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THE GAUSSIAN RANDOM WALK ON THE HEISENBERG GROUP[1]

BY

DAVID R. FISCHER

Introduction

Let G be a locally compact group and let μ be a probability measure on G. μ generates a random walk on G with the crucial property that the number of times that the walk can be expected to visit a Borel subset A of G is $\sum \mu^{*m}(A)$ where the summation runs over all positive integers m, and μ^{*m} denotes the m-fold convolution power of μ. It is known [1] that either $\sum \mu^{*m}(A)$ is finite for every compact set A or else is infinite for every open set A. In the former case the random walk determined by μ is said to be transient and to "wander to ∞". In the latter case the walk is said to be recurrent.

For convenience we now restrict our discussion to the case in which G is a 1-connected nilpotent Lie group. Such groups are diffeomorphic to R^n for some n. If the group actually is R^n then it is a classical theorem (see [2] for instance) that for $n \geq 3$ every random walk on G generated by a measure that is supported by no proper closed subgroup of G is transient. A special case of a theorem proved by Guivarc'h and Keane [3] is that this result is also true for 1-connected nilpotent Lie groups of dimension greater than two.

Let G be a 1-connected nilpotent Lie group, let L denote its Lie algebra, and let exp denote the exponential map from L to G. Then exp is a global diffeomorphism that carries Euclidean measure on L to Haar measure on G [4]. Thus exp carries absolutely continuous probability measures on G to absolutely continuous probability measures on L. If μ is a probability distribution on G and if A is a neighborhood of the origin in L consider the two infinite series

$$\sum \int_A \phi^{*m} \circ \exp (l)\, dl \quad \text{and} \quad \sum \int_A \phi \circ \exp^{*m} (l)\, dl,$$

the former series being associated with the expected number of visits of a random walk on G to a neighborhood of the origin and the latter series being similarly associated with a random walk on the abelian group structure of L. Both series are convergent, but one might expect that, since in order to return to the origin in the non-abelian case one must retrace one's steps in the correct order, the first series should converge more rapidly than the second. In this

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paper we address this question in a particular case—that in which $\phi \circ \exp$ is the Gaussian distribution and G is the Heisenberg group.

Our main results are Theorems 4.1 and 4.5. The former theorem shows that, as in the abelian case, the behavior of the series in question depends only on the behavior of the convolution powers of the density function evaluated at the origin. The latter theorem shows that, asymptotically, the ratio of $\phi^{*m} \circ \exp$ (0, 0, 0) to $\phi \circ \exp^{*m}$ (0, 0, 0) behaves like $m^{-1/2}$. Roughly, this means that the ratio of the probability of returning to the origin at the mth step on the Heisenberg group to the probability of returning to the origin at the mth step on R^3 behaves like $m^{-1/2}$. We thus have a quantitative reflection of the effect of the non-commutative group multiplication on the difficulty of returning to the origin.

One of the main tools in the analysis of random walks on R^n is the Fourier transform. Its value in this situation is that it carries convolution to pointwise multiplication. In the interesting but relatively simple case in which the distribution function ϕ is a Schwartz function one can then apply the Plancherel theorem (inversion theorem) to analyze the series $\sum \int_A \phi^{*m}(x)\, dx$. One might thus expect that in the case in which G is a 1-connected nilpotent Lie group that the representation-theoretic Fourier transform and inversion theorem would prove to be similarly useful. This has not, thus far, been the case. In the non-abelian setting the inversion theorem takes the form, for ϕ a Schwartz function [5],

$$\phi(1) = \int_{\hat{G}} \hat{\phi}(\pi)\, d\pi$$

where $\hat{G}$ is the set of unitary-equivalence classes of irreducible unitary representations of G, $d\pi$ is a (known) positive measure and

$$\hat{\phi}(\pi) = \text{trace } \pi_\phi = \text{trace} \int_G \pi(g)\phi(g)\, dg.$$

While the map $\phi \mapsto \pi_\phi$ is multiplicative, the trace function is not, unless π is one-dimensional. Therefore the non-abelian Fourier transform is not multiplicative.

Since we have been unable to utilize the non-abelian Fourier transform, our analysis of the problem at hand proceeds by real-variable methods. The proofs are computational, using classical theorems. One property of the Gaussian density that makes our calculations manageable is that it is its own (abelian) Fourier transform. This fact is crucial to the proof of Lemma 2.5.

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1. Preliminaries

In this section we list for convenience certain notations and conventions that will be used in the sequel, indicate the means by which convolutions on the Lie group can be lifted to the Lie algebra, and consider briefly the abelian version of our random walk.

N_3 will denote the Heisenberg group. This group can be thought of as the group of 3×3 unipotent matrices, or alternately as R^3 with multiplication given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

The Lie algebra of N_3 is R^3 with bracket operation given by

$$[(x, y, z), (x', y', z')] = (0, 0, xy' - yx').$$

If G is any 1-connected nilpotent Lie group with Lie algebra L then exp is a global diffeomorphism and there is a map $\text{C-H}: L \to G$, called the Campbell-Hausdorff map [4] such that for $l, l' \in L$

$$\begin{aligned}\text{C-H}(l, l') &= \exp^{-1}(\exp l \exp l') \\ &= l + l' + \tfrac{1}{2}[l, l'] + \text{higher order bracket terms.}\end{aligned}$$

Since the group being considered is nilpotent the series terminates, and for N_3 all of the higher order bracket terms vanish.

Recall that exp carries Euclidean measure dl on the Lie algebra to Haar measure dg on G. Let $f \in L^1(G)$ and let $l_i = \exp^{-1} g_i$. Then

$$\begin{aligned}f^{*m}(g_0) &= \int_G f(g_0 g_1^{-1}) f^{*m-1}(g_1)\, dg_1 \\ &\;\;\vdots \\ &= \int_G \cdots \int_G f(g_0 g_1^{-1}) \cdots f(g_{m-2} g_{m-1}^{-1}) f(g_{m-1})\, dg_1 \cdots dg_{m-1} \\ &= \int_L \cdots \int_L f \circ \exp(\text{C-H}(l_0, -l_1)) \\ &\qquad \cdots f \circ \exp(\text{C-H}(l_{m-2}, -l_{m-1})) \\ &\qquad \times f \circ \exp(l_{m-1})\, dl_1 \cdots dl_{m-1}\end{aligned}$$

Thus if one knows C-H explicitly one can carry out all of the integrations necessary to compute convolutions on the group by lifting to the algebra.

We note that for the Lie algebra of N_3,

$$\text{C-H}((x, y, z), (x', y', z')) = (x + x', y + y', z + z' + \tfrac{1}{2}(xy' - yx')).$$

For notational convenience we introduce here certain conventions that will remain in force throughout the remainder of the paper:

(1) All measures on Euclidean space are taken to be Lebesgue measure normalized by multiplication by $(2\pi)^{-n/2}$ where n is the dimension of the space in question.

(2) For any n, $E(x_1, \ldots, x_n) = e^{-i(x_1 + \cdots + x_n)}$.

(3) For any n, $G(x_1, \ldots, x_n) = e^{-(x_1^2 + \cdots + x_n^2)/2}$.

(4) For any n, $F(x_1, \ldots, x_n) = e^{-(x_1 + \cdots + x_n)/2}$.

(5) Sums taken over an empty indexing set are taken to be 0.

(6) Products taken over an empty indexing set are taken to be 1.

(7) $\phi: N_3 \to R$ is the function defined by $\phi \circ \exp(x, y, z) = G(x, y, z)$. We call the random walk on N_3 associated with the measure $\phi\, dg$ the Gaussian random walk on N_3.

We conclude this section by considering briefly the abelian Gaussian random walk, that is, the random walk on R^3 associated with the measure $G(x, y, z)\, dx\, dy\, dz$. It is easily computed, either by direct computation or by utilizing the fact that $\hat{G} = G$, that

$$G^{*m}(x, y, z) = m^{-3/2} G(m^{-1/2}x, m^{-1/2}y, m^{-1/2}z).$$

It follows that, denoting the sphere centered at the origin of radius $k > 0$ by B_k, for every m and for all $(x, y, z) \in B_k$

$$G^{*m}(0, 0, 0) \geq G^{*m}(x, y, z) \geq G(k) G^{*m}(0, 0, 0).$$

Thus for fixed $k > 0$,

$$\int_{B_k} G^{*m}(x, y, z)\, dx\, dy\, dz = O(G^{*m}(0, 0, 0))$$

and

$$G^{*m}(0, 0, 0) = O\left(\int_{B_k} G^{*m}(x, y, z)\, dx\, dy\, dz\right).$$

The question of the transience and rate of "wandering to ∞" of this random walk are thus seen to depend only on the behavior of $G^{*m}(0, 0, 0)$ as $m \to \infty$.

2. Evaluation of $\phi^{*m} \circ \exp^\wedge$

Convolving the function ϕ on N_3 with itself m times and composing with exp, one gets a Schwartz function $\phi^{*m} \circ \exp$ on the Lie algebra of N_3. Our goal in this section is the evaluation of the ordinary Fourier transform of $\phi^{*m} \circ \exp$ as a function on R^3. To do this we define a particular sequence of polynomials, prove some preparatory results concerning their algebraic interrelationships, and then proceed with the evaluation of $\phi^{*m} \circ \exp^\wedge$ by means of a rather long and involved calculation. The major results of this section are Theorem 2.6 and its corollaries.

DEFINITION 2.1. For $m = 0, 1, \ldots$ define polynomials P_m in $R[\gamma]$ recursively by

$$P_0(\gamma) = P_1(\gamma) = 1,$$

$$(2.1) \qquad P_{m+1}(\gamma) = P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma) \quad \text{for } m \geq 1.$$

PROPOSITION 2.2. *For* $m = 1, 2, \ldots,$

$$(2.2) \qquad \begin{aligned} P_{m+1}(\gamma)/P_m(\gamma) = {} & P_m(\gamma)/P_{m-1}(\gamma) \\ & - \gamma^4(16P_{m-1}(\gamma)P_m(\gamma))^{-1}\left(\sum_{i=0}^{m-2} P_i(\gamma)\right)^2 \\ & + \gamma^2 P_{m-1}(\gamma)/(4P_m(\gamma)). \end{aligned}$$

Proof. We proceed by induction on m, the case $m = 1$ being trivial. Assume inductively that the proposition is true for m. We apply Definition 2.1 and the inductive hypothesis and calculate

$$\begin{aligned} P_{m+2}(\gamma) &= P_{m+1}(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m} P_i(\gamma) \\ &= P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m} P_i(\gamma) \\ &= (P_m(\gamma))^{-1}\left[\left(1 + \frac{\gamma^2}{4}\right)P_m^2(\gamma) + \frac{\gamma^2}{2} P_m(\gamma) \sum_{i=0}^{m-1} P_i(\gamma)\right] \\ &= (P_m(\gamma))^{-1}\left[\left(P_m(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-1} P_i(\gamma)\right)^2 - \frac{\gamma^4}{16}\left(\sum_{i=0}^{m-1} P_i(\gamma)\right)^2 + \frac{\gamma^2}{4} P_m^2(\gamma)\right] \\ &= (P_m(\gamma))^{-1}\left[P_{m+1}^2(\gamma) - \frac{\gamma^4}{16}\left(\sum_{i=0}^{m-1} P_i(\gamma)\right)^2 + \frac{\gamma^2}{4} P_m^2(\gamma)\right]. \end{aligned}$$

Dividing both sides of the equality above by $P_{m+1}(\gamma)$ provides the inductive step and completes the proof of the proposition.

The following corollary follows trivially from Definition 2.1.

COROLLARY 2.3. *For* $m = 0, 1, \ldots,$ $P_m \in R[\gamma^2/4]$ *with non-negative coefficients and constant coefficient* 1. *In particular, for every* m *and every real* γ,

$$(2.3) \qquad P_m(\gamma) \geq 1.$$

The next lemma will be used in the proof of Lemma 2.5.

LEMMA 2.4. *For* $m = 0, 1, \ldots$

$$(2.4) \quad (P_m(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma) - \gamma^2(4P_m(\gamma)P_{m+1}(\gamma))^{-1} \cdot \left(\sum_{i=0}^{m-1} P_i(\gamma)\right)^2 = (P_{m+1}(\gamma))^{-1} \sum_{i=0}^{m-1} P_i(\gamma).$$

Proof. We apply Lemma 2.2 and calculate

$$(P_m(\gamma))^{-1}\sum_{i=0}^{m-1}P_i(\gamma)-\gamma^2(4P_m(\gamma)P_{m+1}(\gamma))^{-1}\left(\sum_{i=0}^{m-1}P_i(\gamma)\right)^2$$

$$=(P_m(\gamma)P_{m+1}(\gamma))^{-1}\left[P_{m+1}(\gamma)\sum_{i=0}^{m-1}P_i(\gamma)-\frac{\gamma^2}{4}\left(\sum_{i=0}^{m-1}P_i(\gamma)\right)^2\right]$$

$$=(P_m(\gamma)P_{m+1}(\gamma))^{-1}\left[P_m(\gamma)\sum_{i=0}^{m-1}P_i(\gamma)+\frac{\gamma^2}{4}\left(\sum_{i=0}^{m-1}P_i(\gamma)\right)^2-\frac{\gamma^2}{4}\left(\sum_{i=0}^{m-1}P_i(\gamma)\right)^2\right]$$

$$=(P_m(\gamma))^{-1}\sum_{i=0}^{m-1}P_i(\gamma).$$

LEMMA 2.5. *Fix $m\in\{1,2,\ldots\}$. Let k be an integer with $0\le k\le m$. Then*

$$(2.5)\quad \phi^{*m}\circ\exp^{\wedge}(\alpha,\beta,\gamma)$$

$$=\int_{R^{2(m-k)}}E\left[\sum_{j=0}^{m-k-1}\left(\alpha-\frac{\gamma}{2}\sum_{i=0}^{j-1}y_i\right)x_j,\ \sum_{j=0}^{m-k-1}\left(\beta+\frac{\gamma}{2}\sum_{i=0}^{j-1}x_i\right)y_j\right]$$

$$\times\prod_{j=0}^{m-k-1}F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}x_j^2+(P_k(\gamma))^{-1}\left(\beta\gamma+\frac{\gamma^2}{2}\sum_{i=0}^{j-1}x_i\right)\left(\sum_{i=0}^{k-1}P_i(\gamma)\right)x_j\right]$$

$$\times\prod_{j=0}^{m-k-1}F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}y_j^2-(P_k(\gamma))^{-1}\left(\alpha\gamma-\frac{\gamma^2}{2}\sum_{i=0}^{j-1}y_i\right)\left(\sum_{i=0}^{k-1}P_i(\gamma)\right)y_j\right]$$

$$\times F\left[(P_k(\gamma))^{-1}\left(\sum_{i=0}^{k-1}P_i(\gamma)\right)(\alpha^2+\beta^2)+m\gamma^2\right]$$

$$\times(P_k(\gamma))^{-1}\prod_{i=0}^{m-k-1}dx_i\,dy_i$$

Proof. We proceed by induction on k. Consider first the case $k=0$:

$$\phi^{*m}\circ\exp^{\wedge}(\alpha,\beta,\gamma)$$

$$=\int_{R^3}E(\alpha x_0,\beta y_0,\gamma z_0)\phi^{*m}\circ\exp(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$

$$=\int_{R^{3m}}E(\alpha x_0,\beta y_0,\gamma z_0)\prod_{i=0}^{m-2}$$

$$\times G(x_i-x_{i+1},y_i-y_{i+1},z_i-z_{i+1}+\tfrac{1}{2}(x_{i+1}y_i-x_iy_{i+1}))$$

$$\times G(x_{m-1},y_{m-1},z_{m-1})\prod_{i=0}^{m-1}dx_i\,dy_i\,dz_i.$$

Making several changes of variable the above integral becomes

$$\int_{R^{3m}} E\left[\alpha x_0, \beta y_0, \gamma\left(z_0 + \frac{1}{2}\sum_{i=0}^{m-2}(x_i y_{i+1} - x_{i+1} y_i)\right)\right]$$
$$\times \prod_{i=0}^{m-2} G(x_i - x_{i+1}, y_i - y_{i+1}, z_i - z_{i+1})$$
$$\times G(x_{m-1}, y_{m-1}, z_{m-1}) \prod_{i=0}^{m-1} dx_i\, dy_i\, dz_i.$$

After we integrate with respect to $dz_0 \cdots dz_{m-1}$ and note that $\hat{G} = G$ and $G(x, y, z) = G(x)G(y)G(z)$, the integral becomes

$$\int_{R^{2m}} E\left[\alpha x_0, \beta y_0, \frac{\gamma}{2}\sum_{i=0}^{m-2}(x_i y_{i+1} - x_{i+1} y_i)\right] G(x_{m-1})$$
$$\times \prod_{i=0}^{m-2} G(x_i - x_{i+1}) G(y_{m-1}) \prod_{i=0}^{m-2} G(y_i - y_{i+1}) G(\sqrt{m}\gamma) \prod_{i=0}^{m-1} dx_i\, dy_i.$$

Again making several changes of variable we rewrite the integral as

$$\int_{R^{2m}} E\left[\sum_{j=0}^{m-1}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right) x_j, \sum_{j=0}^{m-1}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right) y_j\right]$$
$$\times \prod_{j=0}^{m-1} F(x_j^2) \prod_{j=0}^{m-1} F(y_j^2) F(m\gamma^2) \prod_{i=0}^{m-1} dx_i\, dy_i.$$

This proves the lemma in the case $k = 0$.

Next, for $0 \le k < m$,

$$\int_{R^{2(m-k)}} E\left[\sum_{j=0}^{m-k-1}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right) x_j, \sum_{j=0}^{m-k-1}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right) y_j\right]$$
$$\times \prod_{j=0}^{m-k-1} F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1} x_j^2\right.$$
$$\left.+ (P_k(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right) x_j\right]$$
$$\times \prod_{j=0}^{m-k-1} F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1} y_j^2\right.$$
$$\left.- (P_k(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right) y_j\right]$$
$$\times F\left[(P_k(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)(\alpha^2 + \beta^2) + m\gamma^2\right](P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx_i\, dy_i$$

$$= \int_{R^{2(m-k)}} E\left[\sum_{j=0}^{m-k-2}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right)x_j, \sum_{j=0}^{m-k-2}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right)y_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}x_j^2\right.$$

$$\left. + (P_k(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)x_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}y_j^2\right.$$

$$\left. - (P_k(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)y_j\right]$$

$$\times F\left[(P_k(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)(\alpha^2 + \beta^2) + m\gamma^2\right]$$

$$\times E\left[\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{m-k-2} y_i\right)x_{m-k-1}, \left(\beta + \frac{\gamma}{2}\sum_{i=0}^{m-k-2} x_i\right)y_{m-k-1}\right]$$

$$\times F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}x_{m-k-1}^2\right.$$

$$\left. + (P_k(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} x_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)x_{m-k-1}\right]$$

$$\times F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}y_{m-k-1}^2\right.$$

$$\left. - (P_k(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} y_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)y_{m-k-1}\right]$$

$$\times (P_k(\gamma))^{-1} \prod_{i=0}^{m-k-1} dx_i\, dy_i.$$

Complete the square in the arguments of the last two F's in the integrand above and the integral becomes

$$\int_{R^{2(m-k)}} E\left[\sum_{j=0}^{m-k-2}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right)x_j, \sum_{j=0}^{m-k-2}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right)y_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[P_{k+1}(\gamma)(P_k(\gamma))^{-1}x_j^2\right.$$

$$\left. + (P_k(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)x_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\Bigg[P_{k+1}(\gamma)(P_k(\gamma))^{-1}y_j^2$$

$$- (P_k(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right) y_j\Bigg]$$

$$\times F\left[(P_k(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)(\alpha^2+\beta^2) + m\gamma^2\right]$$

$$\times E\left[\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{m-k-2} y_i\right) x_{m-k-1}, \left(\beta + \frac{\gamma}{2}\sum_{i=0}^{m-k-2} x_i\right) y_{m-k-1}\right]$$

$$\times F\Bigg[P_{k+1}(\gamma)(P_k(\gamma))^{-1}\Big\{x_{m-k-1}$$

$$+ (2P_{k+1}(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} x_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\Big\}^2\Bigg]$$

$$\times F\Bigg[P_{k+1}(\gamma)(P_k(\gamma))^{-1}\Big\{y_{m-k-1}$$

$$- (2P_{k+1}(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} y_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\Big\}^2\Bigg]$$

$$\times F\Bigg[(4P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\Big\{-\beta^2\gamma^2 - \beta\gamma^3\sum_{i=0}^{m-k-2} x_i$$

$$- \frac{\gamma^4}{4}\left(\sum_{i=0}^{m-k-2} x_i\right)^2 - \alpha^2\gamma^2 + \alpha\gamma^3\sum_{i=0}^{m-k-2} y_i - \frac{\gamma^4}{4}\left(\sum_{i=0}^{m-k-2} y_i\right)^2\Big\}\Bigg]$$

$$\times (P_k(\gamma))^{-1}\prod_{i=0}^{m-k-1} dx_i\, dy_i.$$

Translating x_{m-k-1} and y_{m-k-1} appropriately and performing some algebraic manipulations we rewrite the integral as

$$\int_{R^{2(m-k)}} E\left[\sum_{j=0}^{m-k-2}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right) x_j, \sum_{j=0}^{m-k-2}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right) y_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\Bigg[\Big\{P_{k+1}(\gamma)(P_k(\gamma))^{-1} - \gamma^4(16P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\Big\} x_j^2$$

$$+ \Big\{(P_k(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\sum_{i=0}^{k-1} P_i(\gamma)$$

$$- \gamma^2(4P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\Big\} x_j\Bigg]$$

$$\times \prod_{j=0}^{m-k-2} F\left[\left\{P_{k+1}(\gamma)(P_k(\gamma))^{-1} - \gamma^4(16P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\right\}y_j^2\right.$$

$$- \left\{(P_k(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\right.$$

$$\left.\left. - \gamma^2(4P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\right\}y_j\right]$$

$$\times F\left[\left\{P_k(\gamma))^{-1}\sum_{i=0}^{k-1} P_i(\gamma) - \gamma^2(4P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)\right\}(\alpha^2+\beta^2)\right.$$

$$\left. + m\gamma^2\right]$$

$$\times E\left[\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{m-k-2} y_i\right)\right.$$

$$\left.\times \left\{x_{m-k-1} - (2P_{k+1}(\alpha))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} x_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\right\}\right]$$

$$\times E\left[\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{m-k-2} x_i\right)\right.$$

$$\left.\times \left\{y_{m-k-1} + (2P_{k+1}(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} y_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\right\}\right]$$

$$\times G[P_{k+1}(\gamma)(P_k(\gamma))^{-1}x_{m-k-1}^2,\ P_{k+1}(\gamma)(P_k(\gamma))^{-1}y_{m-k-1}^2)]$$

$$\times (P_k(\gamma))^{-1}\prod_{i=0}^{m-k-1} dx_i\, dy_i.$$

Note that

$$\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{m-k-2} x_i\right)\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} y_i\right) - \left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{m-k-2} y_i\right)\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{m-k-2} x_i\right) = 0$$

and make the substitutions

$$u = [P_{k+1}(\gamma)(P_k(\gamma))^{-1}]^{1/2}x_{m-k-1},\ v = [P_{k+1}(\gamma)(P_k(\gamma))^{-1}]^{1/2}y_{m-k-1};$$

after application of Lemma 2.4 and then integration with respect to $du\,dv$, the integral becomes

$$\int_{R^{2(m-k-1)}} E\left[\sum_{j=0}^{m-k-2}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right)x_j,\ \sum_{j=0}^{m-k-2}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right)y_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[\left\{P_{k+1}(\gamma)(P_k(\gamma))^{-1} - \gamma^4(16P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\right\}x_j^2\right.$$

$$+ \left\{(P_{k+1}(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x\right)\sum_{i=0}^{k-1} P_i(\gamma)\right\}x_j\Bigg]$$

$$\times \prod_{j=0}^{m-k-2} F\left[\left\{P_{k+1}(\gamma)(P_k(\gamma))^{-1} - \gamma^4(16P_k(\gamma)P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)^2\right\}y_j^2\right.$$

$$\left. - \left\{(P_{k+1}(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\sum_{i=0}^{k-1} P_i(\gamma)\right\}y_j\right]$$

$$\times F\left[(P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k-1} P_i(\gamma)\right)(\alpha^2 + \beta^2) + m\gamma^2\right]$$

$$\times F\left[P_k(\gamma)(P_{k+1}(\gamma))^{-1}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{m-k-2} x_i\right)^2\right]$$

$$\times F\left[P_k(\gamma)(P_{k+1}(\gamma))^{-1}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{m-k-2} y_i\right)^2\right]$$

$$\times (P_{k+1}(\gamma))^{-1}\prod_{i=0}^{m-k-2} dx_i\, dy_i.$$

Perform some algebraic manipulations and apply (2.2) and the integral finally becomes

$$\int_{R^{2(m-k-1)}} E\left[\sum_{j=0}^{m-k-2}\left(\alpha - \frac{\gamma}{2}\sum_{i=0}^{j-1} y_i\right)x_j, \sum_{j=0}^{m-k-2}\left(\beta + \frac{\gamma}{2}\sum_{i=0}^{j-1} x_i\right)y_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[P_{k+2}(\gamma)(P_{k+1}(\gamma))^{-1}x_j^2\right.$$

$$\left. + (P_{k+1}(\gamma))^{-1}\left(\beta\gamma + \frac{\gamma^2}{2}\sum_{i=0}^{j-1} x_i\right)\left(\sum_{i=0}^{k} P_i(\gamma)\right)x_j\right]$$

$$\times \prod_{j=0}^{m-k-2} F\left[P_{k+2}(\gamma)(P_{k+1}(\gamma))^{-1}y_j^2\right.$$

$$\left. - (P_{k+1}(\gamma))^{-1}\left(\alpha\gamma - \frac{\gamma^2}{2}\sum_{i=0}^{j-1} y_i\right)\left(\sum_{i=0}^{k} P_i(\gamma)\right)y_j\right]$$

$$\times F\left[(P_{k+1}(\gamma))^{-1}\left(\sum_{i=0}^{k} P_i(\gamma)\right)(\alpha^2 + \beta^2) + m\gamma^2\right]$$

$$\times (P_{k+1}(\gamma))^{-1}\prod_{i=0}^{m-k-2} dx_i\, dy_i.$$

This provides the inductive step and completes the proof of the lemma.

The case $k = m$ of Lemma 2.5 is important. We state this special case as a theorem.

THEOREM 2.6. *For* $m = 1, 2, \ldots,$

$$(2.6)\quad \phi^{*m} \circ \exp^{\wedge}(\alpha, \beta, \gamma) = (P_m(\gamma))^{-1} F\left[(P_m(\gamma))^{-1}\left(\sum_{i=0}^{m-1} P_i(\gamma)\right)(\alpha^2+\beta^2) + m\gamma^2\right].$$

COROLLARY 2.7. *For* $m = 1, 2, \ldots,$

$$(2.7)\quad \phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E[-(P_m(c))^{1/2}xa, -(P_m(c)^{1/2}yb, -zc] \times F\left[(a^2+b^2)\sum_{i=0}^{m-1} P_i(c) + mc^2\right] da\, db\, dc$$

and in particular,

$$(2.8)\quad \phi^{*m} \circ \exp(0, 0, 0) = \int_{R^3} F\left[(a^2+b^2)\sum_{i=0}^{m-1} P_i(c) + mc^2\right] da\, db\, dc.$$

Proof. We apply the Fourier inversion theorem to the Schwartz function $\phi^{*m} \circ \exp^{\wedge}$ to conclude from Theorem 2.6 that

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} (P_m(\gamma))^{-1} E(-x\alpha, -y\beta, -z\gamma) \times F\left[(\alpha^2+\beta^2)(P_m(\gamma))^{-1}\sum_{i=0}^{m-1} P_i(\gamma) + m\gamma^2\right] d\alpha\, d\beta\, d\gamma.$$

Making the substitutions $a = (P_m(\gamma))^{-1/2}\alpha$, $b = (P_m(\gamma))^{-1/2}\beta$, $c = \gamma$ we obtain 2.7.

As an immediate consequence of 2.7 we have

COROLLARY 2.8. *For* $m = 1, 2, \ldots,$ $\|\phi^{*m} \circ \exp\|_\infty = \phi^{*m} \circ \exp(0, 0, 0)$.

3. Some polynomials

In the last section we saw that the behavior of the function $\phi^{*m} \circ \exp$ is intimately related to the behavior of certain polynomials in one variable. In this section we study the polynomials P_m of the previous section and also other polynomials derived from them in some detail. The notation of the previous section remains in force.

PROPOSITION 3.1. *Let* $m = 1, 2, \ldots.$

(1) $P_m \in R[\gamma^2/4]$ *and* $P_m(0) = 1$.

(2) *Let* $x = \gamma^2/4$ *and view* P_m *as a polynomial in* x; *then* $\deg P_m = [m/2]$. *Moreover, if*

$$P_m(x) = 1 + a_1 x + \cdots + a_{[m/2]} x^{[m/2]}$$

then a_i *is a positive integer for* $i = 1, \ldots, [m/2]$. *If* m *is even, then* $a_{m/2} = 1$.

Proof. (1) is a restatement of Corollary 2.3.

To prove (2) we proceed by induction on m, the cases $m = 0, 1$, being trivial. Suppose inductively that $m \geq 1$ and that the proposition has been proved for 0, ..., m. We consider two cases.

Case 1. m odd. We let $x = \gamma^2/4$ and replace $\gamma^2/4$ by x everywhere that $\gamma^2/4$ appears in the expressions for the polynomials $P_i(\gamma)$. Then by Proposition 2.2 and a slight abuse of notation

$$P_{m+1}(x) = P_m(x) + x \sum_{i=0}^{m-1} P_i(x). \tag{3.1}$$

By the inductive hypothesis, $\deg P_m(x) = (m-1)/2 = \deg P_{m-1}(x)$ and, for $i \leq m-2$, $\deg P_i(x) \leq (m-3)/2$ with each of the polynomials $P_i(x)$, $i = 0, \ldots, m$ having positive integer coefficients. We thus see that there are positive integers $a_1, \ldots, a_{(m-1)/2}, b_1, \ldots, b_{(m-3)/2}$, so that

$$P_m(x) + x \sum_{i=0}^{m-2} P_i(x) = 1 + a_1 x + \cdots + a_{(m-1)/2} x^{(m-1)/2}$$

and

$$x P_{m-1}(x) = x + b_1 x^2 + \cdots + b_{(m-3)/2} x^{(m-1)/2} + x^{(m+1)/2}$$

where we have used the inductive hypothesis to conclude that the leading coefficient of $P_{m-1}(x)$ is 1. From 3.1 and the above expressions we see that $P_{m+1}(x)$ is a polynomial of degree $(m+1)/2 = [(m+1)/2]$ with positive integer coefficients and leading coefficient 1.

Case 2. m even. Proceeding as in Case 1 we have

$$P_{m+1}(x) = P_m(x) + x \sum_{i=0}^{m-1} P_i(x)$$

where $\deg P_m(x) = m/2 = [(m+1)/2]$ and $\deg P_i(x) \leq (m-2)/2$ for $i = 0, \ldots, m-1$. The proposition in this case now follows immediately.

This completes the proof of the proposition.

DEFINITION 3.2. For $m, k = 0, 1, \ldots$ define $a_{m,k}$ by

$$P_m(\gamma) = a_{m,0} + a_{m,1}(\gamma^2/4) + \cdots + a_{m,[m/2]}(\gamma^2/4)^{[m/2]} \tag{3.2}$$

and set $a_{m,k} = 0$ for $k > [m/2]$.

Our immediate goal is to obtain a formula for $a_{m,k}$ in terms of m and k.

LEMMA 3.3. *For* $m = 0, 1, \ldots,$ *and* $k = 1, 2, \ldots,$

$$a_{m,k} = \sum_{i=0}^{m-2} (m-i-1) a_{i,k-1}. \tag{3.3}$$

Proof. Since

$$P_m(\gamma) = P_{m-1}(\gamma) + \frac{\gamma^2}{4} \sum_{i=0}^{m-2} P_i(\gamma)$$

for $m \geq 1$ and $P_0(\gamma) = 1$, we see that $a_{m,k} = a_{m-1,k} + \sum_{i=0}^{m-2} a_{i,k-1}$. We fix $k \geq 1$ and proceed by induction on m. For $m = 0$ the lemma is trivial. Assuming inductively that (3.3) holds for m; then

$$\begin{aligned} a_{m+1,k} &= a_{m,k} + \sum_{i=0}^{m-1} a_{i,k-1} \\ &= \sum_{i=0}^{m-2} (m-i-1)a_{i,k-1} + \sum_{i=0}^{m-1} a_{i,k-1} \\ &= a_{m-1,k-1} + \sum_{i=0}^{m-2} (m-i)a_{i,k-1} \\ &= \sum_{i=0}^{m-1} (m-i)a_{i,k-1} \end{aligned}$$

This completes the proof of the lemma.

At this point we recall the classical fact (see [6] for example) that for a non-negative integer p

$$(3.4) \qquad 1^p + 2^p + \cdots + k^p = \frac{k^{p+1}}{p+1} + \frac{k^p}{2} + \sum_{n=2}^{p+1} \frac{B_n}{n} \binom{p}{n-1} k^{p-n+1}$$

where the numbers B_n are the Bernoulli numbers which occur as the coefficients of the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n z^n, \quad |z| < 2\pi.$$

(3.4) is actually a stronger statement than we need. We shall use only the fact that the left side of (3.4) is given by a polynomial in k of degree $p + 1$.

LEMMA 3.4. *For m, $k = 0, 1, \ldots, a_{m,k} = S_k(m)$ where S_k is a polynomial of degree $2k$.*

Proof. We note that $a_{m,0} = 1$ for all m and proceed by induction on k. Assume inductively that the lemma is true for k. By (3.3),

$$a_{m,k+1} = \sum_{i=0}^{m-2} (m-i-1)a_{i,k} = (m-1) \sum_{i=0}^{m-2} S_k(i) - \sum_{i=0}^{m-2} iS_k(i)$$

where S_k is a polynomial of degree $2k$. Applying (3.4) to the summation of the various powers of i occurring in the polynomials $S_k(i)$ and $iS_k(i)$ we see that $a_{m,k+1}$ is given by the difference of two polynomials in m of degree $2k + 2$. Thus there is a polynomial S_{k+1} of degree at most $2k + 2$ such that $a_{m,k+1} = S_{k+1}(m)$.

Since $a_{0,k+1} = \cdots = a_{2k+1,k+1} = 0$ we see that S_{k+1} must in fact be of degree $2k + 2$. This completes the inductive step and thereby the proof of the lemma.

We are now in a position to determine an explicit formula for $a_{m,k}$.

PROPOSITION 3.5. *For $m, k = 0, 1, \ldots,$*

$$a_{m,k} = \frac{1}{(2k)!} \prod_{i=0}^{2k-1} (m - i). \tag{3.5}$$

Proof. By Lemma 3.4, for $k \geq 1$, $a_{m,k} = S_k(m)$ where S_k is a polynomial of degree $2k$ and $S_k(0) = \cdots = S_k(2k - 1) = 0$.
Thus for $k \geq 1$,

$$a_{m,k} = c \cdot m(m - 1) \cdots (m - (2k - 1))$$

for some constant c. By Proposition 3.1

$$1 = a_{2k,k} = c \cdot 2k(2k - 1) \cdots (1) = c(2k)!.$$

Thus $c = 1/(2k)!$. This proves the proposition for $k \geq 1$. For $k = 0$, $S_k \equiv 1$ and the proposition is trivially true.

If we consider the integral of (2.7) which yields $\phi^{*m} \circ \exp(x, y, z)$ we see that we need to determine not only the polynomials $P_i(\gamma)$ but also certain sums of these polynomials.

DEFINITION 3.6. For $m = 0, 1, \ldots$ define polynomials Γ_m by

$$\sum_{i=0}^{m-1} P_i(\gamma) = m + \Gamma_m(\gamma). \tag{3.6}$$

We note that $\Gamma_m(0) = 0$ and that if Γ_m is viewed as an element in $R[\gamma^2/4]$ then $\deg \Gamma_m = [(m - 1)/2]$ for $m \geq 1$, and $\Gamma_0 = 0$.

DEFINITION 3.7. For $m, k = 0, 1, \ldots$ define $\tau_{m,k}$ by

$$\Gamma_m(\gamma) = \tau_{m,0} + \tau_{m,1}\left(\frac{\gamma^2}{4}\right) + \cdots + \tau_{m,[(m-1)/2]}\left(\frac{\gamma^2}{4}\right)^{[(m-1)/2]} \tag{3.7}$$

and set $\tau_{m,k} = 0$ for $k > [(m - 1)/2]$. Note that $\tau_{m,0} = 0$ for all m.

PROPOSITION 3.8. *For $m = 0, 1, \ldots$ and $k = 1, 2, \ldots,$*

$$\tau_{m,k} = \frac{1}{(2k + 1)!} \prod_{i=0}^{2k} (m - i). \tag{3.8}$$

Proof. From (3.6) and (3.7) we see that, for $k \geq 1$,

$$\tau_{m,k} = \sum_{i=0}^{m-1} a_{i,k}, \quad m = 0, 1, \ldots.$$

Thus $\tau_{m,k} = 0$ for $m = 0, \ldots, 2k$. Since, by Proposition 3.4, $a_{i,k}$ is given by a polynomial in i of degree $2k$ we have by equation (3.4) that $\tau_{m,k}$ is given by a polynomial in m of degree $2k + 1$. Thus

$$\tau_{m,k} = c \cdot m(m-1) \cdots (m-2k)$$

for some constant c. Since $\tau_{2k+1,k} = a_{2k,k} = 1$ we have

$$1 = c \cdot (2k+1)(2k) \cdots (1) = c \cdot (2k+1)!.$$

Thus $c = 1/(2k+1)!$ and the proof of the proposition is complete.

COROLLARY 3.9. *For $c \neq 0$,*

$$\lim_{m \to \infty} \frac{1}{m} \Gamma_m(m^{-1/2}c) = \infty. \tag{3.9}$$

Proof. Applying (3.7) and (3.8) we have

$$\begin{aligned}\frac{1}{m} \Gamma_m(m^{-1/2}c) &= \frac{1}{m} \sum_{k=1}^{[(m-1)/2]} \frac{m(m-1) \cdots (m-2k)}{(2k+1)!} \cdot \frac{c^{2k}}{m^k 4^k} \\ &= \sum_{k=1}^{[(m-1)/2]} \frac{m(m-1) \cdots (m-2k)}{(2k+1)!\, m^{k+1}} \left(\frac{c}{2}\right)^{2k}.\end{aligned}$$

For fixed c and k each of the terms in the above sum tend to ∞ individually as $m \to \infty$. This proves the corollary.

4. Asymptotic behavior of the random walk

The notation of the previous sections remains in force.

In Corollary 2.7 we saw that

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E[-(P_m(c))^{1/2}xa, -(P_m(c))^{1/2}yb, -zc] \times F\left[(a^2 + b^2) \sum_{i=0}^{m-1} P_i(c) + mc^2\right] da\, d\, dc.$$

Alternately,

$$\phi^{*m} \circ \exp(x, y, z) = \int_{R^3} E[-(P_m(c))^{1/2}xa, -(P_m(c))^{1/2}yb, -zc] \times F[(m + \Gamma_m(c))(a^2 + b^2) + mc^2]\, da\, db\, dc.$$

Making a change of variables this last integral becomes

$$m^{-3/2} \int_{R^3} E[-(P_m(m^{-1/2}c))^{1/2}m^{-1/2}xa, -(P_m(m^{-1/2}c))^{1/2}m^{-1/2}yb, -m^{-1/2}zc] \times F[m^{-1}(m + \Gamma_m(m^{-1/2}c))(a^2 + b^2) + c^2]\, da\, db\, dc$$

Again making a change of variables we have

$$\phi^{*m} \circ \exp(x, y, z) = m^{-3/2}$$
$$\times \int_{R^3} E\left[-\left(\frac{P_m(m^{-1/2}c)}{m+\Gamma_m(m^{-1/2}c)}\right)^{1/2} xa, -\left(\frac{P_m(m^{-1/2}c)}{m+\Gamma_m(m^{-1/2}c)}\right)^{1/2} yb, -m^{-1/2}zc\right]$$
$$\times \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(a, b, c)\, da\, db\, dc.$$

We integrate with respect to $da\, db$ to conclude that

$$\text{(4.1)} \quad \phi^{*m} \circ \exp(x, y, z) = m^{-3/2} \int_R \frac{m}{m+\Gamma_m(m^{-1/2}c)} E(-m^{-1/2}zc) \times F\left(\frac{P_m(m^{-1/2}c)}{m+\Gamma_m(m^{-1/2}c)} (x^2+y^2)\right) G(c)\, dc$$

and in particular,

$$\text{(4.2)} \quad \phi^{*m} \circ \exp(0, 0, 0) = m^{-3/2} \int_R \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\, dc.$$

In the next theorem we utilize (4.1) and (4.2) to show that, as in the abelian case, the rate at which the Gaussian random walk on N_3 "wanders to ∞" is determined by the behavior of $\phi^{*m} \circ \exp(0, 0, 0)$ as $m \to \infty$.

THEOREM 4.1. *For $k > 0$ let $B_k = \{(x, y, z)\colon (x^2+y^2+z^2)^{1/2} \le k\}$. Then there is a constant, $C_k > 0$, depending only on k so that for $(x, y, z) \in B_k$,*

$$\phi^{*m} \circ \exp(0, 0, 0) \ge \phi^{*m} \circ \exp(x, y, z) \ge C_k \phi^{*m} \circ \exp(0, 0, 0).$$

Hence,

$$\int_{B_k} \phi^{*m} \circ \exp(x, y, z)\, dx\, dy\, dz = O(\phi^{*m} \circ \exp(0, 0, 0))$$

and

$$\phi^{*m} \circ \exp(0, 0, 0) = O\left(\int_{B_k} \phi^{*m} \circ \exp(x, y, z)\, dx\, dy\, dz\right)$$

Proof. The first inequality was established in Corollary 2.8. To establish the second inequality we note that since $\phi^{*m} \circ \exp$ is continuous and strictly positive for all m it is sufficient to find a positive constant C_k that works for all sufficiently large m. To obtain such a C_k we will consider the expression in (4.1) and (4.2) and make some estimates. No effort will be made to obtain the best possible C_k. We will need to make use of certain results that will be obtained later in the course of the proof of Theorem 4.4. No circularity of reasoning will, however, arise.

Fix $k > 0$. Choose $A > 0$ so that for all m,

$$\int_{R\backslash[-A,A]} \frac{m}{m + \Gamma_m(m^{-1/2}c)} G(c)\, dc \le \varepsilon \int_R \frac{m}{m + \Gamma_m(m^{-1/2}c)} G(c)\, dc$$

where $\varepsilon > 0$ is chosen so that $\varepsilon(1 + G(k)/4) < G(k)/8$. Such an A exists since by Theorem 4.4,

$$\liminf_{m\to\infty} \sqrt{m} \int_R \frac{m}{m + \Gamma_m(m^{-1/2}c)} G(c)\, dc > 0$$

whereas by (4.9)

$$\lim_{m\to\infty} \sqrt{m} \int_{R/[-A,A]} \frac{m}{m + \Gamma_m(m^{-1/2}c)} G(c)\, dc = 0$$

for every $A > 0$, and such an A can clearly be found for fixed m.

Since all of the real-valued functions in (4.1), when viewed as functions of c, are even, only Re $(E(-m^{-1/2}zc))$ affects the integral, and hence

$$\begin{aligned}\phi^{*m} \circ \exp(x, y, z) = m^{-3/2} \Bigg(\int_{[-A,A]} & \frac{m}{m + \Gamma_m(m^{-1/2}c)} \cos(m^{-1/2}zc) \\ & \times F\left(\frac{P_m(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} (x^2 + y^2)\right) G(c)\, dc \\ & + \int_{R\backslash[-A,A]} \frac{m}{m + \Gamma_m(m^{-1/2}c)} \cos(m^{-1/2}zc) \\ & \times F\left(\frac{P_m(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} (x^2 + y^2)\right) G(c)\, dc.\end{aligned}$$

Next, we note that by (2.2) and (3.6),

$$\begin{aligned} P_m(m^{-1/2}c) &= P_{m-1}(m^{-1/2}c) + \frac{c^2}{4m} \sum_{i=0}^{m-2} P_i(m^{-1/2}c) \\ &= P_{m-1}(m^{-1/2}c) + \frac{c^2}{4m} (\Gamma_{m-1}(m^{-1/2}c) + m - 1).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{P_m(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} &= \frac{P_{m-1}(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} + \frac{c^2}{4m} \frac{m - 1 + \Gamma_{m-1}(m^{-1/2}c)}{m + \Gamma_m(m^{-1/2}c)} \\ &< 1 + \frac{c^2}{4m}.\end{aligned}$$

Using the above inequality and the fact that $F \le 1$ we see that if m is large enough so that $\cos(m^{-1/2}zc) \ge \frac{1}{2}$ whenever $|z| \le k$ and $|c| \le A$ then

$$\begin{aligned}
&\phi^{*m} \circ \exp(x, y, z) \\
&\geq m^{-3/2}\left(\frac{1}{2}\int_{[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} F\left(\left(1+\frac{c^2}{4m}\right)(x^2+y^2)\right) G(c)\,dc \right.\\
&\qquad \left. - \int_{R\setminus[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\right)\\
&\geq m^{-3/2}\left(G(k)/2 \int_{[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} F(c^2\left(1+\frac{k^2}{4m}\right)\right)dc\\
&\qquad \left. - \int_{R\setminus[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\right).
\end{aligned}$$

Now if m is large enough so that $1 + k^2/4m < 4$, then, using the fact that Γ_m is even and monotone increasing on $[0, \infty)$ and F is even and monotone decreasing on $[0, \infty)$, we have

$$\begin{aligned}
\phi^{*m} \circ \exp(x, y, z) &\geq m^{-3/2}\left(G(k)/2 \int_{[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}2c)} G(2c)\,dc\right.\\
&\qquad \left. - \int_{R\setminus[-A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\right.\\
&= m^{-3/2}\left(G(k)/4 \int_{[-2A,2A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\right.\\
&\qquad \left. - \int_{R\setminus[A,A]} \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\right).
\end{aligned}$$

By our choice of A and ε we now have

$$\begin{aligned}
\phi^{*m} \circ \exp(x, y, z) &\geq \left(\frac{G(k)}{4}(1-\varepsilon) - \varepsilon\right) m^{-3/2} \int_R \frac{m}{m+\Gamma_m(m^{-1/2}c)} G(c)\,dc\\
&\geq (G(k)/4 - \varepsilon(1 + G(k)/4))\phi^{*m} \circ \exp(0, 0, 0)\\
&\geq G(k)/8 \cdot \phi^{*m} \circ \exp(0, 0, 0).
\end{aligned}$$

This completes the proof of the theorem.

The author thanks Jonathan Brezin for pointing out an error in the original proof of Theorem 4.1.

DEFINITION 4.2. For $m = 1, 2, \ldots$ set

$$\begin{aligned}
I_m &= (\phi \circ \exp^{*m}(0, 0, 0))^{-1}\phi^{*m} \circ \exp(0, 0, 0)\\
&= m^{3/2}\phi^{*m} \circ \exp(0, 0, 0)\\
&= \int_R \frac{1}{1 + m^{-1}\Gamma_m(m^{-1/2}c)} G(c)\,dc.
\end{aligned}$$

From the integral defining I_m we see immediately that $I_m < 1$. Thus we see that not only is the non-abelian walk transient but also that it "wanders to ∞" at least as rapidly as does the abelian walk. In fact the non-abelian walk "wanders to ∞" more rapidly than does the abelian walk as one sees by applying Corollary 3.9 and the dominated convergence theorem to conclude that $I_m \to 0$ as $m \to \infty$. It is now natural to inquire about the rate at which I_m tends to 0. This question will be answered rather precisely in Theorem 4.4.

LEMMA 4.3. *For $a > 0$, $c \in R$,*

$$\lim_{m\to\infty} \sum_{k=[am]+1}^{\infty} (mc)^k \frac{1}{(2k+1)!} = 0, \tag{4.3}$$

the limit being uniform on compact subsets of R.

Proof.

$$\sum_{k=[am]+1}^{\infty} \left| (mc)^k \frac{1}{(2k+1)!} \right| \le \sum_{k=[am]+1}^{\infty} \frac{|mc|^k}{(am)^k} \frac{1}{(k+1)!}$$
$$= \sum_{k=[am]+1}^{\infty} (|c|/a)^k \frac{1}{(k+1)!},$$

the latter series tending uniformly to 0 on compacta as $m \to \infty$.

THEOREM 4.4. (a) $\lim_{m\to\infty} m^p I_m = 0$ *for* $p < \frac{1}{2}$.
(b) $0 < \liminf_{m\to\infty} m^{1/2} I_m \le \limsup_{m\to\infty} m^{1/2} I_m < \infty$.
(c) $\lim_{m\to\infty} m^p I_m = \infty$ *for* $p > \frac{1}{2}$.

Proof. Although (a) and (c) follow immediately from (b) we shall prove them independently since little extra work is involved in doing so and the proofs are instructive.

Since Γ_m and G are even functions,

$$m^p I_m = 2 \int_0^\infty \frac{m^p}{1 + m^{-1}\Gamma_m(m^{-1/2}c)} G(c)\,dc.$$

Since Γ_m is a monotone increasing function we have that

$$m^p I_m > \int_0^{m^{-1/2}} \frac{m^p}{1 + m^{-1}\Gamma_m(m^{-1/2}c)} G(c)\,dc \tag{4.4}$$
$$> G(1) m^{p-1/2}(1 + m^{-1}\Gamma_m(m^{-1}))^{-1}.$$

Now, combining (3.7) with (3.8) and setting $\gamma = m^{-1}$ we have

$$m^{-1}\Gamma_m(m^{-1}) = \sum_{k=1}^{[(m-1)/2]} \frac{(m-1)\cdots(m-2k)}{(2k+1)!\,4^k} \frac{1}{m^{2k}}$$
$$< \sum_{k=1}^{[(m-1)/2]} \frac{1}{(2k+1)!\,4^k}.$$

This shows that $m^{-1}\Gamma_m(m^{-1})$ remains bounded as $m \to \infty$. Applying this result to 4.4 we see that

$$\liminf_{m\to\infty} m^{1/2} I_m > 0 \tag{4.5}$$

and

$$\lim_{m\to\infty} m^p I_m = \infty \quad \text{for } p > \tfrac{1}{2}. \tag{4.6}$$

On the other hand,

$$\int_0^{m^{-1/2}} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1} m^p G(c)\, dc < m^{p-1/2}$$

and thus

$$\lim_{m\to\infty} \int_0^{m^{-1/2}} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1} m^p G(c)\, dc = 0 \quad \text{for } p < \tfrac{1}{2} \tag{4.7}$$

and

$$\limsup_{m\to\infty} \int_0^{m^{-1/2}} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1} m^{1/2} G(c)\, dc \le 1. \tag{4.8}$$

To complete the proof of the theorem we must consider the behavior of

$$\int_{m^{-1/2}}^{\infty} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1} G(c)\, dc.$$

Consider the map

$$c \mapsto 1 + m^{-1}\Gamma_m(m^{-1/2}c) = 1 + \sum_{k=1}^{[(m-1)/2]} \frac{(m-1)\cdots(m-2k)}{(2k+1)!}\left(\frac{c^2}{4m}\right)^k.$$

For $m \ge 2$,

$$\begin{aligned} 1 + m^{-1}\Gamma(m^{-1/2}c) &> 1 + \sum_{k=1}^{[m/4]} \frac{(m-1)\cdots(m-2k)}{(2k+1)!}\left(\frac{c^2}{4m}\right)^k \\ &> 1 + \sum_{k=1}^{[m/4]} \frac{(\frac{1}{2}m)^{2k}}{(2k+1)!}\left(\frac{c^2}{4m}\right)^k \\ &= \sum_{k=0}^{[m/4]} \frac{1}{(2k+1)!}\left(\frac{mc^2}{16}\right)^k \end{aligned} \tag{4.9}$$

from whence we see that for all p and all $c \neq 0$,

$$\lim_{m\to\infty} m^{-p} \cdot m^{-1}\Gamma_m(m^{-1/2}c) = \infty,$$

the limit being uniform away from 0. Thus for all p,

$$\lim_{m\to\infty} \int_1^{\infty} m^p (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1} G(c)\, dc = 0. \tag{4.10}$$

We next consider the behavior of the integral

$$\int_{m^{-1/2}}^{1} m^{p}(1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1}G(c)\,dc.$$

Define

$$f(r) = \sum_{k=0}^{\infty} \frac{r^k}{(2k+1)!}, \quad r \in R. \tag{4.11}$$

By (4.9) and Lemma 4.3 we conclude that for K_0 fixed, $0 < K_0 < 1$, and for m sufficiently large,

$$1 + m^{-1}\Gamma_m(m^{-1/2}c) > K_0 f(mc^2/16), \quad c \in [0, 1]. \tag{4.12}$$

Now, for $r > 0$,

$$f(r) = r^{-1/2} \sum_{k=0\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!}. \tag{4.13}$$

We also have

$$\begin{aligned} e^{\sqrt{r}} &= \sum_{k=0}^{\infty} \frac{\sqrt{r}^{k}}{k!} \\ &= 1 + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+2}}{(2k+2)!} + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!} \\ &= 1 + \sqrt{r} \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+2)!} + \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!}. \end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14) we conclude that, for $r > 0$,

$$\begin{aligned} \sqrt{r} f(r) &= \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+1)!} \\ &> \sum_{k=0}^{\infty} \frac{\sqrt{r}^{2k+1}}{(2k+2)!} \\ &= r^{-1/2}(e^{\sqrt{r}} - 1 - \sqrt{r} f(r)) \end{aligned}$$

from whence $f(r) > (r + \sqrt{r})^{-1}(e^{\sqrt{r}} - 1)$. Thus for $c \in [m^{-1/2}, 1]$,

$$f\left(\frac{mc^2}{16}\right) \geq \left(\frac{mc^2}{16} + \frac{\sqrt{m}\,c}{4}\right)^{-1} e^{\sqrt{m}c/4}(1 - e^{-1/4}). \tag{4.15}$$

Setting $K = K_0(1 - e^{-1/4})$ we have, by (4.12) and (4.15),

$$1 + m^{-1}\Gamma_m(m^{-1/2}c) > K\left(\frac{mc^2}{16} + \frac{\sqrt{m}\,c}{4}\right)^{-1} e^{\sqrt{m}c/4} \tag{4.16}$$

for $c \in [m^{-1/2}, 1]$ and m sufficiently large. Thus for large m

$$\int_{m^{-1/2}}^{1} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1}G(c)\,dc$$

$$< K^{-1} \int_{m^{-1/2}}^{1} \left(\frac{mc^2}{16} + \frac{\sqrt{m}\,c}{4}\right) e^{-\sqrt{m}c/4}\,dc$$

$$= K^{-1} \int_{1/4}^{\sqrt{m}/4} (x^2 + x)e^{-x} \cdot 4m^{-1/2}\,dx$$

$$< K^{-1}4m^{-1/2} \int_{0}^{\infty} (x^2 + x)e^{-x}\,dx.$$

Since the last integral above is finite we conclude that

$$\limsup_{m\to\infty} \sqrt{m} \int_{m^{-1/2}}^{1} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1}G(c)\,dc < \infty \tag{4.17}$$

and, for $p < \frac{1}{2}$,

$$\lim_{m\to\infty} m^p \int_{m^{-1/2}}^{1} (1 + m^{-1}\Gamma_m(m^{-1/2}c))^{-1}G(c)\,dc = 0. \tag{4.18}$$

Finally, (4.6) proves (c), (4.7), (4.10) and (4.18) together prove (a), and (4.5), (4.8), (4.10) and (4.17) together prove (b).

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University of Utah
Salt Lake City, Utah