

## GROWTH PROBLEMS FOR A CLASS OF ENTIRE FUNCTIONS VIA SINGULAR INTEGRAL ESTIMATES

BY

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Let  $f$  be an entire function with zeros  $\{z_n\}$ , and let

$$M(r, f) = \max_{\theta} |f(re^{i\theta})|, \quad L(r, f) = \min_{\theta} |f(re^{i\theta})|,$$

$$n(r) = n(r, 0; f) = \sum_{|z_n| \leq r} 1,$$

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

We consider a problem motivated by a classical theorem of Pólya and Valiron; this gives lower bounds for

$$(1) \quad c(f) = \limsup_{r \rightarrow \infty} \frac{n(r)}{\log M(r, f)}$$

for functions of finite nonintegral order  $\rho$ . Let

$$C(\rho) = \inf \{c(f) : f \text{ of order } \rho\}.$$

Then Pólya [6] and Valiron [9] [10] proved, independently, that

$$(2) \quad C(\rho) = \frac{1}{\pi} \sin \pi \rho \quad (0 \leq \rho \leq 1),$$

$$(3) \quad \frac{1}{\pi} |\sin \pi \rho| \geq C(\rho) \geq \frac{|\sin \pi \rho|}{A_0 \{\log \rho + 1\} |\sin \pi \rho| + \pi} \quad (1 < \rho < \infty),$$

for an absolute constant  $A_0$ . The upper estimate in (3) comes from the Lindelöf functions

$$f_{\rho}(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{\sigma}}\right) \exp \left\{ \sum_{k=1}^q \frac{1}{k} \left(\frac{z}{-n^{\sigma}}\right)^k \right\} \quad (\sigma = \rho^{-1}, q = [\rho]),$$

for which  $n(r) \sim r^{\rho}$ ,  $\log M(r, f) \sim \pi |\csc \pi \rho| r^{\rho}$  when  $\rho$  is nonintegral and  $r \rightarrow \infty$ . We conjecture that these  $f_{\rho}$ , having all zeros regularly distributed on a single ray  $\arg z = \text{constant}$ , are extremal for this problem, i.e. that  $C(\rho) = \pi^{-1} |\sin \pi \rho|$ . But not even the order of magnitude of  $C(\rho)$  is known, for  $\rho$  large.

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In this direction, we prove the existence of an absolute constant  $A$  such that

$$(4) \quad c(f) \geq A |\sin \pi \rho| \quad (1 < \rho < \infty)$$

for  $f$  of order  $\rho$ , all of whose zeros lie on a single ray  $\arg z = \pi$ , say. For these  $f$  our proof gives, in fact, an estimate for  $n(r)$  in terms of

$$(5) \quad B(r, f) = \sup_{|\theta| < \pi} |\log f(re^{i\theta})|$$

for a fixed branch of  $\log f(z)$ .

**THEOREM 1.** *Let  $f$  have all zeros real and negative. Then*

$$(6) \quad \limsup_{r \rightarrow \infty} \frac{n(r)}{B(r, f)} \geq A |\sin \pi \rho| \quad (1 < \rho < \infty).$$

We deduce (6) from a known Stieltjes transform representation for  $\log f(z)$ , cf. (1.2) below, by using singular integral estimates to obtain the  $L^p$  norm inequality

$$(7) \quad \left\{ \int_{R_n}^{\infty} \left( \frac{B(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} \leq \left( A \sin \left( \frac{\pi}{p} \right) \right)^{-1} \left\{ \int_{R_n}^{\infty} \left( \frac{n(r)}{r^{q+1}} \right)^p dr \right\}^{1/p}.$$

This is valid for  $q = [\rho]$ ,  $p = (q + 1 - \rho)^{-1} + \varepsilon_n$  and suitable sequences  $\varepsilon_n \rightarrow 0$ ,  $R_n \rightarrow \infty$ .

By iterating (7) we deduce the following:

**COROLLARY.** *Let  $f$  have finite nonintegral order  $\rho$ . If all the zeros of  $f$  lie on  $v$  rays through 0, then*

$$(8) \quad c(f) \geq \frac{A}{v} |\sin \pi \rho|,$$

$$(9) \quad \limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{n(r)} \geq -\frac{v}{A} |\csc \pi \rho|$$

where  $A$  is the constant in (6).

While (4) may remain true for all entire functions of order  $\rho$ , some restriction on the arguments of the zeros is essential in (6) and (9). W. K. Hayman has constructed entire  $f$  of any order  $\rho$  ( $\rho_0 \leq \rho < \infty$ ) with

$$\limsup_{r \rightarrow \infty} \frac{\log L(r, f)}{n(r)} \leq -B_0 \log \rho$$

where  $B_0$  and  $\rho_0$  are positive absolute constants (cf. the estimates on pp. 501–503 of [4]).

Concerning the correct value of  $C(\rho)$ , Valiron [10] asserted that the lower estimate in (3) has the correct order of magnitude for large  $\rho$ , i.e. that

$$(10) \quad c(f) < B_1 / \log \rho$$

for some  $f$  of arbitrarily large half-integral order, but his argument is incorrect. Wahlund [11] claimed to have an example showing that  $C(\rho) < \pi^{-1} |\sin \pi\rho|$  for nonintegral order  $\rho > 1$  but, as far as we know, this example was never published.

From (8) we see that functions which satisfy (10), with  $\rho$  large and half-integral, say, would have to have zeros with fairly complicated sets of arguments. However, these zeros could not be expected to be very uniformly spread throughout the plane, as in Hayman's examples [4], since those  $f$  satisfy  $n(r) > A_1 \log M(r, f)$  ( $r_0 < r < \infty$ ) with  $A_1$  independent of  $\rho$ .

The Pólya-Valiron result (3) was established at a special sequence of  $r$ -values (these days often called the Pólya peaks of  $n(r)$ ), i.e. their proofs give

$$(11) \quad \liminf_{j \rightarrow \infty} \frac{n(r_j)}{\log M(r_j, f)} \geq \frac{|\sin \pi\rho|}{A_0\{\log \rho + 1\}|\sin \pi\rho| + \pi}$$

for any entire  $f$  of order  $\rho$  and  $\{r_j\}$  any sequence satisfying  $r_j \rightarrow \infty$  and

$$(12) \quad n(t) \leq n(r_j) \left(\frac{t}{r_j}\right)^{\rho - \varepsilon_j} \quad (1 \leq t \leq r_j), \quad n(t) \leq n(r_j) \left(\frac{t}{r_j}\right)^{\rho + \varepsilon_j} \quad (t \geq r_j)$$

for some  $\varepsilon_j \rightarrow 0$  ( $j \rightarrow \infty$ ).

Perhaps surprisingly, the bound in (11) does have the correct order of magnitude, as an inequality at the Pólya peaks of  $n(r)$ , even for the functions of Theorem 1: In Section 3 we construct  $f$ , of arbitrarily large order  $\rho$  and with negative zeros only, for which

$$(13) \quad \limsup_{j \rightarrow \infty} \frac{n(r_j)}{\log M(r_j, f)} < \frac{2}{\log \rho}$$

holds for every sequence  $\{r_j\}$  of peaks of  $n(r)$  (or of  $\log M(r, f)$ ). Since these  $f$  satisfy both (4) and (13), and since the second part of (3) is derived from (11), there is no reason to believe that (3) gives a good lower estimate for  $C(\rho)$ .

We mention that our inequalities (6)–(9) have restatements with  $n(r)$  replaced by  $N(r) = \int_0^r n(t) dt/t$ , provided the constant  $A$  is replaced throughout by  $A/\rho$ .

### 1. Proof of Theorem 1

We put  $q = [\rho]$  and assume  $f(0) = 1$ . Then for any  $s > 0$  we can write

$$(1.1) \quad \log f(z) = \log \Pi_s(z) + R_s(z)$$

where

$$\Pi_s(z) = \prod_{t_n > s} \left(1 + \frac{z}{t_n}\right) \exp \left\{ \sum_{k=1}^q \frac{1}{k} \left(\frac{z}{-t_n}\right)^k \right\} \quad (z_n = -t_n < 0),$$

$$|R_s(z)| \leq C \left(\frac{r}{s}\right)^q \log M(2s, f) \quad (|z| = r \geq 2s)$$

for a constant  $C$  (Edrei and Fuchs [1], pp. 296, 313, 314).

After an integration by parts,

$$(1.2) \quad \log \Pi_s(z) = (-1)^q z^{q+1} \int_0^\infty \frac{n(t, 0; \Pi_s)}{t^{q+1}} \frac{dt}{t+z} \quad (|\arg z| < \pi).$$

From (1.2) and (5),

$$(1.3) \quad B(r, \Pi_s) = r^{q+1} \sup_\theta \left| \int_0^\infty \frac{n(t, 0; \Pi_s)}{t^{q+1}} K(r, t, \theta) dt \right| \quad (0 < r < \infty),$$

$$K(r, t, \theta) = e^{-i\theta} \frac{r + t \cos \theta + it \sin \theta}{r^2 + t^2 + 2tr \cos \theta},$$

so that  $B(r, \Pi_s)/r^{q+1}$  can be considered a kind of maximal function. We shall establish the norm estimate

$$(1.4) \quad \int_0^\infty \left( \frac{B(r, \Pi_s)}{r^{q+1}} \right)^p dr \leq \alpha_p \int_0^\infty \left( \frac{n(r, 0; \Pi_s)}{r^{q+1}} \right)^p dr \quad \left( p > \frac{1}{q+1-p} \right),$$

$$(1.5) \quad \alpha_p = \left( A \sin \left( \frac{\pi}{p} \right) \right)^{-p}.$$

Assuming (1.4), we complete the proof of Theorem 1. Recall from (1.1) that  $n(r, 0, \Pi_s) = 0$  for  $t \leq s$ . Thus from (1.1), (1.4) and Minkowski's inequality, we obtain

$$\begin{aligned} \left\{ \int_{2s}^\infty \left( \frac{B(r, f)}{r^{q+1}} \right)^p dr \right\}^{1/p} &\leq \left\{ \alpha_p \int_0^\infty \left( \frac{n(r, 0; \Pi_s)}{r^{q+1}} \right)^p dr \right\}^{1/p} \\ &\quad + \left( \frac{C \log M(2s, f)}{s^q} \right) \left\{ \int_{2s}^\infty \frac{dr}{r^p} \right\}^{1/p} \\ &\leq \left\{ \alpha_p \int_s^\infty \left( \frac{n(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} + B_1(p) \frac{\log M(2s, f)}{s^{q+1-1/p}} \end{aligned}$$

with  $B_1(p) = C(p-1)^{-1/p}$ . Since by Jensen's theorem  $n(2s) < 2 \log M(4s, f)$ , we have

$$(1.6) \quad \left\{ \int_{2s}^\infty \left( \frac{B(r, f)}{r^{q+1}} \right)^p dr \right\}^{1/p} \leq \left\{ \alpha_p \int_{2s}^\infty \left( \frac{n(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} + B_2(p) \frac{\log M(4s, f)}{s^{q+1-1/p}}$$

where  $B_2(p) = B_1(p) + 2\alpha_p^{1/p}$  is bounded for  $p$  bounded away from 1 and  $\infty$ .

By an elementary argument [7, p. 208], there exist  $x_n \rightarrow \infty$ ,  $K_n \rightarrow \infty$  such that

$$(1.7) \quad \log M(r, f) > \frac{1}{2} \log M(x_n, f) \left( \frac{r}{x_n} \right)^p \quad (x_n \leq r \leq K_n x_n).$$

Choose  $s = s_n = x_n/4$ , so that for  $\varepsilon > 0$  and  $p = (q + 1 - \rho)^{-1}(1 + \varepsilon)$  we have, by (5),

$$(1.8) \quad \int_{2s_n}^{\infty} \left( \frac{B(r, f)}{r^{q+1}} \right)^p dr > \int_{x_n}^{K_n x_n} \left( \frac{\log M(r, f)}{r^{q+1}} \right)^p dr > \left( \frac{1 - K_n^{-\varepsilon}}{\varepsilon} \right) \left( \frac{\log M(x_n, f)}{2x_n^{q+1-1/p}} \right)^p.$$

Since we can choose  $\varepsilon$  arbitrarily small, (1.6) implies

$$(1.9) \quad (1 - \eta) \int_{2s_n}^{\infty} \left( \frac{B(r, f)}{r^{q+1}} \right)^p dr \leq \alpha_p \int_{2s_n}^{\infty} \left( \frac{n(r)}{r^{q+1}} \right)^p dr$$

where  $\eta = \eta(\varepsilon) \rightarrow 0$  and  $p \rightarrow (q + 1 - \rho)^{-1}$  when  $\varepsilon \rightarrow 0$ . Thus (1.5) yields (6) and (7).

To deduce the corollary, we write  $f(z) = \prod_{j=1}^v f_j(e^{i\theta_j} z)$  where each  $f_j(z)$  has negative zeros only, and genus  $q$ , and let  $n_j(r) = n(r, 0; f_j)$ ,

$$\sigma(r) = \sum_{j=1}^v \log M(r, f_j), \quad \Sigma(r) = \sum_{j=1}^v B(r, f_j).$$

Then

$$(1.10) \quad \sigma(r) \leq \Sigma(r), \quad |\log |f(re^{i\theta})|| \leq \Sigma(r),$$

and by (1.6) and the Minkowski, Jensen and Hölder inequalities,

$$(1.11) \quad \left\{ \int_{2s}^{\infty} \left( \frac{\Sigma(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} \leq \sum_{j=1}^v \left\{ \alpha_p \int_{2s}^{\infty} \left( \frac{n_j(r)}{r^{q+1}} \right)^p dr + B_2(p)^p \frac{(\log M(4s, f_j))^p}{s^{p(q+1)-1}} \right\}^{1/p} \leq v^{1-1/p} \alpha_p^{1/p} \left\{ \int_{2s}^{\infty} \left( \frac{n(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} + B_2(p) \frac{\sigma(4s)}{s^{q+1-1/p}}.$$

Choose  $x_n, K_n$  to satisfy (1.7) with  $\log M(r, f)$  replaced by  $\sigma(r)$ . Using (1.11) and the first inequality in (1.9), the argument used in (1.8) yields

$$\frac{\sigma(4s_n)}{s_n^{q+1-1/p}} \leq \varepsilon^{1/p} C B_2(p) \left\{ \int_{2s_n}^{\infty} \left( \frac{\sigma(r)}{r^{q+1}} \right)^p dr \right\}^{1/p} \leq \varepsilon^{1/p} B_3(p) \left\{ \int_{2s_n}^{\infty} \left( \frac{n(r)}{r^{q+1}} \right)^p dr \right\}^{1/p}$$

where as before  $p = (q + 1 - \rho)^{-1}(1 + \varepsilon)$ . Thus, (1.11) and the second inequality in (1.10) imply

$$(1 - \eta) \int_{2s_n}^{\infty} \left( \sup_{\theta} \frac{|\log |f(re^{i\theta})||}{r^{q+1}} \right)^p dr \leq v^{p-1} \alpha_p \int_{2s_n}^{\infty} \left( \frac{n(r)}{r^{q+1}} \right)^p dr$$

where  $\eta$  and  $p$  behave as in (1.9). We deduce (8) and (9), in a stronger form than stated: In those inequalities,  $v$  can be replaced by  $v^{\rho-q}$ .

## 2. Proof of (1.4)

For fixed  $s > 0$ , let

$$\phi(r) = n(r, 0; \Pi_s)r^{-q-1}, \quad \Phi(r) = B(r, \Pi_s)r^{-q-1},$$

so that (1.3) becomes

$$\Phi(r) = \sup_{0 < \theta < \pi} \left| \int_0^\infty \phi(t)K(r, t, \theta) dt \right|.$$

We proceed to estimate

$$\|\Phi\|_p = \left\{ \int_0^\infty \Phi(r)^p dr \right\}^{1/p}$$

in terms of the Hardy-Littlewood maximal function

$$M\phi(r) = \sup_{\varepsilon > 0} \frac{1}{2\varepsilon} \int_{|t-r| < \varepsilon} \phi(t) dt$$

and the maximal Hilbert transform

$$H^*\phi(r) = \sup_{\varepsilon > 0} \left| \int_{|t-r| > \varepsilon} \frac{\phi(t)}{t-r} dt \right|.$$

We shall prove

$$(2.1) \quad \Phi(r) \leq 12M\phi(r) + H^*\phi(r) + 10 \int_0^\infty \frac{\phi(t)}{t+r} dt.$$

Assuming (2.1), we deduce

$$(2.2) \quad \|\phi\|_p \leq 12\|M\phi\|_p + \|H^*\phi\|_p + A(p)\|\phi\|_p$$

for  $(q+1-\rho)^{-1} < p < \infty$ ; here  $A(p) = 10\pi \csc(\pi/p)$  by a simple application of Minkowski's inequality (cf. [8, p. 271]).

By a classical estimate (e.g. [8, p. 7]),

$$(2.3) \quad \|M\phi\|_p \leq \frac{10p}{p-1} \|\phi\|_p \quad (p > (q+1-\rho)^{-1}).$$

We also have

$$(2.4) \quad \|H^*\phi\|_p \leq A_2 \left( \frac{p}{p-1} + p \right) \|\phi\|_p \quad (p > (q+1-\rho)^{-1})$$

for an absolute constant  $A_2$ ; this can be derived for example when  $p \geq 2$  from the estimate of [8, p. 67] together with (2.3), and for  $1 < p < 2$  from [8, pp. 42, 21, 22]. (Or, compare the estimates in [12, pp. 256–258] for an equivalent [15, p. 256] problem.) Since

$$p \leq \frac{p}{p-1} \leq \pi \csc\left(\frac{\pi}{p}\right) \quad (1 < p \leq 2), \quad \frac{p}{p-1} \leq p \leq \pi \csc\left(\frac{\pi}{p}\right) \quad (p \geq 2),$$

(2.1)–(2.4) yield (1.4).

It remains to prove (2.1). We have from (1.3) that

$$\Phi(r) \leq \sum_{j=1}^3 \left( \sup_{0 < \theta < \pi} \left| \int_0^\infty \phi(t) K_j(r, t, \pi - \theta) dt \right| \right)$$

where, if we let  $D = (r - t)^2 + 2tr(1 - \cos \theta)$ ,

$$K_1 = \frac{r - t}{D}, \quad K_2 = \frac{t(1 - \cos \theta)}{D}, \quad K_3 = \frac{t \sin \theta}{D}.$$

Now  $K_2 < 4/(r + t)$  and, if we let  $\varepsilon^2 = 2r^2(1 - \cos \theta)$ , then

$$K_1 = \frac{r - t}{(r - t)^2 + \varepsilon^2} + \Delta_1 = Q + \Delta_1$$

where  $0 \leq \Delta_1 \leq 2/(r + t)$ . We estimate  $\int_0^\infty \phi Q$  in terms of  $H^*\phi$ ,  $M\phi$  and

$$\phi * P_\varepsilon(r) = \int_0^\infty \phi(t) P_\varepsilon(r - t) dt, \quad P_\varepsilon(r) = \varepsilon(r^2 + \varepsilon^2)^{-1},$$

by observing that

$$\sup_{\varepsilon > 0} \left| \int_0^\infty \phi(t) Q dt \right| \leq \sup_{\varepsilon > 0} \left| \int_{|t-r| \geq \varepsilon} \phi Q \right| + \sup_{\varepsilon > 0} \left| \int_{|t-r| < \varepsilon} \phi Q \right| = T_1 + T_2$$

where, since  $\phi \geq 0$  and  $0 < \theta < \pi$ ,

$$T_1 \leq H^*\phi(r) + 2 \sup_{\varepsilon > 0} \phi * P_\varepsilon(r), \quad T_2 \leq 2M\phi(r).$$

In  $K_3$ , set  $\psi = \frac{1}{2}\theta$  and  $\varepsilon = 2r \sin \psi$  so that

$$K_3 \leq \frac{2t \sin \psi}{(r - t)^2 + 4rt \sin^2 \psi} = P_\varepsilon(r - t) + \Delta$$

where

$$|\Delta| \leq \frac{2 \sin \psi |r - t|}{(r - t)^2 + 4r^2 \sin^2 \psi}.$$

Thus

$$\sup_{0 < \psi < \frac{1}{2}\pi} \left| \int_0^\infty \phi(t) \Delta dt \right| \leq \sup_{0 < \varepsilon < 2r} \left\{ \int_{|t-r| < \varepsilon} + \int_{|t-r| \geq \varepsilon} \right\} \phi |\Delta|$$

where

$$|\Delta| \leq \frac{1}{r} \leq \frac{4}{r + t} \quad (|t - r| < \varepsilon < 2r)$$

and, since  $\varepsilon = 2r \sin \psi$ ,

$$\begin{aligned} \int_{|t-r| \geq \varepsilon} \phi |\Delta| &\leq \frac{1}{2r} \int_0^{(r-\varepsilon)^+} \phi + \frac{1}{2r} \int_{r+\varepsilon}^{3r} \phi + \int_{3r}^\infty \phi(t) \frac{\varepsilon}{r(t-r)} dt \\ &\leq 4 \int_{|t-r| \geq \varepsilon} \frac{\phi(t)}{r+t} dt. \end{aligned}$$

Summing up,

$$\Phi(r) \leq 3 \sup_{\varepsilon>0} \phi * P_\varepsilon(r) + 2M\phi(r) + H^*\phi(r) + 10 \int_0^\infty \frac{\phi(t)}{r+t} dt.$$

Now (2.1) follows from the fact [8, p. 62] that  $\sup_{\varepsilon>0} \phi * P_\varepsilon(r) \leq \pi M\phi(r)$ .

### 3. An example

As far as we know, earlier studies comparing the growth of  $n(r)$  [or  $N(r)$ ] with that of  $\sup_\theta |f(re^{i\theta})|$  have always, explicitly or implicitly, computed this growth at the Pólya peaks of  $n(r)$  [or  $N(r)$ ]. This is true of the papers of Valiron [9], Pólya [6] and Wahlund [11], as well as of more recent papers of Williamson [14] and Fuchs [2]. We give an example showing that the order of magnitude in  $\rho$  of the classical estimate (11) cannot be improved at the peaks (of  $n(r)$ ,  $N(r)$  or  $\log M(r, f)$ ).

This example is related to a question raised by Williamson in [14]. The method of [14] gives a sharp estimate

$$(3.1) \quad \log M(r_j, f) \leq \{\pi\rho/|\sin \pi\rho| + o(1)\}N(r_j) \quad (j \rightarrow \infty)$$

at peaks  $\{r_j\}$  of  $N(r)$ , when  $f$  has negative zeros only and satisfies Williamson's additional assumption  $A$  [14, p. 500] on the location of the points  $z$  where  $|f(z)| = M(|z|, f)$ . In [13], Wheeler shows that  $A$  does not hold for all  $f$  with negative zeros. This fact also follows from our examples (3.5) below, since (3.1) itself fails for these  $f$ .

In [2], Fuchs proves an inequality implying

$$(3.2) \quad \sup_\theta \log |f(r_j e^{i(\theta+\beta)})f(r_j e^{i(\theta-\beta)})| \leq \{2\pi\rho/|\sin \pi\rho| + o(1)\}N(r_j) \quad (j \rightarrow \infty)$$

for certain  $\beta$  near  $\pi/2\rho$  and  $\{r_j\}$  any sequence of peaks of  $N(r)$ . Our estimates (3.7)–(3.8) for the functions (3.5) show that one cannot expect (3.2) to remain valid for  $\beta$  near 0. (Compare also [3], where Fuchs proves a version of (3.2) not uniform in  $\theta$ , but for which the restriction on  $\beta$  is removed.)

Our examples are defined as follows: Let  $\rho > 4$  be given and let  $\{b_k\}$  be a sequence with  $0 < b_k < b_{k+1}$  ( $k \geq 1$ ), and  $b_k \rightarrow \infty$  so that

$$(3.3) \quad \sum_{j=1}^{k-1} b_j^\rho = o(b_k^{1/2}) \quad (k \rightarrow \infty).$$

Let  $q = [\rho]$  and define  $v(t) = 0$  ( $0 \leq t \leq b_1$ ),  $b_0 = 0$  and

$$(3.4) \quad v(t) = \begin{cases} v(2b_{k-1}) + (t - b_k)t^{\rho-1} & \text{if } b_k \leq t \leq 2b_k, k \geq 1 \\ v(2b_k) & \text{if } 2b_k \leq t < b_{k+1}, k \geq 1. \end{cases}$$

We can now define

$$(3.5) \quad f(z) = \prod_{n=1}^\infty E\left(\frac{z}{-a_n}, q\right)$$



where  $E(z, q) = (1 - z) \exp(\sum_{k=1}^q z^k/k)$  and the sequence  $\{a_n\}$  is determined by  $0 < a_n < a_{n+1}$  ( $n \geq 1$ ) and  $n(t) = n(t, 0; f) = [v(t)]$ . Let  $r_k = 2b_k$  ( $k \geq 1$ ). Then  $\{r_k\}$  is a sequence of peaks of  $n(t)$ , as defined in (12). Further, a little calculation shows that any sequence  $\{R_k\}$  of peaks of  $n(t)$  must satisfy

$$(3.6) \quad 1 \leq r'_k/R_k \leq 1 + o(1) \quad (k \rightarrow \infty)$$

for some subsequence  $\{r'_k\}$  of  $\{r_k\}$ , say  $\{r'_k\} = \{r_k\}$ . We shall prove

$$(3.7) \quad \liminf_{k \rightarrow \infty} \log M(R_k, f)/n(R_k) > 2^{-1/2} \log \rho - 3.$$

Thus, for  $\rho$  sufficiently large, these  $f$  satisfy (13). Further, since (3.4) and (3.6) imply

$$(3.8) \quad n(r)/N(r) \rightarrow A(\rho)$$

for  $r$  near  $R_k$  and  $k \rightarrow \infty$ , with  $A(\rho) > 2\rho$  for  $\rho_0 < \rho < \infty$ , it is clear that these  $f$  fail to satisfy (3.1). Also, if  $\{t_k\}$  is any sequence of peaks of  $\log M(r, f)$ , it is easy to see, from (3.4) and (12), that (13) remains valid with  $r_k$  replaced by  $t_k$ .

To prove (3.7), let  $|z| = R_k$  and write

$$\log |f(z)| = \int_{b_k}^{r_k} \log \left| E\left(\frac{z}{-t}, q\right) \right| dn(t) + S_k(z)$$

where, by (3.3),  $S_k(z)/n(R_k) \rightarrow 0$  ( $k \rightarrow \infty$ ). Writing  $R_k = R$ ,  $r_k = r$  and  $z = Re^{i\theta}$ ,

$$(3.9) \quad \log |f(z)| = n(r) \log \left| E\left(\frac{z}{-r}, q\right) \right| + (-1)^q \int_{b_k}^r n(t) \left(\frac{r}{t}\right)^{q+1} \times \frac{r \cos q\theta + t \cos (q+1)\theta}{r^2 + t^2 + 2tr \cos \theta} dt - o(n(r))$$

when  $k \rightarrow \infty$ , uniformly in  $\theta$ . We now put  $\theta = \pi - \pi/4q$  and  $t = Rs$  in (3.9), and obtain

$$\liminf_{k \rightarrow \infty} \frac{\log M(R_k, f)}{n(R_k)} \geq \cos \frac{\pi}{4} \int_{1/2}^1 \frac{2s-1}{s^{q+1-\rho}} \frac{1-s}{(1-s)^2 + q^{-2}} ds + \log |E(e^{\pi i/4q}, q)|.$$

Here the integral is bounded below by

$$\int_{1/2}^1 (2s-1) \frac{1-s}{(1-s)^2 + q^{-2}} ds = 2 \int_{1/2}^1 \left\{ \int_0^{1-s} \frac{x dx}{x^2 + q^{-2}} \right\} ds > \log \left(\frac{q}{2}\right) - 1.$$

Since  $\log |E(e^{\pi i/4q}, q)| > -1$ , (3.7) follows at once.

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