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MULTIPLIERS OF THE DIRICHLET SPACE

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Introduction

This paper deals with the space of analytic functions on the unit disc in the complex plane for which $\sum n^{\alpha}|a_n|^2$ is finite, where $\{a_n\}$ represent the Taylor coefficients and α is a real number. For $\alpha = 1$ this space can alternately be described by demanding that the Dirichlet integral $\iint |f'|^2\, dx\, dy$ be finite. We also consider real variable analogs of these spaces on the circle and on Euclidean space. In $\mathbf{R}^n$, these are the fractional Sobolev space L_{α}^{p}.

Our principal result is a characterization of the pointwise multipliers of these spaces. Various authors have studied properties of these multipliers and determined sufficient conditions, see [24], [5], [14], [15], [16], and in the complex case [20], [25].

Denote by D_α the space of analytic functions for which the norm

$$\left\{\sum_{n=0}^{\infty} (1+n^2)^{\alpha}|a_n|^2\right\}^{1/2}$$

is finite.

THEOREM A. *An analytic function $f(z)$ multiplies D_α $(0 < \alpha \le \frac{1}{2})$ if and only if f is bounded on $|z| < 1$ and there is a constant A such that*

$$\iint_{\cup S(I_j)} |f'|^2(1-|z|)^{1-2\alpha}\, dx\, dy \le A \operatorname{Cap}_\alpha(\cup I_j)$$

for all finite disjoint collections of subarc $\{I_j\}$ on the circle.

Here $S(I)$ denotes the "square" in the disc with side I and $\operatorname{Cap}_\alpha(\cdot)$ denotes an appropriate (Bessel) capacity depending on α. For $\alpha = \frac{1}{2}$ (Dirichlet space) the classical logarithmic capacity may be used.

For multipliers of $L_\alpha^p(\mathbf{R}^n)$ and also a boundary characterization on D_α we have the following result:

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THEOREM B. *A function f multiplies $L_\alpha^p(\mathbf{R}^n)$ for $0 < \alpha < 1$, $\alpha p \le n$, and $p \ge 2$ if and only if f is in L^∞ and there is a constant A such that*

$$\int_E \left| \int_{\mathbf{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^{n+2\alpha}} \, dy \right|^{p/2} dx \le A \operatorname{Cap}_{\alpha,p}(E)$$

for all Borel subsets E. Here $\operatorname{Cap}_{\alpha,p}$ is the α, p-Bessel capacity on $\mathbf{R}^n$.

The idea used in proving these results is a generalization of the Carleson measures used in the study of the Hardy spaces combined with the strong type capacity inequality of V. G. Maz'ya and D. R. Adams. In Section 1 we develop the connection with Carleson measures and in Section 2 we prove our principal results. In Section 3 we investigate a connection between the multipliers of the Hardy space H^2 into the Bergman space B^2 and the functions of bounded mean oscillation, BMO, of John and Nirenburg. Finally, in Section 4 we show by means of a counterexample that the sets used in Theorem A and Theorem B can not simply be intervals.

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1. Carleson measures on D_α

Denote by $\|f\|_\alpha$ the norm $\{\sum_{n=0}^\infty (1+n^2)^\alpha |a_n|^2\}^{1/2}$ of a function $f(z) = \sum_0^\infty a_n z^n$ in D_α. Using Parseval's relation it is easy to show that for $\alpha < 0$ (see [25, Lemma 2]),

$$\iint_{|z|<1} |f(z)|^2 (1 - |z|)^{-1-2\alpha} \, dx \, dy$$

is equivalent to $\|f\|_\alpha^2$. Also, it is obvious that f is in D_α if and only if f' is in $D_{\alpha-1}$. As a result, f is in D_α for $\alpha < 1$ if and only if

$$\iint_{|z|<1} |f'(z)|^2 (1 - |z|)^{1-2\alpha} \, dx \, dy$$

is finite. Also, $\|f\|_\alpha^2$ is equivalent to the above plus $|f(0)|^2$.

Thus we see three spaces of particular interest, namely, $\alpha = -\frac{1}{2}$ is the Bergman space of analytic functions in $L^2(dx\, dy)$ of the unit disc, $\alpha = \frac{1}{2}$ is the Dirichlet space of analytic functions with $\iint |f'|^2 \, dx\, dy$ finite, and $\alpha = 0$ is the Hardy space H^2.

Let $M(D_\alpha, D_\beta)$ be the collection of all functions f which multiply, D_α into D_β, i.e., fg is in D_β for all g in D_α. In [25], $M(D_\alpha, D_\beta)$ is characterized for certain ranges of α and β. The known cases are as follows:

(i) $\beta \le \alpha < 0$. Then $M(D_\alpha, D_\beta)$ consists of all analytic functions (in $|z| < 1$) which satisfy the growth condition $|f(z)| = 0(1 - r)^{\beta - \alpha}$.

(ii) $\alpha > \frac{1}{2}$ and $\beta \leq \alpha$. Then $M(D_\alpha, D_\beta) = D_\beta$.
(iii) $\beta > \alpha$. Then $M(D_\alpha, D_\beta)$ contains only the zero function.

The difficulty with the range $0 \leq \alpha \leq \frac{1}{2}$ is that the derivative enters into the definition of the norm and that the functions in D_α are not in general bounded. For $\alpha > \frac{1}{2}$, the functions in D_α are bounded (an easy calculation) which partially explains why D_α is an algebra in this case. A natural candidate for $M(D_\alpha) = M(D_\alpha, D_\alpha)$ for $0 < \alpha \leq \frac{1}{2}$ is simply the space of bounded functions in D_α. This is shown to be false in [25].

DEFINITION. A positive Borel measure on the open unit disc is an α-Carleson measure provided there is a constant c satisfying $\int |g|^2 \, d\mu \leq c\|g\|_\alpha^2$ for all g in D_α.

For $\alpha = 0$ ($D_0 = H^2$), Carleson [3] characterized these measures and applied them in the solution of the Corona Theorem. Carleson measures were also important in Fefferman and Stein's duality theory for H^1 (see [8]).

THEOREM 1.1. (a) *$f \in M(D_\alpha, D_\beta)$ for $\beta < 0$ if and only if f is analytic and*

$$|f|^2(1 - |z|)^{-1-2\beta} \, dx \, dy$$

is an α-Carleson measure.

(b) *$f \in M(D_\alpha, D_\beta)$ for $0 \leq \beta < \alpha \leq \frac{1}{2}$ if and only if f is analytic and*

$$|f'|^2(1 - |z|)^{1-2\beta} \, dx \, dy$$

is an α-Carleson measure.

(c) *$f \in M(D_\alpha)$ for $0 \leq \alpha \leq \frac{1}{2}$ if and only if f is bounded, analytic, and*

$$|f'|^2(1 - |z|)^{1-2\alpha} \, dx \, dy$$

is an α-Carleson measure.

(d) *$M(D_\alpha, D_\beta) = D_\beta$ for $\beta \leq \alpha$ and $\alpha > \frac{1}{2}$.*

Proof. By Lemma 3 [25], f is in $M(D_\alpha, D_\beta)$ if and only if the corresponding multiplication operator is bounded. Using this and the equivalent norm for D_β yields (a).

For (b) we take f in $M(D_\alpha, D_\beta)$ and g in D_α then we must bound $\|f'g\|_{\beta-1}$ by a multiple of $\|g\|_\alpha$. By Schwarz's inequality we have

$$\|f'g\|_{\beta-1} \leq \|(fg)'\|_{\beta-1} + \|fg'\|_{\beta-1}.$$

The first term is dominated by $\|fg\|_\beta$ which can be replaced by $\|g\|_\beta$ since f is a bounded multiplication operator. By the obvious inclusion relation $\|g\|_\beta \leq \|g\|_\alpha$. For the second term we use the fact that $f(z) = O(1 - |z|)^{\beta-\alpha}$, see Theorem 1 [25], to show that this term is also dominated by $\|g'\|_{\alpha-1}$ or $\|g\|_\alpha$.

To prove the converse we use Corollary 1.5 which we prove later. As a result, if f gives rise to an α-Carleson measure then

$$|f(z)| \leq |f(0)| + c(1 - |z|)^{\beta-\alpha}$$

for some constant c. By a similar argument to the above we show that $\|fg\|_\beta$ is dominated by $\|g\|_\alpha$ and hence f is in $M(D_\alpha, D_\beta)$.

The proof of (c) is the same as (b) except we can not conclude f is bounded as a consequence of the α-Carleson measure property.

To prove (d) we use the fact that $M(D_\alpha)$ is contained in $M(D_\beta)$ whenever $\alpha \geq \beta$, see Theorem 3 [25]. Now it suffices to prove $M(D_\alpha) = D_\alpha$ for $\alpha > \frac{1}{2}$. This follows since $D_\alpha \cdot D_\beta = M(D_\alpha) \cdot D_\beta \subset M(D_\beta) \cdot D_\beta \subset D_\beta$ and hence $D_\beta \subset M(D_\alpha, D_\beta)$ by the inclusion relation whenever $\alpha \geq \beta$ and $\alpha > \frac{1}{2}$. Since the function 1 is in all the D_α spaces we trivially obtain the opposite inclusion $M(D_\alpha, D_\beta) \subset D_\beta$.

We first note the obvious fact that H^∞ (the space of bounded analytic functions in $|z| < 1$) is contained in $M(D_\alpha)$ for $\alpha \leq 0$. If $\frac{1}{2} < \alpha \leq 1$ and $f, g \in D_\alpha$ then $\|(fg)'\|_{\alpha-1} \leq \|f'g\|_{\alpha-1} + \|fg'\|_{\alpha-1}$ is finite since f and g are both bounded. For $\alpha > 1$, we assume by induction that the result holds for $\frac{1}{2} < \alpha \leq n$ and that $n < \alpha \leq n+1$. Let $f, g \in D_\alpha$ then $g \in D_n = M(D_n) \subset M(D_{\alpha-1})$ so gf' is in $D_{\alpha-1}$. Similarly, $g'f$ is in $D_{\alpha-1}$ and hence fg is in D_α. Thus, $D_\alpha \subset M(D_\alpha)$ for $\alpha > \frac{1}{2}$ and the proof is complete.

Although the result (d) is known we included its proof since it used the same ideas as the other parts. We also remark that α-Carleson measures played no role for $\alpha > \frac{1}{2}$ since in this case they are trivially characterized as finite measures.

With regard to the norm of the multiplication operators in the above theorem it is not difficult to determine that the norm in part (a) is comparable to the smallest possible constant in the definition of α-Carleson measures. The same thing holds for the other ranges of α and β provided we add $|f(0)|$ in case (b) and $\|f\|_\infty$ in case (c).

The preceding theorem shows that a characterization of α-Carleson measures is needed. We start with Carleson's well known characterization for H^2 or in our case D_0. Let I be a subarc on the unit circle and define

$$S(I) = \{z \colon z/|z| \in I,\ 1 - |I| \leq |z| < 1\}$$

where $|I|$ denotes the normalized arc length of I. A positive measure μ on $|z| < 1$ is a 0-Carleson measure if and only if $\mu(S(I)) = O(|I|)$.

We remark that Theorem 1.1 applied in the case $\alpha = \beta = 0$ yields

$$\iint_{S(I)} |f'|^2(1 - |z|)\, dx\, dy = O(|I|)$$

whenever f is in H^∞. This fact was first proved by Fefferman [7] and in fact the analytic functions in BMO are characterized by the above relation.

THEOREM 1.2. *A positive Borel measure μ on $|z| < 1$ is an α-Carleson measure for $\alpha \leq 0$ if and only if $\mu(S(I)) = O(|I|^{1-2\alpha})$.*

Proof. Let $\alpha < 0$, the case $\alpha = 0$ was just discussed. Let m_α be the measure $(1 - |z|)^{-1-2\alpha}\, dx\, dy$ on $|z| < 1$ and observe that $m_\alpha(S(I))$ is comparable to $|I|^{1-2\alpha}$. For $u \in L^1(dm_\alpha)$ put

$$M_\alpha[u](z) = \sup_I M_\alpha(S(I))^{-1} \int_{S(I)} |u|\, dm_\alpha$$

where the supremum is taken over all arcs $I \supset I_z$. Here I_z is the arc centered at $z/|z|$ and $|I_z| = 1 - |z|$.

Suppose μ is a measure satisfying the hypothesis then there is a constant c such that $\mu(S(I)) \leq cM_\alpha(S(I))$. Now it follows that the weak type (1, 1) condition $\mu\{M_\alpha[u] > s\} \leq cs^{-1} \int |u|\, dm_\alpha$ holds for some constant c. See Duren's book [6, Chapter 9.5] for details of a similar argument. Since M_α is obviously a sublinear operator of type (∞, ∞), we use the Marcinkiewicz interpolation theorem [26, Chapter XII] to conclude that M_α is a bounded sublinear operator mapping $L^2(m_\alpha)$ into $L^2(\mu)$.

Now the novelty of this proof is that M_α is not a maximal function as is usually the case. However, when applied to analytic or harmonic functions u it follows that $|u| \leq cM_\alpha[u]$ for some constant c. Assuming this fact we see that if $g \in D_\alpha$ then

$$\begin{aligned}
\int |g|^2\, d\mu &\leq c \int |M_\alpha[g]|^2\, d\mu \\
&\leq c \int |g|^2\, dm_\alpha \\
&\leq c \iint |g|^2 (1 - |z|)^{-1-2\alpha}\, dx\, dy \\
&= c\|g\|_\alpha^2.
\end{aligned}$$

Here the constant c changes with each inequality but does not depend on the function g. Thus, μ is an α-Carleson measure.

To show that M_α has the desired maximal property we use the mean value property for harmonic functions. Fix $|w| < 1$ and let D_w denote the disc centered at w with radius $\frac{1}{2}(1 - |w|)$. Let I be the subarc centered at $w/|w|$ and such that $|I| = 2(1 - |w|)$ or 1 whichever is smaller. If u is harmonic then

$$\begin{aligned}
|u(w)| &\leq c(1 - |w|)^{-2} \iint_{D_w} |u|\, dx\, dy \\
&\leq c(1 - |w|)^{2\alpha-1} \iint_{D_w} |u|(1 - |z|)^{-1-2\alpha}\, dx\, dy \\
&\leq cm_\alpha(S(I))^{-1} \iint_{S(I)} |u|\, dm_\alpha \\
&\leq cM_\alpha[u](w).
\end{aligned}$$

In order to prove the converse we use the notation

$$(\beta)_n = \beta(\beta + 1) \cdots (\beta + n - 1)$$

and note that for $\beta > 0$, $(\beta)_n/n!$ is comparable to $n^{\beta-1}$, see [12, Chapter 5.4]. For $|w| < 1$, let $g_w(z) = [(1 - \bar{w}z)^{1-2\alpha}]^{-1}$. By the Binomial Theorem $g_w(z) = \sum_{n=0}^{\infty} (1 - 2\alpha)_n (n!)^{-1} (\bar{w}z)^n$ and hence

$$\begin{aligned}
\|g_w\|_\alpha^2 &= \sum_{n=0}^{\infty} (1 + n^2)^\alpha \left| \frac{(1 - 2\alpha)_n}{n!} w^{-n} \right|^2 \\
&\le c \sum_{n=0}^{\infty} (1 + n^2)^{-\alpha} |w|^{2n} \\
&\le c \sum_{n=0}^{\infty} \frac{(1 - 2\alpha)_n}{n!} |w|^{2n} \\
&= c(1 - |w|^2)^{2\alpha - 1}.
\end{aligned}$$

If μ is an α-Carleson measure for $\alpha < 0$, in fact $\alpha < \frac{1}{2}$, then it follows that

$$\int |1 - \bar{w}z|^{-2(1-2\alpha)} \, d\mu(z) \le c(1 - |w|^2)^{2\alpha - 1}.$$

If I is a subarc we pick w centered in $S(I)$ and observe that $1 - |w|^2$ and $|1 - \bar{w}z|$ for z in $S(I)$ are both comparable to $|I|$. The above inequality then becomes $\mu(S(I)) = O(|I|^{1-2\alpha})$. The proof is complete.

We remark that the particular case $\alpha = -\frac{1}{2}$ of Theorem 1.2 had been previously attained by other methods, see [11].

We also observe that the necessity portion of the proof is valid for $\alpha < \frac{1}{2}$. For $\alpha = \frac{1}{2}$ a slight modification is required.

COROLLARY 1.3. *If μ is an α-Carleson measure for $\alpha \le \frac{1}{2}$ then*

$$\mu(S(I)) = O \begin{cases} |I|^{1-\alpha} & \text{if } \alpha < \frac{1}{2} \\ \left(\log \dfrac{2}{|I|}\right)^{-1} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

Proof. The case $\alpha = \frac{1}{2}$ is proved in a similar manner using the functions

$$\log (1 - \bar{w}z)^{-1} = \sum_{n=1}^{\infty} n^{-1} (\bar{w}z)^n.$$

COROLLARY 1.4. *An analytic function f in the unit disc belongs to $M(D_\alpha, D_\beta)$ for $\beta, \alpha < 0$ if and only if $f(z) = O(1 - r)^{\beta - \alpha}$. Moreover, for $\alpha < \beta < 0$ the zero function is the only multiplier.*

Proof. This result was mentioned earlier, however, we demonstrate the use of Theorems 1.1 and 1.2. If f is analytic then the mean value property yields

$$|f(w)|^2 \le c(1 - |w|)^{2\beta-1} \iint\limits_{S(2I_w)} |f|^2(1 - |z|)^{-1-2\beta}\, dx\, dy.$$

The result follows easily from this inequality and the two theorems. If $\alpha < \beta$ then f is tend to zero as $|z|$ tends to 1 and hence f is identically zero by the maximum principle.

COROLLARY 1.5. *If $\beta < 0$ then f is in $M(D_0, D_\beta)$ if and only if f is analytic and $\iint_{S(I)} |f|^2(1 - |z|)^{-1-2\beta}\, dx\, dy$ is $O(|I|)$.*

COROLLARY 1.6. *If $|f'|^2(1 - |z|)^{1-2\beta}\, dx\, dy$ is an α-Carleson measure for $0 \le \beta < \alpha \le \frac{1}{2}$ then $|f(z)| \le |f(0)| + c(1 - |z|)^{\beta-\alpha}$.*

Proof. By Corollary 1.3 and the mean value property we obtain

$$|f'(w)| = O(1 - |w|)^{\beta-\alpha-1}.$$

Integrating f' along $[0, w]$ gives the desired result.

We have previously observed that a function in $M(D_\alpha)$ must be a bounded member of D_α. For $\alpha \le 0$ or $\alpha > \frac{1}{2}$ these necessary conditions are also sufficient. In [25] an example is given of a nonmultiplying bounded function in $D_{1/2}$ by cleverly prescribing its power series coefficients. A somewhat easier example can be obtained by using Corollary 1.3.

Consider the functions

$$f(z) = \frac{\sin(\log \delta(1 - z)^{-1})}{(\log \delta(1 - z)^{-1})^{1/2+\varepsilon}}, \qquad g(z) = \frac{\cos(\log \delta(1 - z)^{-1})}{(\log \delta(1 - z)^{-1})^{1/2+\varepsilon}}$$

where $0 < \varepsilon < \frac{1}{2}$ and $0 < \delta < \frac{1}{2}$. Both functions are bounded, in fact continuous on $|z| \le 1$, and members of $D_{1/2}$. This last fact can be seen by noticing that

$$|f'(z)|^2 + |g'(z)|^2 = O(|1 - z|^{-2}(\log |1 - z|^{-1})^{-1-2\varepsilon})$$

and using the integral norm condition for $D_{1/2}$. Using the equation $\sin^2 z + \cos^2 z = 1$ and some manipulation we get

$$|1 - z|^{-2}(\log |1 - z|^{-1})^{-1-2\varepsilon} \le c[|f'(z)|^2 + |g'(z)|^2 + |1 - z|^{-2}(\log |1 - z|^{-1})^{-3-2\varepsilon}]$$

for some constant c and $|1 - z|$ small. Integrating this relation over $S(I)$ where I is a small arc centered at $z = 1$ we get

$$(\log |I|^{-1})^{-2\varepsilon} \le c\left[\iint\limits_{S(I)} |f'|^2 + |g'|^2\, dx\, dy\right].$$

Since $2\varepsilon < 1$, Corollary 1.3 shows that f and g can not both multiply $D_{1/2}$. Actually, neither function multiplies.

For $0 < \alpha < \frac{1}{2}$, analysis similar to the above shows that

$$f(z) = \exp\,(A_\beta(1-z)^{-2\beta}),$$

where $A_\beta = \exp\,(i(1+2\beta)\pi/2)$, is a bounded function in D_α but not a multiplier whenever $\alpha(2\beta+1) < \frac{1}{2}$ and $\beta > 0$. In fact, $f \notin M(D_t, D_t)$ for all $t > 0$.

The following theorem shows that bounded functions in $D_{1/2}$ are in some sense close to multiplying $D_{1/2}$.

THEOREM 1.7. *The following relations hold:*

(a) $D_{1/2} \cap H^\infty \subset M(D_\beta)$ *whenever* $\beta < \frac{1}{2}$,
(b) $D_{1/2} \subset M(D_\beta, D_\gamma)$ *whenever* $\gamma < \beta \leq \frac{1}{2}$,
(c) $D_\alpha \subset M(D_\beta, D_{\alpha+\beta-1/2})$ *whenever* $\alpha, \beta < \frac{1}{2}$.

Proof. Let $g(z) = \sum a_n z^n$ then for $\alpha < \frac{1}{2}$ Schwarz's inequality yields

$$|g(z)| \leq \|g\|_\alpha \{\textstyle\sum (n^2+1)^{-\alpha} |z|^{2n}\}^{1/2} \leq c\|g\|_\alpha (1-|z|)^{\alpha-1/2}.$$

Similarly, for $\alpha = \frac{1}{2}$ we obtain $|g(z)| \leq c\|g\|_{1/2}\{\log\,(1-|z|)^{-1}\}^{1/2}$. The proof of each part is similar and we prove (c) to demonstrate the method. Let f be in D_α and g be in D_β then the growth condition on f, g imply that

$$\begin{aligned}\|(fg)'\|_{\alpha+\beta-3/2} &\leq \|fg'\|_{\alpha+\beta-3/2} + \|f'g\|_{\alpha+\beta-3/2}\\ &\leq c[\|g'\|_{\beta-1} + \|f'\|_{\alpha-1}]\\ &\leq c[\|g\|_\beta + \|f\|_\alpha].\end{aligned}$$

Thus, $fg \in D_{\alpha+\beta-1/2}$ and the proof is complete.

We remark that part (a) is the analog of Theorem 3.2 [24] which concerns multipliers of fractional Sobolev spaces.

If we take $\alpha = \beta = 0$ in Theorem 1.7 we get the curious fact that $D_0 \subset M(D_0, D_{-1/2})$, i.e., H^2 is contained in the Bergman space B^4. More generally, Schwarz's inequality shows that $B^4 \subset M(H^2, B^2)$. Both of these facts can easily be deduced from Corollary 1.5. In fact using Corollary 1.5 we see that the function $f(z) = (1-z)^{-1/2}$ is in $M(H^2, B^2)$ but not B^4.

2. Characterization of α-Carleson measures for $0 < \alpha \leq \frac{1}{2}$

Theorem 1.2 and Corollary 1.3 suggest that an α-Carleson measure for $0 < \alpha \leq \frac{1}{2}$ is determined by the condition $\mu(S(I)) = O(|I|^{1-2\alpha})$ or $\mu(S(I)) = O(\log\,|I|^{-1})^{-1}$ for $\alpha = \frac{1}{2}$. However, for $\alpha > 0$ a change occurs. Specifically, suppose an arc I is subdivided into n equal subarcs $\{I_j\}$. Estimating $\mu(\bigcup_j S(I_j))$ in the obvious way we get $n^{2\alpha}|I|^{1-2\alpha}$. But $S(I)$ is a larger set, yet $|I|^{1-2\alpha}$ is the estimate for $\mu(S(I))$. This difficulty is the crux of the problem.

Let $k_\alpha(\theta) = |\theta|^{\alpha-1}$ for $|\theta| \le \pi, 0 < \alpha < 1$ and extend k_α periodically for all θ. It can be shown that the Fourier coefficients $\hat{k}_\alpha(n)$ are of the form $a_n(1+n^2)^{-\alpha/2}$ where $0 < \delta \le a_n \le \delta^{-1}$ for all n. The space H^2 is determined by its Taylor coefficients or as the space of harmonic extensions of function in $L^2(T)$ (T denotes the unit circle) for which $\hat{f}(n) = 0$ for $n = -1, -2 \ldots$. See [10, Chapter 3] for details.

The Fourier coefficients of the convolution of two periodic integrable functions

$$f * g(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t) g(t)\, dt$$

satisfies the relation $(f * g)^\wedge(n) = \hat{f}(n)\hat{g}(n)$. Using this we see a natural isomorphism between H^2 and D_α; namely, the linear operator taking the boundary function of f in H^2 into the harmonic extension of $f * k_\alpha$. The harmonic extension of a function f in $L^2(T)$ is given by the Poisson integral of f, denoted by $P[f](z)$, where

$$\begin{aligned} P[f](z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta - t)(1 - r^2)}{|1 - ze^{-it}|^2}\, dt \qquad (z = re^{i\theta}) \\ &= \sum_{-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}. \end{aligned}$$

LEMMA 2.1. *A function g in D_α for $0 < \alpha < 1$ if and only if $g = P[f * k_\alpha]$ where f is the boundary function of a function in H^2. Moreover, $\|g\|_\alpha$ and $\|f\|_2$ are comparable.*

Proof. *If $f(z) = \sum_{n=0}^{\infty} b_n z^n$ then*

$$g(z) = P[f * k_\alpha](z) = \sum_{n=0}^{\infty} a_n b_n (1 + n^2)^{-\alpha/2 z^n}$$

and

$$\delta^2 \|f\|_2^2 \le \sum_{n=0}^{\infty} (1 + n^2)^\alpha |a_n b_n (1 + n^2)^{-\alpha/2}|^2 \le \delta^{-2} \|f\|_2^2.$$

The middle term is $\|g\|_\alpha^2$ and the result follows.

LEMMA 2.2. *A positive measure μ is an α-Carleson measure for $0 < \alpha < 1$ if and only if there is a constant c such that*

$$\text{(1)} \qquad \int_{|z|<1} |P[f * k_\alpha]|^2\, d\mu \le c \|f\|_2^2$$

for all $f \ge 0$ in $L^2(T)$.

Proof. Assume that (1) holds. Since the Poisson kernel and k_α are both positive it follows that (1) holds for all f in $L^2(T)$. Applying (1) for f in H^2 and using Lemma 2.1 shows that μ is an α-Carleson measure.

Conversely, if μ is an α-Carleson measure then (1) holds for all f in H^2. But the general function f in $L^2(T)$ can be written $f = f_1 + \bar{f}_2$ where $\{f_i\}$ are in H^2 and $\|f\|_2^2 = \|f_1\|_2^2 + \|f_2\|_2^2$. Thus, (1) holds for all f in $L^2(T)$.

We now turn to the capacitary notions we will need. There are various classical capacities which can be employed in this problem, however, the natural candidates are the Bessel capacities introduced in [2]. We will be using the notation and definitions in [18]. The Bessel kernel of order $\alpha > 0$ in $\mathbf{R}^n$ is denoted by g_α. We will not explicitly indicate the dimension. Its definition and properties can be found in [2, p. 416]. Briefly, the kernel g_α is positive, in $L^1(\mathbf{R}^n)$, decays exponentially at infinity, and for x near the origin behaves asymptotically as

$$g(x) \sim \begin{cases} c_\alpha |x|^{\alpha-n} & \text{if } 0 < \alpha < n, \\ c_n \log |x|^{-1} & \text{if } \alpha = n, \\ c_\alpha & \text{if } \alpha > n. \end{cases}$$

For $1 < p < \infty$ the Bessel capacity $B_{\alpha,p}$ is defined by

$$B_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \geq 0,\ g_\alpha * f \geq 1 \text{ on } E\}$$

where the infimum over the empty set is taken to be infinity. For $n = 1$, the capacity $B_{\alpha,2}$ is closely related to the D_α space. In particular, for intervals I in $\mathbf{R}^1$ which are sufficiently small Lemmas 7, 8 [18] show that

$$B_{\alpha,2}(I) \sim \begin{cases} |I|^{1-2\alpha} & \text{if } 0 < \alpha < \frac{1}{2}, \\ (\log |I|^{-1})^{-1} & \text{if } \alpha = \frac{1}{2}, \end{cases}$$

where $\sim$ simply indicates comparability.

For subsets E of the unit circle, let $\tau(E)$ denote the corresponding subset in the interval $[-\pi, \pi]$ obtained by the natural identification of this interval with the line.

THEOREM 2.3. *Let $0 < \alpha \leq \frac{1}{2}$. A measure μ is an α-Carleson measure if and only if there is a constant c such that*

$$\mu\left(\bigcup_{j=1}^{n} S(I_j)\right) \leq c B_{\alpha,2}\left(\tau\left(\bigcup_{j=1}^{n} I_j\right)\right) \tag{2}$$

whenever $I_1, \ldots, I_n$ are disjoint arcs on T.

We remark that for $\alpha = \frac{1}{2}$ the capacity $B_{1/2,2}$ in the theorem can be replaced by the classical logarithmic capacity in $\mathbf{R}^2$. This follows from consideration given in [2, p. 426–7].

To prove Theorem 2.3 we introduce a capacity on the circle. For $E \subset T$, for $0 < \alpha \le \frac{1}{2}$ let

$$\gamma_\alpha(E) = \inf\{\|f\|_2^2 : f \ge 0, f \in L^2(T), k_\alpha * f \ge 1 \text{ on } \tau(E)\}.$$

The capacity γ_α is monotone and subadditive, see Theorem 1 [18]. We also claim that $\gamma_\alpha(E)$ is comparable to $B_{\alpha,2}(E)$ and that there is a constant c such that the strong type capacitary inequality

$$(3) \qquad \int_0^\infty \gamma_\alpha(\{e^{i\theta} \in T : k_\alpha * f > t\})\, dt^2 \le c\|f\|_\alpha^2$$

holds for all nonnegative functions f in $L^2(T)$. Assuming these facts for the moment, we prove the theorem.

Proof of Theorem 2.3. Assume that μ is an α-Carleson measure. Let $E = \bigcup_{j=1}^n I_j$ and f be a test function for E, i.e., $f \ge 0$ and in $L^2(T)$ with $k_\alpha * f \ge 1$ on $\tau(E)$. Thus, $k_\alpha * f \ge \chi_{I_j}$ for each j. It follows that $P[\chi_{I_j}](z) \ge \frac{1}{4}$ for z in $S(I_j)$. Thus, $P[k_\alpha * f] \ge \frac{1}{4}$ on $\bigcup_{j=1}^n S(I_j)$ and since μ is an α-Carleson measure, Lemma 2.2 yields

$$\mu(\cup S(I_j)) \le 16 \int |P[k_\alpha * f]|^2\, d\mu \le 16c\|f\|_2^2.$$

Since f is arbitrary, $\mu(\cup S(I_j))$ is dominated by $\gamma_\alpha(\cup I_j)$ or $B_{\alpha,2}(\tau(\cup I_j))$.

Now assume that μ satisfies (2) for all finite disjoint collection of arcs. Let $f \ge 0$ be a function in $L^2(T)$ and set $u = P[k_\alpha * f]$. The nontangential maximal function u^* is a function on the circle whose value at $e^{i\theta}$ is obtained by taking the supremum of $|u(z)|$ over a cone centered at $e^{i\theta}$. Precisely, let $\Gamma(e^{i\theta})$ be the convex hull formed by the circle of radius r $(0 < r < 1)$ centered at the origin and the point $e^{i\theta}$. Then $u^*(e^{i\theta}) = \sup_{z \in \Gamma(e^{i\theta})} |u(z)|$. We will assume that r is large enough so that z is in $\Gamma(e^{i\theta})$ for all $e^{i\theta} \in 2I_z$. This is clearly possible.

Next let $M[g]$ denote the Hardy–Littlewood maximal function of g in $L^2(T)$, i.e., $M[g](e^{i\theta}) = \sup |I|^{-1} \int_I |g|$ where I is a subarc centered at $e^{i\theta}$. The basic facts concerning these maximal functions are that $M[g]$ is bounded on $L^2(T)$, in fact L^p for $1 < p < \infty$, and $P[g]^* \le cM[g]$ for some constant c, see [23, Chapter 7].

Let $t \ge 0$ and K be a compact subset of $A_t = \{z : |u(z)| > t\}$. By compactness there are finitely many points $z_1, \ldots, z_n$ in K such that $\{S(2I_{z_j})\}$ covers K. The union of the closed arcs $\{2I_{z_j}\}$ can be expressed as the disjoint union of arcs $\{J_i\}$. Clearly, each arc $2I_{z_j}$ is contained in one of the arcs J_i and hence $K \subset \bigcup_j S(J_j)$. In addition, our assumption on the nontangential maximal function force $2I_{z_j}$ to be contained in $\{u^* > t\}$ and hence the same thing is true for each J_i. It now follows that $\mu(K) \le c\gamma_\alpha\{u^* > t\}$ and by the regularity of μ we have $\mu(A_t) \le c\gamma_\alpha\{u^* > t\}$.

Observe that $M[k_\alpha * f] \leq k_\alpha * M[f]$. By the above consideration, (3), and the L^2-boundedness of the Hardy-Littlewood maximal function we have

$$\begin{aligned}\int |P[k_\alpha * f]|^2\, d\mu &= \int_0^\infty \mu(A_t)\, dt^2 \\ &\leq c \int_0^\infty \gamma_\alpha\{u^* > t\}\, dt^2 \\ &\leq c \int_0^\infty \gamma_\alpha\{M[k_\alpha * f] > t\}\, dt^2 \\ &\leq c \int_0^\infty \gamma_\alpha\{k_\alpha * M[f] > t\}\, dt^2 \\ &\leq c\|M[f]\|_2^2 \\ &\leq c\|f\|_2^2.\end{aligned}$$

Thus, μ is an α-Carleson measure. This completes the proof modulo our preliminary assumptions.

We remark that the basic approach used in the above proof is suggested by Stein's proof of the original Carleson measure problem [23, p. 236].

LEMMA 2.4. *There is a constant c such that for each nonnegative function f in $L^2(T)$ there is a nonnegative function h in $L^2(-\infty, \infty)$ satisfying $\|h\|_2 \leq c\|f\|_2$ and $k_2 * f \leq g_\alpha * h$ on $[-\pi, \pi]$.*

Proof. Let $f \geq 0$ be in $L^2(T)$. Put $h_0 = f$ on $[-2\pi, 2\pi]$ and zero elsewhere. Since k_α and g_α are comparable on $[-\pi, \pi]$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)k_\alpha(y)\, dy \leq c\int_{-\infty}^{\infty} h_0(x-y)g_\alpha(y)\, dy.$$

Taking $h = ch_0$ the result follows.

Note that we are using the same norm notation for functions on the lines and periodic function on the circle. This should not cause confusion.

LEMMA 2.5. *For subsets $E \subset T$, $\gamma_\alpha(E)$ is comparable to $B_{\alpha,2}(\tau(E))$.*

Proof. Lemma 2.4 proves that $B_{\alpha,2}(\tau(E)) \leq c\gamma_\alpha(E)$. In order to prove a lower bound it suffices to consider sets for which $B_{\alpha,2}(E)$ is small. Accordingly, let $\varepsilon > 0$ and assume that $B_{\alpha,2}(E) < \varepsilon$. Then there is a test function h with $\|h\|_2 < \varepsilon$. Since g_α decays exponentially at infinity Schwarz's inequality guarantees that $\int_{|y| \geq \pi} h(x-y)g_\alpha(y)\, dy < \frac{1}{2}$ provided ε is sufficiently small. But h is a test function and hence we must have $\int_{|y| < \pi} h(x-y)g_\alpha(y)\, dy \geq \frac{1}{2}$ whenever x is in $\tau(E)$. Now g_α can be replaced by k_α to yield $\int_{|y| < \pi} h(x-y)k_\alpha(y)\, dy \geq c^{-1} > 0$ whenever x is in $\tau(E)$.

Finally, if we take f_1 to be the periodic extension of h on $[-2\pi, 0]$ and f_2 to be the periodic extension of h on $[0, 2\pi]$ then $f = c(f_1 + f_2)$ will be a test function for $\gamma_\alpha(E)$. Thus, $\gamma_\alpha(E) \leq \|f\|_2^2 \leq c^2\|h\|_2^2$ and hence $\gamma_\alpha(E) \leq c^2 B_{\alpha,2}(\tau(E))$. We now have $\gamma_\alpha(E)$ and $B_{\alpha,2}(\tau(E))$ comparable and the proof is complete.

By the preceding lemmas we see that the capacitary inequality (3) is a consequence of the inequality

$$\int_0^\infty B_{\alpha,p}\{g_\alpha * h > t\}\, dt^p \leq c\|h\|_p^p \tag{4}$$

for $0 < \alpha \leq \frac{1}{2}$, $p = 2$, and h an arbitrary nonnegative function in $L^p(\mathbf{R}^n)$. D. R. Adams has proved (4) with $0 < \alpha < 1$, $p \geq 2$, and $\alpha p < n$, see [1, p. 139]. In addition, for $0 < \alpha < 1$, $p = 2$, and $\alpha = n/2$ inequality (4) appears in the same place in an equivalent form using Besov spaces. Adams' result is a generalization of the strong capacitary inequality of V. A. Maz'ya [27]. Thus, the proof of Theorem 2.2 is complete.

Up to this point we have been viewing the boundary function of elements in D_α, $0 < \alpha < 1$, as convolutions. However, using Parseval's relation we may characterize these functions directly. Let f be in $L^2(T)$ and define

$$\mathscr{D}_\alpha f(\theta) = \left\{\int_{-\pi}^{\pi} \frac{|f(\theta - t) - f(\theta)|^2}{|e^{it} - 1|^{1+2\alpha}}\, dt\right\}^{1/2}.$$

LEMMA 2.6. *A function f in H^2 is in D_α for $0 < \alpha < 1$ if and only if $\mathscr{D}_\alpha f \in L^2(T)$. Moreover, $\|f\|_\alpha$ is comparable to $\|f\|_\alpha + \|\mathscr{D}_\alpha f\|_2$.*

The proof is simple and is essentially in [2, p. 402]. See also Theorem 3c [5].

By combining Theorems 1.1, 1.2 and 2.3 we obtain a complete characterization of the spaces $M(D_\alpha, D_\beta)$. We explicitly record the most interesting case along with a boundary characterization.

THEOREM 2.7. *The following are equivalent for $0 < \alpha \leq \frac{1}{2}$:*

(a) $f \in M(D_\alpha)$.

(b) $f \in H^\infty$ *and* $\iint_{\cup S(I_j)} |f'|^2(1 - |z|)^{1-2\alpha}\, dx\, dy = O(B_{\alpha,2}(\tau(\cup I_j)))$ *for all finite disjoint collections of subarcs* $\{I_j\}_{j=1}^n$.

(c) $f \in H^\infty$ *and the boundary function satisfies* $\int_E (\mathscr{D}_\alpha f)^2\, dx = O(B_{\alpha,2}(E))$ *for all compact subsets* $E \subset T$.

The equivalence of (a) and (b) is clear. We postpone the equivalence of (a) and (c) since a similar result will hold in a more general setting.

Let $L_\alpha^p(\mathbf{R}^n)$ be the space of functions of the form $g_\alpha * f$ where f is in $L^p(\mathbf{R}^n)$ and take $\|g_\alpha * f\|_{\alpha,p}$ to be $\|f\|_p$. We then have the well known space of Bessel potentials introduced in [2]. Since $\hat{g}_\alpha(x) = (1 + |x|^2)^{-\alpha/2}$, see [2, p. 417], a calculation with Plancheral's theorem shows that f is in L_α^2 if and only if

$\int |\hat{f}(x)|^2(1+|x|^2)^\alpha\,dx$ is finite. Thus, $L^2_\alpha(\mathbf{R}^1)$ is the real variable analog of D_α. In fact we can define $L^p_\alpha(T)$ to be the space of functions of the form $f * k_\alpha$ where f is in $L^p(T)$ and use the similar norm. Then $L^2_\alpha(T)$ for $0<\alpha<1$ will consist of all f in $L^2(T)$ with $\sum_{-\infty}^{\infty}(1+n^2)^\alpha|\hat{f}(n)|^2$ finite. The exceptional sets (in the sense of [2]) for L^p_α potentials are the sets of $B_{\alpha,p}$ capacity zero. This connection best explains the appearance of Bessel capacities in the D_α multiplier problem.

The multiplier problem for $L^p_\alpha(\mathbf{R}^n)$ was studied in [24] and for $L^2_\alpha(T)$ in [5], [14], and [16]. In particular, functions in $M(L^p_\alpha)$ are bounded, the inclusion relation $M(L^p_\alpha)\subset M(L^p_\beta)$ holds for $0<\beta\le\alpha$ and $L^p_\alpha\subset M(L^p_\alpha)$ for $\alpha p>n$. Here $\alpha=n/p$ is the critical index just as $\alpha=\frac{1}{2}$ was the case in the D_α setting. However, in the Euclidean space setting the constant functions are not in $L^p(\mathbf{R}^n)$ and hence we do not have the inclusion $M(L^p_\alpha)\subset L^p_\alpha$. But multipliers are locally in L^p_α, see [24, Chapter 3]. We also note that the analog of Lemma 2.6 holds for $0<\alpha<1$ and $p\ge 2$ with

$$\mathscr{D}_\alpha f(y)=\left\{\int \frac{|f(x-y)-f(\alpha)|^2}{|y|^{n+2\alpha}}\,dy\right\}^{1/2}$$

(see [22]).

Lemma 2.8. *Let μ be a positive Borel measure on $\mathbf{R}^n$. A necessary and sufficient condition that*

$$\int |f|^p\,d\mu\le c\|f\|^p_{\alpha,p}\quad (f\in L^p_\alpha) \tag{5}$$

where $0<\alpha<1$, $p\ge 2$, and $\alpha p\le n$ is that $\mu(E)=O(B_{\alpha,p}(E))$ hold for all compact subsets $E\subset\mathbf{R}^n$.

Proof. The necessity proof is trivial. The sufficiency proof, at least for $\alpha p<n$, is an immediate consequence of (4). We indicate shortly a method of extending (4) to the critical index $\alpha=n/p$.

We remark that this lemma is the analog for Bessel capacities of a special case of Theorem 4 [1].

Theorem 2.9. *Assume that $p\ge 2$, $0<\alpha<1$, and $\alpha p\le n$. A function f is in $M(L^p_\alpha)$ if and only if f is in $L^\infty(\mathbf{R}^n)$ and $\int_E|\mathscr{D}_\alpha f|^p\,dx=O(B_{\alpha,p}(E))$ for all compact subsets E in $\mathbf{R}^n$.*

Proof. By the analog of Lemma 3.1 [24] for $\mathscr{D}_\alpha$ we see that f is in $M(L^p_\alpha)$ if and only if f is in L^∞ and $\|g\mathscr{D}_\alpha f\|_p\le c\|g\|_{\alpha,p}$ for all g in L^p_α. The proof is completed by applying Lemma 2.8.

We remark that the analog of Theorem 2.9 with essentially the same proof holds for D_α. Thus, Theorem 2.7 is also proved.

We now sketch the method for extending the capacitary inequality.

Claim. Inequality (4) holds for all $p \geq 2$ and $0 < \alpha < 1$.

Proof. The proof uses the same techniques in [1], however, by using the $\mathcal{D}_\alpha$ characterization for Bessel potentials we can proceed directly.

Let H be a smooth nondecreasing function on $(-\infty, \infty)$ which is zero for $x < \frac{1}{2}$ and one for $x > 1$. Put $H_j(x) = 2^j H(2^{-j}x)$ for integers j. Observe that $2^{-j}H_j \circ f \geq 1$ on the set $\{f > 2^j\}$ and hence $B_{\alpha,p}\{f > 2^j\} \leq 2^{-jp}\|H_j \circ f\|_{\alpha,p}^p$. Then,

$$\begin{aligned}\int_0^\infty B_{\alpha,p}(f > t)\, dt^p &\leq c \sum_{j=-\infty}^{\infty} 2^{jp} B_{\alpha,p}(f > 2^j) \\ &\leq c \sum_{-\infty}^{\infty} \|H_j, f\|_{\alpha,p}^p \\ &\leq c \sum_{-\infty}^{\infty} \{\|H_j \circ f\|_p^p + \|\mathcal{D}_\alpha(H_j \circ f)\|_p^p\}.\end{aligned}$$

Now an easy calculation shows that $\sum_{-\infty}^{\infty} \|H_j \circ f\|_p^p \leq c\|f\|_p^p$. Since $p \geq 2$,

$$\begin{aligned}\sum_{-\infty}^{\infty} \|\mathcal{D}_\alpha(H_j \circ f)\|_p^p &\leq \int \left\{\int \frac{|\sum (H_j \circ f(x+y) - H_j \circ f(x)|^2}{|y|^{n+2\alpha}}\, dy\right\}^{p/2} dx \\ &\leq c\|\mathcal{D}_\alpha f\|_p^p.\end{aligned}$$

This last inequality follows from the fact that

$$\left|\sum_{-\infty}^{\infty} H_j \circ f(x) - \sum_{-\infty}^{\infty} H_j \circ f(y)\right| \leq c|f(x) - f(y)|$$

holds for all x, y. The hardest case is $2^j \leq f(y) < 2^{j+1} \leq f(x) < 2^{j+2}$ in which case

$$\begin{aligned}\sum_{-\infty}^{\infty} H_j \circ f(x) - \sum_{-\infty}^{\infty} H_j \circ f(y) &= 2^{j+2}H(2^{-j-2}f(x)) \\ &\quad + 2^{j+1} - 2^{j+1}H(2^{-j-1}f(y)) \\ &= 2^{j+2}[H(2^{-j-2}f(x)) - H(2^{-1})] \\ &\quad + 2^{j+1}[H(1) - H(2^{-j-1}f(y))] \\ &\leq \sup H'\{[f(x) - 2^{j+1}] + [2^{j+1} - f(y)]\} \\ &\leq \sup H'(f(x) - f(y))\end{aligned}$$

by the Mean Value Theorem.

We remark that a result similar in nature to Theorem 2.7 for the multipliers on the space L_1^2 is in [19]. The space L_1^2 is the classical Sobolev space of functions in L^2 whose weak derivative of order one are also in L^2. The author is indebted to A. Torchinsky for bringing this reference to our attention.

A nice application of these results is obtained by using a lower bound for the capacity of a set. In our case, we have that the capacity of a Borel set is no smaller than a multiple of the capacity of the ball of identical volume. For Borel sets of small diameter this fact is established in [18, Theorem 20, Lemmas 7, 8]. For sets with large diameters we use a different technique.

LEMMA 2.10. *There is a constant c such that, for all Borel sets E,*

(a) $B_{\alpha,p}(E) \geq c|E|^{(n-\alpha p)n^{-1}}$ *for* $\alpha p < n$ *and* $|E| \leq 1$,
(b) $B_{\alpha,p}(E) \geq c(\log|E|^{-1})^{1-p}$ *for* $\alpha p = n$ *and* $|E| \leq \frac{1}{2}$,
(c) $B_{\alpha,p}(E) \geq c|E|$

Proof. Let m_E be Lebesque measure restricted to E. Denote by B the ball centered at the origin for which $|B| = |E|$. If f is a measurable function, denote by $f^{\#}$ the equimeasurable symmetric decreasing rearrangement of f. In other words, f and $f^{\#}$ have the same distribution function: $|\{|f| > t\}| = |\{f^{\#} > t\}|$ for all $t > 0$. By a theorem of Riesz-Sobolev [21] we have

$$\begin{aligned}
\|g_\alpha * m_E\|_{p'} &= \sup_{\|f\|_p=1} \left|\iint f(x)\chi_E(y)g_\alpha(x-y)\,dy\,dx\right| \\
&\leq \sup_{\|f\|_p=1} \iint |f(x)|\chi_E(y)g_\alpha(x-y)\,dy\,dx \\
&\leq \sup_{\|f\|_p=1} \iint f^{\#}(x)\chi_E^{\#}(y)g_\alpha^{\#}(x-y)\,dy\,dx \\
&\leq \|g_\alpha * \chi_B\|_{p'}
\end{aligned}$$

Following the calculation in [18, p. 285] we obtain

$$\|g_\alpha * m_E\|_{p'} \leq c\begin{cases} |E|(\log|E|^{-1})^{1/p'} & \text{if } \alpha p = n, \\ |E|^{1-(n-\alpha p)/np} & \text{if } \alpha p < n. \end{cases}$$

Finally, the desired lower bounds are obtained by applying Theorem 14 [18]. Part (c) is elementary.

Let W^p denote the weak L^p space of functions for which the distribution function $\lambda(t) = |\{|f| > t\}|$ satisfies $\lambda(t) \leq ct^{-p}$. The multiplier problem involves determining when $\|gh\|_p \leq c\|g\|_{\alpha,p}$ holds for certain functions $h(=\mathcal{D}_\alpha f)$. For $1 < p < n/\alpha$, R. Strichartz has shown using interpolation methods that this relation holds whenever h is in $W^{n/\alpha}$, see Theorem 3.6 [24]. A similar result is in [15].

Using our results we obtain a direct proof of this result. More importantly, we obtain a generalization to the critical index $\alpha p = n$. An elementary calculation shows that f is W^p if and only if $f^{\#}(x) \leq c|x|^{-n/p}$.

THEOREM 2.11. *If $2 \leq p \leq n/\alpha$, $0 < \alpha < 1$, and*

$$f^{\#}(x) = O\begin{cases} |x|^{-\alpha} & \text{if } \alpha p < n, \\ |x|^{-\alpha}(\log |x|^{-1})^{-1} & \text{if } \alpha p = n \end{cases}$$

for small $|x|$ then there is a constant c such that $\|gf\|_p \leq c\|g\|_{\alpha,p}$.

Proof. By the Riesz-Sobolev theorem used earlier, $\int f(g * \phi) \leq \int f^{\#}(g^{\#} * \phi)$ holds for nonnegative f, g, ϕ where ϕ is symmetric decreasing. By a standard regularization argument we obtain $\int fg \leq \int f^{\#}g^{\#}$.

Assume $\alpha p = n$ and let E be a compact subset. Let B be the ball centered at the origin with $|B| = |E|$. If $|E| \leq \frac{1}{2}$ then by the above symmetrization result we have

$$\begin{aligned} \int_E |f|^p \, dx &\leq \int_B (f^{\#})^p \, dx \\ &\leq c \int_0^r (\log t^{-1})^{-p} \frac{dt}{t} \\ &\leq c(\log |E|^{-1})^{1-p} \\ &\leq cB_{\alpha,p}(E). \end{aligned}$$

The last step is Lemma 2.10. For $|E| > \frac{1}{2}$, we easily obtain

$$\int_E |f|^p \, dx \leq c|E| \leq cB_{\alpha,p}(E).$$

For $\alpha p < n$ the same argument works.

The growth of $f^{\#}$ near the origin is dependent on the decay of the distribution function λ_f. However, the decay in λ_f for $f^{\#}$ satisfying the growth condition $|x|^{-\alpha}(\log |x|^{-1})^{-1}$ is not obvious.

Let ϕ, Ψ be increasing functions on $[t_0, \infty)$ which tend to infinity with t. Assume that $\phi \circ \Psi(t)$ and $\Psi \circ \phi(t)$ are both comparable to t for large t, i.e.,

$$0 < a \leq \frac{\Psi \circ \phi(t)}{t}, \qquad \frac{\phi \circ \Psi(t)}{t} \leq a^{-1},$$

for some $a > 0$ and t large. In addition, assume that ϕ grows slowly enough so that $\phi(ct) = O(\phi(t))$ for each $c > 0$.

LEMMA 2.12. *Under the above conditions, $\lambda(t) = O(\phi(t)^{-n})$ for large t if and only if $f^{\#}(x) = O(\Psi(|x|^{-1}))$ for all small $|x|$.*

Proof. If f is measurable then there is a nonincreasing function $r(t)$ satisfying $\lambda(t) = |B_{r(t)}|$ where B_r is a ball of radius r. We then have $f^{\#}(x) = \sup \{t\colon |x| \leq r(t)\}$.

Assume $\lambda_f(t) = O(\phi(t)^{-n})$ for t large then $r_{cf}(t) \le \phi(t)^{-1}$ for t large and some constant c. Hence

$$\begin{aligned} cf^{\#}(x) = (cf)^{\#}(x) &\le \sup \{t\colon \phi(t) \le |x|^{-1}\} \\ &\le \sup \{t\colon at \le \Psi(|x|^{-1})\} \\ &\le a^{-1}\Psi(|x|^{-1}) \end{aligned}$$

holds for all sufficiently small $|x|$.

Conversely, if $f^{\#}(x) = O(\phi(|x|^{-1}))$ for $|x|$ small then

$$\begin{aligned} \lambda_f(t) &= |\{f^{\#} > t\}| \\ &\le |\{x\colon \Psi(|x|^{-1}) > ct\}| \\ &\le |\{x\colon a^{-1}|x|^{-1} > \phi(ct)\}| \\ &= O(\phi(t)^{-n}) \end{aligned}$$

for all large t.

Obviously, the above lemma can be applied to the function $\phi(t) = t^{\alpha}$ and $\Psi(t) = t^{1/\alpha}$. However, a more interesting example is supplied by $\phi(t) = (t \log t)^{1/\alpha}$ and $\Psi(t) = t^{\alpha}(\log t)^{-1}$. A straightforward calculation shows that ϕ, Ψ satisfy the hypothesis of Lemma 2.12 and hence we have proved the following theorem.

THEOREM 2.13. *Let $0 < \alpha < 1$, $p \ge 2$, and $\alpha p \le n$. If a function f satisfies*

$$\lambda(t) = O\begin{cases} t^{-n/\alpha} & \text{if } \alpha p < n, \\ (t \log t)^{-n/\alpha} & \text{if } \alpha p = n, \end{cases}$$

for all large t then $\|gf\|_p \le c\|g\|_{\alpha,p}$.

In order for $\|gf\|_p < \infty$ for all g in L^p_α either f must be bounded or its large values must be controlled in size or location. We remark that if f and all its rearrangements satisfy $\|gf\|_p \le c\|g\|_{\alpha,p}$ where $\alpha p < n$ then $\lambda_f(t) = O(t^{-n/\alpha})$ for large t. Using Lemma 7 [18] and Lemma 2.7 applied to $f^{\#}$ we obtain $\int_{|x| \le r} (f^{\#})^p \, dx = O(r^{n-\alpha p})$. Since $f^{\#}$ is radially decreasing we get

$$f^{\#}(x) = O(|x|^{-\alpha}) \quad \text{or} \quad \lambda_f(t) = O(t^{-n/\alpha}).$$

We now give a few applications to multipliers. A fairly direct calculation shows that a function f in H^∞ which satisfies a Lipschitz condition of order β is in $M(D_\alpha)$ for $\alpha < \beta$. However, this fact is particularly transparent in view of Theorem 2.9 since $\mathscr{D}_\alpha f$ is bounded in this case.

An extension of this idea is obtained by using the L^p modules of continuity, i.e.,

$$\omega_p(f, \delta) = \sup_{|y| \le \delta} \left\{ \int |f(x-y) - f(x)|^p \, dx \right\}^{1/p}.$$

THEOREM 2.14. *Let $q > p \geq 2$, $q \geq n\alpha^{-1}$, and $0 < \alpha < 1$. If $f \in L^\infty(\mathbf{R}^n)$ and*

$$\int_0^1 (\omega_q(f, t))^2 \frac{dt}{t^{1+2\alpha}} < \infty \tag{6}$$

then $f \in M(L^p_\alpha)$. The same result holds for D_α provided $p = 2$, $n = 1$, and $f \in H^\infty$.

Proof. We apply Minkowski's inequality to obtain

$$\int_E |\mathscr{D}_\alpha f|^p\, dx \leq c\|f\|^p_\infty |E| + \int_E \left\{\int_{|y|\leq 1} \frac{|f(x-y)-f(x)|^2}{|y|^{n+2\alpha}}\, dy\right\}^{p/2} dx$$

$$\leq c|E| + \left[\int_{|y|\leq 1} \left\{\int_E |f(x-y)-f(x)|^p\, dx\right\}^{2/p} \frac{dy}{|y|^{n+2\alpha}}\right]^{p/2}.$$

By the Holder inequality

$$\int_E |f(x-y)-f(x)|^p\, dx \leq |E|^{1-p/q}(\omega_q(f, |y|))^p$$

and hence

$$\int_E |\mathscr{D}_\alpha f|^p\, dx \leq c|E| + |E|^{1-p/q}\left\{\int_0^1 \omega_q^2(f, t) \frac{dt}{t^{1+2\alpha}}\right\}^{p/2}.$$

If $\alpha p < n$ then $1 - p/q \geq (n - \alpha p)n^{-1}$. Thus

$$\int_E |\mathscr{D}_\alpha f|^p\, dx \leq c\begin{cases}|E| & (|E| \geq 1)\\ |E|^{(n-\alpha p)n^{-1}} & (|E| < 1)\end{cases}$$

and hence by Lemma 2.10 and Theorem 2.9 imply that $f \in M(L^p_\alpha)$. For $\alpha p = n$ we have $0 < 1 - p/q < 1$ and the result follows. For $\alpha p > n$, we see that $\int_Q |\mathscr{D}_\alpha f|^p\, dx \leq c$ where Q is any unit cube. It follows that f is uniformly locally in $L^p_\alpha(E)$ and hence $f \in M(L^p_\alpha)$ by Corollary 2.2 [24].

The proof for D_α is similar except that Minkowski's inequality is not used and the case $\alpha > \frac{1}{2}$ uses Lemma 2.6 and the fact that $D_\alpha = M(D_\alpha)$.

COROLLARY 2.15. *Condition* (6) *in Theorem* 2.14 *can be replaced with the condition $\omega_q(f, \delta) \leq c\delta^{\alpha+\varepsilon}$ for $0 < \delta \leq 1$ and some $\varepsilon > 0$.*

In [16] a sufficient condition is given for $L^2_\alpha(T)$ multipliers where $0 < \alpha < \frac{1}{2}$. The condition is expressed in terms of the p-variation ($1 \leq p < \infty$) of a function. Here the p-variation is defined for $f \in L^\infty(T)$ by

$$V_p[f] = \sup\left\{\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^p\right\}^{1/p}$$

where the supremum is taken over all finite sets $t_0 < t_1 < \cdots < t_n$ and $t_n - t_0 \leq 2\pi$. We now show that this result is a corollary of Theorem 2.13. For simplicity we prove this result for $L^2_\alpha(-\infty, \infty)$.

LEMMA 2.16. *If f is in $L^\infty(-\infty, \infty)$ then $\omega_p(f, \delta) \le (2\delta)^{1/p} V_p[f]$.*

Proof. By a standard regularization argument we may assume that f is continuous. Let N be a positive integer and let $x_k = ky$. Then there are points x_k^* between x_k and x_{k+1} such that

$$\int_{|x| \le N|y|} |f(x+y) - f(x)|^p \, dx = |y| \sum |f(x_k^* + y) - f(x_k^*)|^p.$$

Since the sum can be estimated by $2V_p^p[f]$ the result follows.

COROLLARY 2.17. *If $f \in L^\infty(\mathbf{R}^1)$ $(L^\infty(T), H^\infty(T))$ and $V_q[f] < \infty$ for some $q > 2$ then f multiplies $L_\alpha^2(\mathbf{R}^1)$ $(L_\alpha^2(T), D_\alpha)$ for $\alpha < q^{-1}$.*

Proof. This is an immediate consequence of Corollary 2.18 and the fact that $\omega_q(f, \delta) \le C\delta^{1/q}$.

3. A boundary value characterization of $M(H^2, B^2)$

Returning to the characterization of $M(D_0, D_\beta)$ for $\beta < 0$ we see that $f' \in M(D_0, D_\beta)$ if and only if

$$\iint_{S(I)} |f'|^2 (1 - |z|)^{-1-2\beta} \, dx \, dy = O(|I|)$$

As remarked earlier, the case $\beta = -1$ has the alternate boundary value description that f is in BMO.

In this section we give a boundary value characterization somewhat similar to that of BMO for f' in $M(D_0, D_\beta)$ with $-1 < \beta < 0$. The boundary characterization is also similar to the classical theorem that certain growth rates on the derivative are equivalent to a Lipschitz condition on the boundary.

First we observe that $f' \in M(D_0, D_\beta)$ implies that f has a continuous extension to the closed unit disc, in fact f satisfies a Lipschitz condition. Using the estimation procedure of Theorem 1.2 we see that $|f'(z)| = O((1 - |z|)^\beta)$ and hence by the classical result of Hardy–Littlewood [13], f extends to a continuous function which satisfies a Lipschitz condition of order $1 + \beta$ on the boundary.

THEOREM 3.1. *Let $f \in H^\infty$ and $-1 < \beta < 0$. A necessary and sufficient condition that $f' \in M(D_0, D_\beta)$ is that*

$$\int_I \int_I \frac{|f(\theta) - f(t)|^2}{|e^{i\theta} - e^{it}|^{3+2\beta}} \, d\theta \, dt = O(|I|) \tag{1}$$

hold for all subarcs I on the unit circle.

We first remark that the Dini type condition (1) is somewhat similar, but stronger than Lip $1+\beta$. In fact, functions f in Lip $1+\beta$, $-1<\beta<0$, are characterized by the relation

$$\iint_{II} \frac{|f(\theta)-f(t)|^2}{|I|^{3+2\beta}}\, d\theta\, dt = O(|I|). \tag{2}$$

Obviously, functions in Lip $1+\beta$ satisfy relation (2). Conversely, if f satisfies (2) then letting $f_I = |I|^{-1}\int_I f\,dt$ we get

$$\begin{aligned}\frac{1}{|I|}\int_I |f-f_I| &\le \frac{1}{|I|^2}\iint_{II} |f(\theta)-f(t)|\, d\theta\, dt\\ &\le \left[\frac{1}{|I|^2}\iint_{II} |f(\theta)-f(t)|^2\, d\theta\, dt\right]^{1/2}\\ &= O(|I|^{1+\beta}).\end{aligned}$$

By Theorem [17] we see that f satisfies a Lipschitz condition of order $1+\beta$. Since $|e^{i\theta}-e^{it}| \le |I|$ for $e^{i\theta}, e^{it} \in I$ we see that (1) implies (2).

We remark that condition (2) and generalizations to other moduli of continuity can be found in [9]. We also observe that (2) with $\beta=-1$ is equivalent to *BMO*.

LEMMA 3.2. *Let I, J be arcs centered at $e^{i\theta_0}$ with $|J|\ge 3|I|$ and let f be a function in $L^1(T)$ with u denoting its Poisson extension to $|z|<1$. For $0<\alpha<1$ there is a constant c_α, independent of f, I, and J such that*

$$\begin{aligned}&\iint_{S(I)} |\nabla u|^2(1-|z|)^{1-2\alpha}\, dx\, dy\\ &\quad\le c_\alpha\left[\iint_{II}\frac{|f(\theta)-f(t)|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\, d\theta\, dt + |I|^{3-2\alpha}\left(\int_{|t|\ge 1/3|J|} |f(t+\theta_0)-f_J|\,\frac{dt}{t^2}\right)\right]^2\end{aligned}$$

Proof. Assume without loss of generality that $\theta_0=0$ and let ϕ be a function which is one on $1/3J$, supp $\phi\subset 2/3J$, and $|\phi(\theta)-\phi(t)|\le c(|\theta-t|/|J|)$ for all θ, t. Now

$$f=(f-f_J)\phi+(f-f_J)(1-\phi)+f_J=f_1+f_2+f_3$$

and we take u_i to be the Poisson extension of f_i. Since f_3 is constant we have $|\nabla u|^2$ is dominated by $|\nabla u_1|^2+|\nabla u_2|^2$.

For $z=re^{i\theta}$ in $S(I)$,

$$|\nabla u_2(z)| \le c\int \frac{|f_2(t)|}{(1-r)^2+(\theta-t)^2}\, dt \le c\int_{|t|\ge 1/3|J|} |f(t)-f_J|\,\frac{dt}{t^2}$$

and hence

$$\iint_{S(I)} |\nabla u_2|^2(1-|z|)^{1-2\alpha}\,dx = O\left(|I|^{3-2\alpha}\left(\int_{|t|\geq 1/3|J|} |f-f_J|\,\frac{dt}{t^2}\right)^2\right).$$

Now for the integral over $S(I)$ of $|\nabla u|^2$ we replace $S(I)$ with the entire disc and using the equivalent Dirichlet norm we have an upper bounded given by $\|f_1\|_2^2 + \|\mathscr{D}_\alpha f_1\|_2^2$.

Since supp $\phi \subset J$ it follows that

$$\|f_1\|_2^2 \leq \int_J |f-f_J|^2 \leq \iint_{JJ} \frac{|f(\theta)-f(t)|^2}{|J|}\,d\theta\,dt \leq c\iint_{JJ} \frac{|f(\theta)-f(t)|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\,d\theta\,dt.$$

Thus it suffices to consider the term $\|\mathscr{D}_\alpha f_1\|_2^2$.

Since supp $\phi \subset 2/3J$ we put

$$\|\mathscr{D}_\alpha f_1\|_2^2 = \iint_{\theta,t\in J} + \iint_{\substack{\theta\notin J\\ t\in 2/3J}} + \iint_{\substack{t\notin J\\ \theta\in 2/3J}} \frac{|f_1(\theta)-f_1(t)|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\,d\theta\,dt$$

$$= A + B + C.$$

For A we observe that

$$|f_1(\theta)-f_1(t)| \leq |f(\theta)-f(t)| + c|J|^{-1}|e^{i\theta}-e^{it}|\,|f(t)-f_J|$$

since ϕ satisfies a Lipschitz condition. Thus we need only estimate

$$\frac{1}{|J|^2}\iint_{JJ} \frac{|f(t)-f_J|^2}{|e^{i\theta}-e^{it}|^{2\alpha-1}}\,d\theta\,dt \leq \frac{1}{|J|^2}\int_J |f(t)-f_J|^2\left\{\int_J \frac{d\theta}{|e^{i\theta}-e^{it}|^{2\alpha-1}}\right\}dt$$

$$\leq c|J|^{-2\alpha}\int_J |f(t)-f_J|^2\,dt$$

$$\leq c\iint_{JJ} \frac{|f(\theta)-f(t)|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\,d\theta\,dt$$

as was done above. The B and C terms are handled similarly.

Proof of Theorem 3.1. Assume that f is in H^∞ and satisfies (1). Put $\alpha = 1+\beta$ then $-1-2\beta = 1-2\alpha$. We apply Lemma 3.2 with $J = 3I$ and use the fact that f is Lip α to obtain

$$\iint_{S(I)} |f'|^2(1-|z|)^{-1-2\beta}\,dx\,dy = O(|I|)$$

and hence f is in $M(D_0, D_\beta)$.

Conversely, suppose f' is in $M(D_0, D_\beta)$. Fix an arc I which we assume is centered at $z = 1$. Put $r = 1 - |I|$ and $g(z) = (1-r)(1-rz)^{-1}$.

Since f is Lip α, setting a equal to the value of f at $z = 1$, a computation shows that

$$\iint |g'(f-a)|^2(1-|z|)^{1-2\alpha}\,dx\,dy$$

$$= \sum \iint_{S(2^nI)\setminus S(2^{n-1}I)} |g'(f-a)|^2(1-|z|)^{1-2\alpha}\,dx\,dy$$

$$= O\sum\left(\frac{|I|}{2^n}\right)$$

$$= O(|I|).$$

In addition, $\|g\|_0^2 = O(|I|)$ and since $f' \in M(D_0, D_\beta)$ we have

$$\iint |f'g|^2(1-|z|)^{1-2\alpha}\,dx\,dy = O(|I|)$$

Putting these two estimates together we see that, for $h = (f-a)g$,

$$\iint |h'|^2(1-|z|)^{-1-2\alpha} = O(|I|).$$

Hence $\|h\|_\alpha^2 = O(|I|)$. Finally, it follows from the inequality

$$\iint_{II} \frac{|f(\theta)-f(t)|^2\,|g(t)|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\,d\theta\,dt$$

$$\le c\left[\|h\|_\alpha^2 + \iint_{II} \frac{|g(\theta)-g(t)|^2\,|f(\theta)-a|^2}{|e^{i\theta}-e^{it}|^{1+2\alpha}}\,d\theta\,dt\right]$$

and the fact that f is Lip α that (1) holds.

4. The counterexample

In this section we prove the existence of a continuous analytic function f in $D_{1/2}$ with

$$\sup_I B_{1/2,2}(I)^{-1}\left\{\iint_{S(I)} |f'|^2\,dx\,dy + \int_I |\mathscr{D}_{1/2} f|^2\,dx\right\} < \infty \tag{1}$$

yet f is not a multiplier on $D_{1/2}$. The supremum is taken over arcs I. In other words, in our characterization of multipliers proper behavior on squares or intervals is not sufficient.

LEMMA 4.1. *If f is in $H^\infty(T)$, $0 < \alpha \le \frac{1}{2}$, and $\int_I |\mathscr{D}_\alpha f|^2 = O(B_{\alpha,2}(I))$ then*

$$\iint\limits_{S(I)} |f'|^2(1 - |z|)^{1-2\alpha}\, dx\, dy = O(B_{\alpha,2}(I)).$$

Proof. This fact follows from Lemma 3.2 with $J = 3I$ if $0 < \alpha < \frac{1}{2}$ and $|J| = |I|^{1/2}$ if $\alpha = 1$.

Let $\lambda_0 = 1$ and λ_m be $\frac{1}{2}$ raised to the 2^m power for $m = 1, 2, \ldots$. Denote by A_n the set of n-triples whose entries are zero or one and let $x_\alpha = \sum_{i=1}^n \alpha_i(1 - \lambda_i)\lambda_0 \cdots \lambda_{i-1}$ for α in A_n. Put I_α equal to the interval centered at x_α and of length $\lambda_1 \cdots \lambda_n$. Also, denote this length by δ_n. Let E_n be the union of these intervals for α in A_n. The set E_n is a disjoint union of 2^n intervals whose lengths are roughly $\frac{1}{2}$ raised to the 2^n power and whose positions are modeled after those of the Cantor "middle thirds" set.

For simplicity the construction will take place on the line because dilations are easier to describe. We take $\phi(x) = (1 - ix)^{-1}$ then ϕ extends to be bounded and analytic for z in the upper half plane. Finally, we put

$$f_n(x) = \sum_{\alpha \in A_n} 2^{-n/2}\phi_n(x - x_\alpha)$$

where $\phi_n(x) = \phi(x\delta_n^{-1})$. Clearly, f_n is bounded, continuous, and analytic. We claim that there are positive constants A, a independent of n such that

(i) $\|f_n\|_\infty \le A$,
(ii) $\int_I |\mathscr{D}_{1/2} f_n|^2\, dx \le AB_{1/2,2}(I)$,
(iii) $\int_{E_n} |\mathscr{D}_{1/2} f_n|^2\, dx \ge a > 0$.

THEOREM 4.2. *There exists a function f in $H^\infty(T)$ which is continuous and satisfies* (1) *but is not a multiplier of D_1.*

Proof. If we assume that continuous functions in $H^\infty(T)$ which satisfy (1) are multipliers then the closed graph theorem would imply that

(2)

$$\int_E |\mathscr{D}_{1/2} f|^2\, dx \le cB_{1/2,2}(E)\left[\|f\|_\infty + \sup_I B_{1/2,2}(I) \int_I |\mathscr{D}_{1/2} f|^2\, dx\right]$$

holds for all such functions where c is independent of f. We have used Lemma 4.1 in the above.

Let $\Psi(t) = (\log 1/t)^{-1}$ for t small and denote by H_Ψ the corresponding Hausdorff measure, see [4, Chapter 2] for details. Observe that $H_\Psi(E_n) \le 2^n(\log \delta_n^{-1}) \le c < \infty$. Thus, the set $E = \cap E_n$ satisfies $H_\Psi(E) < \infty$. By Theorem 21 [18] we must have $B_{1/2,2}(E) = 0$ and by the outer regularity of $B_{1/2,2}$ [18, Theorem 1] we see that $\lim_{n\to\infty} B_{1/2,2}(E_n) = 0$.

We now use the sequence of functions $\{f_n\}$ to produce a contradiction. By the properties of these functions we see that (2) becomes $aB_{1/2,2}(E_n)^{-1} \leq 2cA$ which is impossible.

We now complete the proof by verifying properties (i), (ii), and (iii). To prove (i) we must show that $g_n(x) = 2^{-n/2}S_n \sum_\alpha \{S_n^2 + (x - x_\alpha)^2\}^{1/2}$ is bounded independent of α and n. For a given x there is a closest point x_α and a next closest point $x_{\alpha'}$. Due to the extremely rapid decrease of δ_n we see that for n sufficiently large $|x - x_\beta| \geq 2^{n/2}\delta_n^{1/2}$ holds for all other indices x. As a result we see that

$$g_n(x) \leq \delta_n^{1/2} + 2^{1-n/2}\delta_n(\delta_n^2 + (x - x_\alpha)^2)^{-1/2}.$$

Now let $\tilde{I}_\alpha$ be the interval centered at x_α with $|\tilde{I}_\alpha| = 2^{1-n/2}S_n^{1/2}$ then

$$g_n(x) = 0 \begin{cases} \delta_n^{1/2} & \text{if } x \notin \cup \tilde{I}_\alpha, \\ \dfrac{2^{-n/2}\delta_n}{(\delta_n^2 + (x - x_\alpha)^2)^{1/2}} & \text{if } x \in \cup \tilde{I}_\alpha \end{cases}$$

where x_α is the closest point to x. In particular $\|g_n\|_\infty = O(2^{-n/2})$ so (i) is satisfied.

To prove (ii) we observe that Schwarz's inequality yields

$$\mathscr{D}_{1/2} f(x) \leq \sum 2^{-n/2}\mathscr{D}_{1/2}\phi_n(x - x_\alpha).$$

But a computation reveals that

$$\mathscr{D}_{1/2}\phi(x)^2 = \pi(1 + x^2)^{-1}$$

and hence

$$\mathscr{D}_{1/2} f(x) \leq 2^{-n/2}(\pi\delta_n)^{1/2} \sum \{\delta_n^2 + (x - x_\alpha)^2\}^{-1/2} = \pi^{1/2}\delta_n^{-1/2}g_n(x).$$

Combining this with our previous estimate on g_n we get

$$\mathscr{D}_{1/2} f(x) = 0 \begin{cases} 1 & \text{if } x \notin \cup \tilde{I}_\alpha, \\ \dfrac{2^{-n/2}\delta_n^{1/2}}{(\delta_n^2 + (x - x_\alpha)^2)^{1/2}} & \text{if } x \in \cup \tilde{I}_\alpha. \end{cases} \tag{2}$$

Case 1. $|I| \leq \frac{1}{2}\delta_{n-1}$. Since I can intersect at most one set $\tilde{I}_\alpha$ the integral over I, using our dominating function in (2), becomes largest if I is centered at x_α for some α. For $|I| \leq \delta_n$ we have

$$|I|(\log |I|^{-1}) \leq \delta_n(\log \delta_n^{-1}) \leq C2^n\delta_n$$

and hence $2^{-n}\delta_n^{-1}|I| \leq CB_{1/2,2}(I)$. Thus, using (2) we get

$$\int_I |\mathscr{D}_{1/2} f|^2 \leq c2^{-n}\delta_n^{-1}|I| \leq cB_{1/2,2}(I).$$

For $n \leq |I| \leq |\tilde{I}_\alpha|$ we also have

$$\int_I |\mathscr{D}_{1/2} f|^2 \leq c \int_0^{|I|} \frac{2^{-n}\delta_n}{\delta_n^2 + x^2}\, dx \leq c2^{-n} \leq cB_{1/2,2}(I).$$

If $|I| \geq |\tilde{I}_\alpha|$ then we use the same estimate combined with the fact that $|\mathscr{D}_{1/2} f| \leq 1$ on $I \backslash \tilde{I}_\alpha$ and $|I| \leq B_{1/2,2}(I)$ to obtain the same result.

Case 2. $|I| \geq \frac{1}{2}\delta_{n-1}$. We have just observed that the portion of I outside of $\cup \tilde{I}_\alpha$ causes no difficulty. On the other hand, if we let M denote the number of indices α for which $I \cap \tilde{I}_\alpha \neq \emptyset$ then by the first case we have

$$\int_{I \cap (\cup \tilde{I}_\alpha)} |\mathscr{D}_{1/2} f|^2 \leq \sum \int_{I \cap \tilde{I}_\alpha} |\mathscr{D}_{1/2} f|^2 \leq cM2^{-n}.$$

Now it is clear from the construction of E_n that $M \leq 2^j$ if $|I| \leq \delta_{n-j}$. Hence if $\delta_{n-j+1} < |I| \leq \delta_{n-j}$ with $j = 1, 2, \ldots, n-1$ it follows that

$$M \leq 2^j \leq C2^n(\log \delta_{n-j+1}^{-1})^{-1} \leq C2^n(\log |I|^{-1})^{-1} \leq C2^n B_{1/2,2}(I).$$

If $|I| \geq \delta_1$ the above conclusion is trivially true and hence we have $M = O(2^n B_{1/2,2}(I))$. Combining this estimate with our previous inequality we see that (ii) is proved.

To prove (iii) we notice that

$$\int_{E_n} |\mathscr{D}_{1/2} f|^2\, dx \geq \sum_\alpha \int_{I_\alpha} |\mathscr{D}_{1/2} f|^2 \geq 2^n \min_\alpha \left(\int_{I_\alpha} |\mathscr{D}_{1/2} f|^2 \right)$$

so we must show that $\int_I |\mathscr{D}_{1/2} f|^2 \geq C2^{-n}$. Fix α_0 and let

$$h = 2^{-n/2}\phi_n(x - x_0) \quad \text{and} \quad k = f - h.$$

Then $\mathscr{D}_{1/2} h \leq \mathscr{D}_{1/2} f + \mathscr{D}_{1/2} k$. Now by a change of variables we get

$$\int_{I_{\alpha_0}} |\mathscr{D}_{1/2} h|^2 = 2^{-n} \int_{|x| \leq 1/2} |\mathscr{D}_{1/2}\phi(x)|^2\, dx = \pi 2^{-n} \int_{|x| \leq 1/2} \frac{dx}{1 + x^2} \geq 2^{-n}.$$

By estimating $\mathscr{D}_{1/2} k$ as before we see that it is bounded independent of n and α_0 and hence $\int_{I_{\alpha_0}} |\mathscr{D}_{1/2} k|^2 = O(\delta_n)$. Since δ_n tends to zero at a faster rate than 2^{-n} we have established (iii) and the proof is complete.

5. Concluding remarks

It has come to my attention that V. G. Maz'ya had published the results of Lemma 2.8 and Theorem B, see [28], at about the time of the writing of this manuscript. In addition, Maz'ya and T. O. Shaposhnikova have recently obtained some further multiplier results.

Finally, Björn Dahlberg has recently extended the strong type capacity inequality for $p > 1$. It follows that by modifying $\mathscr{D}_\alpha$ in Theorem 2.9 a similar result holds for all $p > 1$.

References

1. D. R. Adams, *On the existence of capacitary strong type estimates in* $\mathbf{R}^n$, Arkiv Math., vol. 14 (1976), pp. 125–140.
2. N. Aronszajn and K. T. Smith, *Theory of Bessel potentials I*, Ann. Inst. Fourier, Grenoble, vol. 11, (1961), pp. 385–475.
3. L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math., vol. 76 (1962), pp. 547–559.
4. ———, *Selected problems in exceptional sets*, Van Nostrand, Princeton, N.J., 1967.
5. A. Devinatz and I. I. Hirschman, *Multiplier transformations on* $l^{2,\alpha}$, Ann. of Math., vol. 69 (1959), pp. 575–587.
6. P. L. Duren, *Theory of* H^p *Spaces*, Academic Press, New York, 1970.
7. C. L. Fefferman, *Characterization of bounded mean oscillation*, Bull. Amer. Math. Soc., vol. 77 (1971), pp. 587–588.
8. C. L. Fefferman and E. M. Stein, H^p *spaces of several variables*, Acta Math., vol. 129 (1972), pp.137–193.
9. A. M. Garcia, E. Rodemich and H. Rumsey, Jr., *A real variable lemma*, Indiana J. Math., vol. 20 (1970), pp. 565–578.
10. K. Hoffman, *Banach space of analytic functions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
11. W. W. Hastings, *A Carleson measure theorem for Bergman spaces*, Proc. Amer. Math. Soc., vol. 52 (1975), pp. 237–241.
12. G. H. Hardy, *Divergent series*, Oxford, at the Clarendon, 1949.
13. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals II*, Math. Zeitschr., vol. 34 (1932), pp. 403–459.
14. I. I. Hirschman, *On multiplier transformations*, Duke Math. J., vol. 26 (1959), pp. 221–242.
15. ———, *On multiplier transformations II*, Duke Math. J., vol. 28 (1961), pp. 45–56.
16. ———, *On multiplier transformations III*, Proc. Amer. Math. Soc., vol. 13 (1962), pp. 851–857.
17. N. G. Meyers, *Mean oscillation over cubes and Holder continuity*, Proc. Amer. Math. Soc., vol. 15 (1964), pp. 717–721.
18. ———, *A theory of capacities for potentials of functions in Lebesque classes*, Math. Scand., vol. 26 (1970), pp. 255–292.
19. B. Rizzi, *Moltiplicatori di* $H^{1,2}(S)$, Math. Anuali, Ferrara Univ., vol. 14–17 (1969–72).
20. A. L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys, vol. 13 (1974), pp. 51–128.
21. S. L. Sobolev, *On a theorem of functional analysis*, Mat. Sb. (N.S.), vol. 4 (46) (1938), pp. 471–497; English transl., Amer. Math. Soc. transl. (2) vol. 34, pp. 39–68.
22. E. M. Stein, *The characterization of functions arising as potentials*, Bull. Amer. Math. Soc., vol. 67 (1961), pp. 102–104.
23. ———, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
24. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech., vol. 16 (1967), pp. 1031–1060.
25. G. D. Taylor, *Multipliers on* D_α, Trans. Amer. Math. Soc., vol. 123 (1966), pp. 229–240.
26. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London, 1959.
27. V. G. Maz'ya, *On certain integral inequalities for functions of many variables*, J. Soviet Math., vol. 1 (1973), pp. 205–234.
28. ———, *On capacitary estimates of the strong type for the fractional norm*, Zap. Sem. LOMI Leningrad, vol. 70 (1977), pp. 161–168.

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