{|z - a|}{|1 - \bar{a}z|} < r\right\}.$$

Then:

(4) *If* $z \in D(a, \eta)$ *and* $2\eta/(1 + \eta^2) \le r < 1$ *then*

$$D(a, \eta) \subseteq \Delta(z, r);$$

(5) *There exist constants* C_r *depending only on* r *such that*

$$\frac{(1 - |a|)^2}{C_r} \le m(\Delta(a, r)) \le C_r(1 - |a|)^2.$$

To prove (4) simply estimate $|z - z'|/|1 - \bar{z}z'| \leq 2\eta/(1 + \eta^2)$ for z, $z' \in D(a, \eta)$. This estimate is simplified by the fact that the maximum occurs at $z = a + \eta(1 - a)$ and $z' = a - \eta(1 - a)$ when $0 < a < 1$.

Proving (5) is an easy estimate of the diameter of $\Delta(a, r)$. Estimates like those used in the proof of (2′) (using (4) in one direction) show that (2′) is equivalent to:

(2″) *There exist $\delta_1 > 0$ and η_1 with $0 < \eta_1 < 1$ such that*

$$m(G \cap \Delta(a, \eta_1)) > \delta_1 m(\Delta(a, \eta_1)), \qquad a \in U.$$

IV. Proof of the main theorem

The proof that (1)⇒(2) is relatively simple. We actually prove (1)⇒(2″). Take $0 < \eta_1 < 1$ so that

$$\frac{1}{\pi} \iint_{|z| < \eta_1} 1 \, dm > 1 - \frac{1}{2C}.$$

Using the change of variables $z \to (z - a)/(1 - \bar{a}z)$ we get

$$\frac{1}{\pi} \iint_{U - \Delta(a, \eta_1)} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm < \frac{1}{2C}.$$

We have used the fact that the Jacobian is just

$$\left| \frac{d}{dz} \frac{z - a}{1 - \bar{a}z} \right|^2.$$

Applying (1) to the function $f(z) = (1 - |a|^2)^{2/p}/(1 - \bar{a}z)^{4/p}$ we get

$$\frac{1}{\pi} \iint_G \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm \geq \frac{1}{C} \left(\frac{1}{\pi} \iint_U \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm \right) = \frac{1}{C}$$

whence

$$\frac{1}{\pi} \iint_{G \cap \Delta(a, \eta_1)} \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm \geq \frac{1}{C} - \frac{1}{2C} = \frac{1}{2C}. \tag{6}$$

It is easy to verify that

$$\frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} \leq \frac{4}{(1 - |a|)^2}.$$

Combining this with (6) we get

$$m(G \cap \Delta(a, \eta_1) \geq \frac{1}{8C} (1 - |a|)^2$$

which, because of (5), gives (2″).

The implication $(2') \Rightarrow (1)$ is the difficult one to prove. It requires three lemmas. Throughout we assume that δ_0 and η are given by $(2')$ and fixed. All constants used depend only on η and p unless explicitly stated otherwise. In particular they do not depend on the function f. We use the convention that the letter C denotes a constant which may differ from one occurrence to the next. Since η is fixed we abbreviate $D(a, \eta)$ by $D(a)$. If the analytic function f is given and $0 < \lambda < 1$ we define the set

$$E_\lambda(a) = E_\lambda(f, a) = \{z \in D(a) \colon |f(z)| > \lambda |f(a)|\}$$

and the operator

$$B_\lambda f(a) = \frac{1}{m(E_\lambda(a))} \iint_{E_\lambda(a)} |f|^p \, dm.$$

Note that

$$B_\lambda f(a) \geq \frac{1}{m(D(a))} \iint_{D(a)} |f|^p \, dm \geq |f(a)|^p$$

because $|f|^p$ is subharmonic. Henceforth it will be assumed $p = 1$, the proof of the general case can be obtained with only minor modifications on replacing $|f|$ by $|f|^p$.

LEMMA 1. *Let f be analytic in U and $a \in U$. Then*

$$\frac{m(E_\lambda(a))}{m(D(a))} \geq \frac{\log \dfrac{1}{\lambda}}{\log \dfrac{B_\lambda f(a)}{|f(a)|} + \log \dfrac{1}{\lambda}}. \tag{7}$$

Proof. For this lemma the size and placement of the disk are immaterial so we assume $a = 0$ and $D = D(a)$ has area $m(D) = 1$. Applying Jensen's inequality and elementary estimates we have

$$\begin{aligned}
\log |f(0)| &\leq \iint_D \log |f| \, dm \\
&= \iint_{D - E_\lambda(0)} + \iint_{E_\lambda(0)} \\
&\leq [1 - m(E_\lambda(0))] \log \lambda |f(0)| + m(E_\lambda(0)) \frac{1}{m(E_\lambda(0))} \iint_{E_\lambda(0)} \log |f| \, dm \\
&\leq [1 - m(E_\lambda(0))] \log \lambda |f(0)| + m(E_\lambda(0)) \log B_\lambda f(0)
\end{aligned}$$

The last inequality is due to the concavity of log. If we subtract log $|f(0)|$ from both sides we get

$$0 \leq [1 - m(E_\lambda(0))] \log \lambda + m(E_\lambda(0)) \log \left(\frac{B_\lambda f(0)}{|f(0)|} \right).$$

Notice that

$$\log \lambda < 0 \quad \text{and} \quad \log \left(\frac{B_\lambda f(0)}{|f(0)|} \right) > 0.$$

Solving for $m(E_\lambda(0))$ we get

$$m(E_\lambda(0)) \geq \frac{\log \frac{1}{\lambda}}{\log \frac{B_\lambda f(0)}{|f(0)|} + \log \frac{1}{\lambda}}$$

as required. QED.

The purpose of this lemma is to show eventually that $E_\lambda(a)$ takes up a large enough fraction of $D(a)$ to include some of $G \cap D(a)$. This will not be true for all $a \in U$ since $B_\lambda f(a)/|f(a)|$ may be very large. Therefore we use Lemmas 2 and 3 to show that the set of a's where $B_\lambda f(a)/|f(a)|$ is not too large is sufficient.

LEMMA 2. *Let* $\varepsilon > 0$ *and* $f \in A^1$. *Define the set*

$$A = \left\{ a \in U \colon |f(a)| < \frac{\varepsilon}{m(D(a))} \iint_{D(a)} |f| \, dm \right\}.$$

There is a constant C depending only on η *such that*

$$\iint_A |f| \, dm \leq C\varepsilon \iint_U |f| \, dm.$$

Proof. For $a \in A$ we have

$$|f(a)| \leq \varepsilon \iint_U |f(z)| \frac{1}{m(D(a))} \chi_{D(a)}(z) \, dm(z).$$

Integrate over $a \in A$ and use Fubini's Theorem on the right to obtain

$$\iint_A |f(a)| \, dm(a) \leq \varepsilon \iint_U |f(z)| \left[\iint_A \frac{1}{m(D(a))} \chi_{D(a)}(z) \, dm(a) \right] dm(z).$$

The proof will be done if it can be shown that the bracketed expression is suitably bounded. Using (4) with $r = 2\eta/(1 + \eta^2)$ we can write

$$\chi_{D(a)}(z) \leq \chi_{\Delta(a,r)}(z) = \chi_{\Delta(z,r)}(a),$$

the last equality because $z \in \Delta(a, r)$ precisely when $a \in \Delta(z, r)$. Thus the bracketed integral is dominated by

$$\iint_{\Delta(z,r)} \frac{1}{m(D(a))}\, dm(a). \tag{8}$$

However, if $a \in \Delta(z, r)$, $m(D(a)) \geq (1/C)(1 - |z|)^2$ for a suitable constant. Combining this with (5) we see that (8) is bounded by a constant independent of z. QED.

The only use made of Lemma 2 is in the proof of the following. If $p \neq 1$, in addition to changing $|f|$ to $|f|^p$, ε^3 should be changed to $\varepsilon^{1+2/p}$ in Lemma 3. We assume from now on that $\lambda < \frac{1}{2}$.

LEMMA 3. *Let* $0 < \varepsilon < 1$ *and* $f \in A^1$. *Define the set*

$$B = \{a \in U\colon |f(a)| < \varepsilon^3 B_\lambda f(a)\}.$$

Then there is a constant C *depending only on* η *(and* p*) such that*

$$\iint_B |f|\, dm \leq C\varepsilon \iint_U |f|\, dm$$

Proof. Write

$$\iint_B |f|\, dm = \iint_{B\cap A} |f|\, dm + \iint_{B-A} |f|\, dm.$$

The first integral is estimated by Lemma 2. For the second integral, use Fubini's Theorem as before to obtain

$$\iint_{B-A} |f|\, dm \leq \varepsilon^3 \iint_U |f(z)| \iint_{B-A} \frac{1}{m(E_\lambda(a))} \chi_\lambda(a, z)\, dm(a)\, dm(z),$$

where $\chi_\lambda(a, z)$ is the characteristic function of $E_\lambda(a)$ evaluated at z. Again, we need only show the inner integral is suitably bounded. Since $\chi_\lambda(a, z) \leq \chi_{D(a)}(z)$, we can invoke the argument in Lemma 2 provided we can show

$$\frac{1}{m(E_\lambda(a))} \leq \frac{C}{\varepsilon^2 m(D(a))}$$

whenever $a \notin A$. We do this by showing that any disk D centered at a contains a concentric disk D' of area $(1/C)\varepsilon^2 m(D)$ with the following property. Whenever f is analytic and

$$|f(a)| \geq \varepsilon \frac{1}{m(D)} \iint_D |f|\, dm.$$

Then $|f(z)| > \frac{1}{2}|f(a)| > \lambda |f(a)|$ on D'.

Without loss of generality we take $a = 0$, $D = U$, and

$$(1/\pi) \iint_U |f| \, dm = 1.$$

Our hypothesis then is $|f(0)| \geq \varepsilon$. There is a constant $C > 1$ (depending only on p) such that $|f(z)| < C$ on the set $|z| = \frac{1}{2}$. Assuming $|z| < \frac{1}{4}$ we have

$$2\pi |f(z) - f(0)| \leq \left| \int_{|t| = 1/2} f(t)\left(\frac{1}{t - z} - \frac{1}{t}\right) dt \right| \leq C \cdot 8\pi |z|.$$

Choosing $|z| < \varepsilon/8C$ we see that $|f(z)| > |f(0)| - \varepsilon/2 > \frac{1}{2}|f(0)|$ on a disk about zero of area $\pi(\varepsilon/8C)^2$. Translating this back to the disk $D(a)$ shows that $E_\lambda(a)$ contains a disk of area $(\varepsilon^2/C)m(D(a))$ whenever $a \notin A$. This is just what we needed. QED.

Let $F = U - B = \{a \in U\colon |f(a)| \geq \varepsilon^3 B_\lambda f(a)\}$. If we now choose ε so small that $\varepsilon C < \frac{1}{2}$ (a choice which depends only on η and p) we have

$$\iint_U |f| \, dm < 2 \iint_F |f| \, dm. \tag{9}$$

For $a \in F$ we have $B_\lambda f(a)/|f(a)| \leq 1/\varepsilon^3$. Therefore, if we choose λ less than ε^{6/δ_0} we get, from (7),

$$\frac{m(E_\lambda(a))}{m(D(a))} > \frac{(2/\delta_0) \log (1/\varepsilon^3)}{\log (1/\varepsilon^3) + (2/\delta_0) \log (1/\varepsilon^3)}$$
$$> 1 - \frac{\delta_0}{2}.$$

Consequently, (2′) implies, for $a \in F$,

$$m(G \cap E_\lambda(a)) > \tfrac{1}{2}\delta_0 m(D(a)) \tag{10}$$

where the choice of λ depended only on η, δ_0, and p.

We are now in a position to complete the proof. Because of (10) we have

$$\frac{1}{m(D(a))} \iint_G \chi_{D(a)}(z) |f(z)| \, dm \geq \tfrac{1}{2}\delta_0 \lambda |f(a)|, \quad a \in F.$$

We integrate this over F, using Fubini's Theorem on the left, to obtain

$$\iint_G |f(z)| \left[\iint_F \frac{1}{m(D(a))} \chi_{D(a)}(z) \, dm(a) \right] dm(z) \geq \tfrac{1}{2}\delta_0 \lambda \iint_F |f| \, dm.$$

The integral in brackets can then be treated exactly as in Lemma 2 and the right hand side can be estimated from below using (9). This gives

$$C \iint_G |f(z)| \, dm(z) \geq \tfrac{1}{4}\delta_0 \lambda \iint_U |f| \, dm.$$

This completes the proof of $(2') \Rightarrow (1)$.

V. Remarks

It is unfortunate that the proof (Lemmas 2 and 3) requires $f \in A^p$ initially. That is, it does not follow from this proof that $\iint_U |f|^p \, dm$ will be finite when $\iint_G |f|^p \, dm$ is. This is a defect I have been unable to remedy.

The main theorem can be extended to some weighted A^p spaces without much difficulty. In particular we have the following if $\alpha > -1$.

$$\iint_U |f(z)|(1 - |z|^2)^\alpha \, dm(z) \leq C \iint_G |f(z)|(1 - |z|^2)^\alpha \, dm(z)$$

if and only if condition (2) *of the main theorem holds.*

The crucial properties of the weight $w(z) = (1 - |z|^2)^\alpha$ are the following:

$$w(a) \leq C \inf \{w(z) \colon z \in D(a, \eta)\}$$

and

$$w\left(\frac{z-a}{1-\bar{a}z}\right) = \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2}\right)^\alpha (1 - |z|^2)^\alpha.$$

The first property is needed in Lemmas 2 and 3. It implies, for example, that

$$\iint_A |f(a)|^p w(a) \, dm(a) \leq C\varepsilon \iint_U |f(z)|^p w(z) \iint_A \frac{1}{m(D(a))} \chi_{D(a)}(z) \, dm(a) \, dm(z).$$

whence the proof of Lemma 2 can proceed as before. The second property is useful in proving the implication $(1) \Rightarrow (2'')$, which goes much like the case $w \equiv 1$.

We conclude with two questions.

(1) What are necessary and sufficient conditions on a weight function w in order to satisfy

$$\iint_U |f|^p \, dm \leq C \iint |f|^p w \, dm?$$

The main theorem settles the case $w = \chi_G$.

(2) Can $H^p(w)$ in Corollary 2 be replaced by

$$A^p(w) = \{f \text{ analytic in } U \text{ such that } \iint |f|^p w \, dm < +\infty\}?$$

This would be the case if the main theorem could be proved without the hypothesis $f \in A^p$.

Added in proof. Question (2) can be settled in the negative. The hypothesis $f \in A^p$ is a necessary part of the main theorem: There exists a set G satisfying condition (2) and a function f analytic in U such that $\iint_G |f|^p \, dm < +\infty$ but $\iint_U |f|^p \, dm = +\infty$.

References

1. W. W. HASTINGS, *A Carleson measure theorem for Bergman spaces*, Proc. Amer. Math. Soc., vol. 52 (1975), pp. 237–241.
2. G. MCDONALD and C. SUNDBERG, *Toeplitz operators on the disc*, Indiana Univ. Math. J., vol. 28 (1979), pp. 595–611.

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