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FINITE GROUPS WITH ISOMORPHIC GROUP ALGEBRAS

BY

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Suppose that G and H are finite groups with isomorphic group algebras RG and RH over the ring R of integers in some finite algebraic extension of the rationals.

In [7] Passman established a bijective correspondence between the set of normal subgroups of G and that of H which preserves many natural operations and properties defined on these sets. Some of Passman's results, however, depend on nilpotency conditions.

In this paper we provide generalizations of Passman's result in two directions. Namely, we remove the nilpotency condition and replace R by a larger class of rings.

Our result is as follows.

THEOREM. *Let $KG \cong KH$ where K is any integral domain of characteristic 0 in which no rational prime divisor of $|G|$ is invertible. Then there exists an isomorphism between the lattice of normal subgroups of G and that of H which preserves the following.*

(a) *the commutation of any two normal subgroups,*
(b) *normal abelian sections and the isomorphism class of normal abelian sections,*
(c) *the order and period of normal sections.*

In fact, the corresponding normal sections have the same number of elements of any given order.

COROLLARY. *The above isomorphism preserves the following.*

(i) *nilpotency, solvability, class of nilpotency and the derived length of N, N being an arbitrary normal subgroup of G (in particular, the Fitting subgroup of N);*
(ii) *a central series of N consisting of normal subgroups of G and the isomorphism class of corresponding factors (in particular, the upper central series and the lower central series of N and any central series of G);*
(iii) *the derived series of N, the chief series of G and the isomorphism class of corresponding factors;*

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(iv) *the group $C^n(N)$ generated by all nth powers of elements of N and the group $C_n(N)$ generated by all elements of N whose order divides n.*

The special case of (a) and (b) when $K = Z$ was proved by Whitcomb [13] (see also [9]). When $K = R$ Passman [7] proved part of (c) and, when G is nilpotent, he has obtained (a) and part of (b). Obayashi [6] proved part of (b) for $K = R$. For other various special cases of the above theorem and corollary refer to [3], [6], [7], [11] and [12].

1. The setting

Before proceeding with the proof we shall describe the notation, recall the definitions and record some elementary properties of group rings. Let G be a finite group and let S be a commutative ring with unit. The augmentation ideal $I(S, G)$ is the kernel of the homomorphism from the group ring SG to S induced by collapsing G to the unit group. We shall write $I(G)$ instead of $I(S, G)$ when there is no danger of confusion. For J the ideal of SG the set $G \cap (1 + J)$ of elements x in G for which $x - 1$ in J is a normal subgroup of G. Throughout, N and M are normal subgroups of G and in the group ring KG K will always stand for an integral domain of characteristic 0 in which no rational prime divisor of $|G|$ is invertible. In particular, K can be one of the following rings:

$$R,\ Z_{(G)} = \{a/b \mid a, b \in Z, \quad (b, |G|) = 1\},$$

and, when G is a p-group, the ring of p-adic integers.

If C and D are subsets of SG, define the Lie bracket (C, D) as the subgroup of the additive group of SG generated by all $(c, d) = cd - dc$, c in C, d in D.

Let λ: $G \to H$ be a group epimorphism and let $\bar{\lambda}$: $SG \to SH$ be the group ring epimorphism which is the extension of λ by S-linearity. Then

$$\operatorname{Ker} \bar{\lambda} = SG \cdot I(N) = \left\{x \in SG \,\middle|\, x \sum_{y \in N} y = 0\right\} \quad \textit{where } N = \ker \lambda.$$

As a consequence of this we have $G \cap (1 + SG \cdot I(N)) = N$. That $SG \cdot I(N)$ is generated by $\{x - 1 \mid x \in X\}$, where X is a generating set for N (see [4]) implies

$$(1) \qquad SG \cdot I(N \cdot M) = SG \cdot I(N) + SG \cdot I(M)$$

and the same fact together with the identity $(a - 1, b - 1) = ba(a^{-1}b^{-1}ab - 1)$, $a, b \in G$ implies

$$(2) \qquad SG \cdot I([N, M]) = SG \cdot (I(N), I(M)).$$

We shall write $KG = KH$ for H being a normalized group basis of KG (i.e. H is a group basis consisting of elements having augmentation 1).

Elements of particular interest in KG are the class sums. These are the sum of all the group elements in any given class of G. We will denote by C_g the class containing $g \in G$. That $KG = KH$ implies existence of a bijective correspondence between the conjugacy classes of G and those of H such that the corresponding classes have identical class sums was proved by Berman [2] for the case $K = Z$ and for the case $K = R$ the same result was proved by Glauberman (see [7]), Poljak [8] and Saksonov [12]. The general case was established by Saksonov [11].

2. Preliminary results

We shall need the following result due to Saksonov.

LEMMA 1 [11]. *Let $KG \cong KH$. Then there exists a bijective correspondence* $*$ *be tween the conjugacy classes of G and those of H such that:*

(3) *The corresponding classes have identical class sums.*
(4) *For any $g \in G$, $C_g^* = C_h$ implies $(C_g n)^* = C_h n$, $h \in H$.*
(5) *Every group which consists of normalized units of KG is a K-linearly independent set.*

We next need to calculate the normal subgroup of G attached to the product of ideals $KG \cdot I(N)$ and $KG \cdot I(M)$.

LEMMA 2. *Let $J_1 = KG \cdot I(N)$ and $J_2 = KG \cdot I(M)$. Then*

$$G \cap (1 + J_1 J_2) = (N \cap M)'.$$

Proof. It follows from [1, Theorem 10] that $G \cap (1 + J_1 J_2) = (N \cap M)^K$ where the right hand side is the intersection of the kernels of all homomorphisms of $N \cap M$ into the additive group of K-modules. In particular, the case $N = M = G$ yields

$$G \cap (1 + I^2(K, G)) = G^K.$$

Therefore it suffices to prove that $G \cap (1 + I^2(K, G)) = G'$. By taking the case $n = 2$ in Theorem 2.1 of [2] we see that $G \cap (1 + I^2(K, G)) = G'$ whenever $T_p(G \bmod G') = G'$ for all primes p for which $p^e K = p^{e+1} K$ for some e, $T_p(G \bmod G')$ being the subgroup of G generated by all elements of G some pth power of which is in G'. It is clear that $T_p(G \bmod G') = G'$ whenever $p \nmid |G|$. If p is a prime such that $p^e K = p^{e+1} K$ for some e then $p^e(1 - px) = 0$ for some $x \in K$ and since K has no zero divisors, p is a unit in K. This shows that $p \nmid |G|$ and completes the proof.

With these preliminaries settled, we are now ready to prove the theorem.

3. Proof of the theorem

We may assume without loss of generality that $KG = KH$. Let $N = \bigcup_{g \in \wedge} C_g$ and let $N^* = \bigcup_{g \in \wedge} C_g^*$. It follows from (3) that

$$\sum_{x \in N} x = \sum_{t \in N^*} y \quad \text{whence} \quad \left(\sum_{y \in N^*} y \right)^2 = |N| \sum_{y \in N^*} y.$$

Therefore N^* is closed with respect to multiplication, proving that $N^* \lhd H$. It is immediate that $N \to N^*$ determines the desired bijective correspondence which preserves order, inclusion and intersection. That $(N \cdot M)^* = N^* \cdot M^*$ will follow from (1) and the following property:

(6) If J is any ideal of KG then $G \cap (1 + J) = N$ implies $H \cap (1 + J) = N^*$.

To prove (6), we have only to remark that $N \subseteq 1 + J$ if and only if $J \supseteq KG \cdot I(N)$.

Let J be the ideal of KG generated by (J_1, J_2) where $J_1 = KG \cdot I(N)$, $J_2 = KG \cdot I(M)$. We next claim that

$$G \cap (1 + J) = [N, M] \tag{7}$$

from which (a) will follow by virtue of (6). To prove (7) we first observe that $KG \cdot I([N, M]) \leq J$ by virtue of (2). Therefore we may assume that $[N, M] = 1$ in which case $J \leq J_1 J_2$. The desired assertion is now a consequence of Lemma 2.

Let π: $KG \to KG_1$ where $G_1 = G/N$ be the canonical homomorphism. Be-$M \geq N'$ the application of (a) yields the first part of (b). Since two abelian groups having the same number of elements of any given order are isomorphic [5, p. 95] the application of (c) yields the second part of (b). We are now left to prove (c).

Let π: $KG \to KG_1$ where $G_1 = G/N$ be the canonical homomorphism. Because of (6), $KG_1 = KH_1$ where $H_1 = \pi(H)$. Let $(M/N)^*$ be the normal subgroup of H_1 which corresponds to M/N. It is a consequence of (4) that M/N and $(M/N)^*$ have the same number of elements of any given order. Thus to complete the proof it suffices to show that

$$(M/N)^* \cong M^*/N^*. \tag{8}$$

To prove (8), consider the sequence of group ring homomorphisms

$$KG \xrightarrow{\pi} KG_1 \xrightarrow{\alpha} KG_2$$

where $G_2 = G/M$ and $\alpha(gN) = gM$ for any $g \in G$. Then $\ker \alpha\pi = KG \cdot I(M)$, $\ker \alpha = KG_1 \cdot I(M/N)$ and for any $h \in H$, $(\alpha\pi)(h - 1) = \alpha(\pi(h) - \bar{1})$ where $\bar{1}$ is the identity element of KG_1. It follows from (6) that

$$H \cap (1 + KG \cdot I(N)) = N^*$$

whence

$$\begin{aligned}(M/N)^* &= H_1 \cap (\bar{1} + \ker \alpha)\\ &= \{\pi(h) \mid \alpha(\pi(h) - \bar{1}) = 0\}\\ &= \{\pi(h) \mid h \in H \cap (1 + \mathrm{Ker}\ \alpha\pi)\}\\ &= \{\pi(h) \mid h \in M^*\}\\ &= \pi(M^*).\end{aligned}$$

Since π induces the group epimorphism $\pi'\colon M^* \to \pi(M^*)$ and since

$$\ker \pi' = M^* \cap (H \cap (1 + KG \cdot I(N))) = M^* \cap N^* = N^*,$$

the proof is complete.

To prove the corollary we have only to remark that (i), (ii), and (iii) are direct consequences of (a) and (b) while (iv) follows from (4).

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