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ABELIAN SUBGROUPS OF $\mathrm{Aut}_k\,(k[X, Y])$ AND APPLICATIONS TO ACTIONS ON THE AFFINE PLANE

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Introduction

In this study, we apply some theorems of group theory to study algebraic and non-algebraic actions of algebraic groups on the affine plane. The main object of study is the group $\mathrm{Aut}_k\,(k[X, Y])$, which is denoted by $GA_2(k)$. When k is a field this group has a decomposition as an amalgamated free product $A *_B E$ of groups (§1). Since $GA_2(k)$ is (up to anti-isomorphism) the group of algebraic isomorphisms of the affine plane $\mathbf{A}^2(k)$, any action of a commutative algebraic group G on $\mathbf{A}^2(k)$ gives rise to a group homomorphism $G \to GA_2(k)$, the image of which homomorphism is then an abelian subgroup of $GA_2(k)$. Abelian subgroups of any amalgamated free product $A *_B E$ may be understood group theoretically, up to conjugacy, in terms of the groups A and E, and the containments $B \subset A$ and $B \subset E$, using certain results from combinatorial group theory, especially a theorem of Moldavanski (see 0.5). These essential facts are laid out in §0. By these means we are able to give a classification of any action of an algebraic group on the affine plane, up to equivalence (3.10 and 3.11).

The main theorem of §4 (Theorem 4.9) explicitly describes, up to equivalence, actions of the n-dimensional vector group G_a^n on the plane, as long as the field k is infinite. This generalizes the results of R. Rentschler and M. Miyanishi ([11] and [8]), which describe actions of G_a on the affine plane.

In Section 5, we employ these methods to give another proof of Gutwirth's theorem [5], which describes, up to equivalence, actions of the n-dimensional torus G_m^n on the plane. Again, we assume only that the field k is infinite. (Certain generalizations of this theorem involving faithful actions of tori on n-space can be found in [2] and [4].)

The writer is indebted to Professor Hyman Bass, who suggested these group theoretic methods as a means of describing actions of groups on the plane.

0. Some facts about subgroups of amalgamated free products

0.1. *Notation.* When G is a group and H is a subgroup of G, a right coset of G modulo H is an element of the coset space on which G acts on the right. Hence if $g \in G$, Hg is a right coset. If $h \in G$, we conjugate h by g by writing $h^g = g^{-1}hg$. Also, for $H \subset G$, we write H^g for $g^{-1}Hg$. We write

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$\langle h, g\rangle$ for the subgroup of G generated by h and g. Similarly, we write $\langle H, g\rangle$ for the subgroup generated by H and g.

Let A, E, and B be groups. Given monomorphisms $B \to A$ and $B \to E$, we can form $G = A *_B E$, the free product of A and E, amalgamated over B.

Let I, resp. J, be a system of non-trivial right coset representatives of A, resp. E, modulo B. (Here, and from now on, we identify B as a subgroup of A, and of E.) Let W be the set of all words spelled using elements of I and J in alternating fashion, including an empty word x. Given a word $w \in W$, let $|w|$ denote the element of G obtained by multiplying the letters of w in order. The map $B \times W \to G$ defined by $(b, w) \mapsto b\,|w|$ is a bijection (see §1 of [12]). Given an element $g \in G$, we call the corresponding element $(b, w) \in B \times W$ the *normal form* of g. Thus each element of G has a unique normal form. We define the *length* of $g \in G$ to be the length of the word w. This is independent of I and J

Let $g \in G$ with normal form (b, w). We say that g is *cyclicly reduced* if w is non-empty, and if w begins with an element of I and ends with an element of J, or vice versa. Every element of G is either (1) conjugate in G to an element of A or E, or (2) conjugate to a cyclicly reduced element; and (1) and (2) are mutally exclusive. (See 1.3 of [12].) If g is cyclicly reduced, then clearly length $(g^d) = d$ length (g). Hence g is of infinite order.

0.2. There are some facts from combinatorial group theory which will be exploited to obtain the main results of this paper. These results have to do with the classification of abelian subgroups of a group G which is an amalgamated free product $A *_B E$. First I shall state the main group-theoretic theorem, due to Moldavanski [11], which describes how such subgroups can occur.

0.3. *Theorem–Definition* (Moldavanski). *Suppose $G = A *_B E$, and suppose H is an abelian subgroup of G. Precisely one of the following situations holds.*

(1) *H is conjugate (in G) to a subgroup of A, or H is conjugate to a subgroup of E.*

(2) *H is not conjugate to any subgroup of A or E. There exists an infinite nested chain of subgroups $H_0 \subset H_1 \subset \cdots \subset H_{i-1} \subset H_i \subset \cdots$ such that $H = \bigcup_{i=0}^{\infty} H_i$, and such that each H_i is conjugate in G to a subgroup of B. (The chain is necessarily non-stationary.)*

(3) *$H = F \times \langle g\rangle$, where F is conjugate to a subgroup of B, and g is not conjugate to any element in A or E. Hence g is of infinite order.*

We say that H is an abelian subgroup of type 1, 2, *or* 3 *accordingly.*

0.4. This theorem has been proved by combinatorial group theoretic methods similar to those used in proving similar theorems for free groups. The reader is referred to Karrass and Solitar's very lucid treatment [7], in which this comes as a corollary to a much more general theorem (Theorem 6 of [7]).

I will merely outline a proof which uses the Bass–Serre theory of groups acting on trees, as presented in [12], and which provides a method for the explicit construction of subgroups of type II and III. There are four main lemmas (0.21, 0.22, 0.25, and 0.26), from which 0.3 follows immediately. The lemmas themselves will be important for my purposes because they provide a method for determining whether or not certain types of abelian subgroups can occur, and, if so, how to construct them.

0.5. Let $(\mathcal{G}, X)$ be the following graph of groups [12, §3]:

$$v_1 \bullet\!\!\xrightarrow[t]{}\!\!\bullet\, v_2$$

where $\mathcal{G}_{v_1} = A$, $\mathcal{G}_{v_2} = E$, $\mathcal{G}_t = B$ and the monomorphisms $\mathcal{G}_t \to \mathcal{G}_{v_1}$, $\mathcal{G}_t \to \mathcal{G}_{v_2}$ are those given in 0.1. (Referring to the notation of [12], we have written t to represent both the edges t and $\bar{t}$.) The graph X is a tree, and the fundamental group $\pi_1(\mathcal{G}, X, X)$ of $(\mathcal{G}, X)$ [12, §5] is the group G.

0.6. As in [12, §4], we construct the tree $\tilde{X} = \tilde{X}(\mathcal{G}, X, X)$; and we have an action (on the right) of G on $\tilde{X}$ and a projection $p: \tilde{X} \to X$ which induces an isomorphism $\tilde{X}/G \to X$. In addition, we have a section $s: X \to \tilde{X}$ such that $G_{s(v_1)}$ (the stabilizer in G of $s(v_1)) = A$, $G_{s(v_2)} = E$, and $G_{s(t)} = B$.

0.7. The tree $\tilde{X}$ can be realized in the following way. Let $\mathcal{W}$ be the set consisting of all the non-empty words in W, together with two formally "empty" words x_I and x_J. The vertices in $\tilde{X}$ correspond bijectively to elements of $\mathcal{W}$. Given a word $w \in \mathcal{W}$, we write $v(w)$ for the corresponding vertex in $\tilde{X}$. We formally declare that x_I begins with an element of J, and that x_J begins with an element of I. Now, given two words w, $w' \in \mathcal{W}$, not both empty, the vertices $v(w)$ and $v(w')$ are connected by an edge in $\tilde{X}$ if and only if w' is obtained from w by dropping the first letter, or vice versa. In addition, the vertices $v(x_I)$ and $v(x_J)$ are connected by an edge. Whenever vertices v and v' in $\tilde{X}$ are connected by an edge, we write $t(v, v')$ for the edge connecting them. If $v = v(w)$, $v' = v(w')$, with w, $w' \in \mathcal{W}$, we will also write $t(w, w')$ for $t(v, v')$. We will say that a vertex $v(w)$ of X is of type A if w begins with an element of J, and of type E if w begins with an element of I.

0.8. We now describe the action of G on $\tilde{X}$. The action will be transitive on vertices of type A and transitive on vertices of type E. Note that, for any $g \in G$, we can write $g = a\,|w|$ with $a \in A$, and w a word in $\mathcal{W}$ beginning with an element of J, and this expression is unique. Similarly, we can write $g = e\,|w'|$ where $e \in E$ and $w' \in \mathcal{W}$ begins with an element of I. Given $g \in G$ and $v(w)$ a vertex in X, then $v(w) \cdot g$ is defined as follows. If $v(w)$ is a vertex of type I, write $|w|\,g = a\,|w'|$ with $a \in A$, $w' \in \mathcal{W}$, and let $v(w) \cdot g = v(w')$. If $v(w)$ is a vertex of type J, write $|w|\,g = e\,|w'|$ with $e \in E$, $w' \in \mathcal{W}$ and let $v(w) \cdot g = v(w')$.

0.9. Obviously, each edge in $\tilde{X}$ connects a vertex of type A and a vertex of type E. Given any segment

$$v \bullet\!\!\longrightarrow\!\!\bullet\, v'$$

with v of type A and v' of type E, there exists $g \in G$ such that $v \cdot g = x_I$ and $v' \cdot g = x_J$. (For example, suppose $v' = v(w)$, where $w \in \mathcal{W}$ begins with an element of I, and $v = v(ew)$, where $e \in J$. Then take $g = |w|^{-1}$.)

0.10. The stabilizers of the vertices in $\tilde{X}$ are the conjugates of A and E in G. The stabilizers of edges are the conjugates of B in G. More precisely, given a vertex $v(w)$ in $\tilde{X}$, then one sees that

$$G_{v(w)} = \begin{cases} A^{|w|} & \text{if } v(w) \text{ is of type } A, \\ E^{|w|} & \text{if } v(w) \text{ if of type } E. \end{cases}$$

Suppose $h \in J$. Let w be a word beginning with an element of I. Upon letting $v = v(w)$, $v' = v(hw)$, then v and v' are connected by an edge $t(v, v')$, and $G_{t(v,v')} = B^{|hw|}$. The same is true if $h \in I$ and w is a word beginning with an element of J. The stabilizer of the edge connecting $v(x_I)$ and $v(x_J)$ is B.

0.11. The projection $p: \tilde{X} \to X$ sends v to v_1 if v is of type A, and to v_2 if v is of type E. We choose the section $s: X \to \tilde{X}$ so that $s(v_1) = v(x_I)$ and $s(v_2) = v(x_J)$.

0.12. Let H be a subgroup of G. Then H acts on $\tilde{X}$ by restriction, and we form the graph of groups $(\mathcal{H}, Y)$ where $Y = \tilde{X}/H$, as in §5 of [12]. Upon choosing a maximal subtree T in Y we form the fundamental group $\pi_1(\mathcal{H}, Y, T)$ [12, §5]. There exists a section s $Y \to \tilde{X}$ such that $s|_T$ is a morphism of graphs. Such a section s induces an isomorphism $\tilde{s}: \pi_1(\mathcal{H}, Y, T) \to H$ such that for any vertex v in Y and for any edge e in T, the isomorphism carries $\mathcal{H}_v$ onto $H_{s(v)}$, and $\mathcal{H}_e$ onto $H_{s(e)}$ [12, §5].

0.13. Employing the technique, and using the terminology, introduced in [1], we can choose a maximal filtering forest D in Y, and form a reduced graph of groups $(\mathcal{H}', Y')$. The vertices in Y' correspond to the connected components in D. The edges in Y' correspond to edges in Y which aren't in D. If T is a maximal subtree of Y which contains D, then T corresponds to a maximal subtree T' in Y', and there is a canonical isomorphism

$$\pi_1(\mathcal{H}, Y, T) \to \pi_1(\mathcal{H}', Y', T').$$

The graph $(\mathcal{H}', Y')$ contains no directed edges.

0.14. Suppose H is a subgroup of $G = A *_B E$, and that $(\mathcal{H}, Y)$, D, and $(\mathcal{H}', Y')$ are as in 0.13. Let us further assume that the graph Y' consists of one point with no edges. Clearly this happens precisely when $D = Y$, i.e., $(\mathcal{H}, Y)$ is a filtering tree of groups. Therefore we have $\pi_1(\mathcal{H}, Y, Y) = \pi_1(\mathcal{H}', Y', Y') \cong H$.

0.15. If the filtering tree of groups $(\mathcal{H}, Y)$ has a maximal vertex v, then the map

$$\mathcal{H}_v \to \pi_1(\mathcal{H}, Y, Y) \cong H$$

is an isomorphism (see 1.6 of [1]). If we lift v to a vertex v' in $\tilde{X}$, then $\mathcal{H}_v$ is identified with the stabilizer $H_{(v')}$, and so $H_{(v')} = H$. But $H_{(v')} = H \cap G_{(v')}$. By

0.10, we see that H is a subgroup of either a conjugate of A or a conjugate of E.

0.16. Conversely, if H is a subgroup of either a conjugate of A or a conjugate of E, then there is a vertex v' of $\tilde{X}$ which is fixed by the action of H on $\tilde{X}$. If v' projects to v in Y, and if T is a maximal subtree of Y, then the map $\mathcal{H}_v \to \pi_1(\mathcal{H}, Y, T)$ is an isomorphism. It follows that $T = Y$ (otherwise there would be projections $\pi_1(\mathcal{H}, Y, T)$ onto Z which, when restricted to $\mathcal{H}_v$, were trivial), and that $(\mathcal{H}, Y)$ is a filtering tree of groups, v being a maximal vertex.

0.17. *Now suppose that H is a subgroup of G, and that $(\mathcal{H}, Y)$ is a filtering tree of groups with no maximal vertex. This will be the standing assumption in* 0.17–0.21. In this case there exists in Y an infinite directed geodesic

(1)
$$\overset{v_{-1}}{\bullet}\!\!-\!\!-\!\!\overset{v_0}{\bullet}\!\!-\!\!-\!\!\overset{v_1}{\bullet}\!\!-\!\!-\!\!\overset{v_2}{\bullet}\!\!-\!\!-\!\!\overset{v_3}{\bullet}\!\!-\!\!-\cdots$$

with no maximal vertex, and

$$\varinjlim \mathcal{H}_{v_i} = \pi_1(\mathcal{H}, Y, Y) \cong H.$$

Upon choosing a section s of the projection $p: \tilde{X} \to Y$, we can choose the isomorphism $\pi_1(\mathcal{H}, Y, Y) \to H$ so as to identify $\mathcal{H}_{v_i}$ with $H_{s(v_i)} = H \cap G_{s(v_i)}$.

0.18. Recalling the construction of $\tilde{X}$ (0.6), we have $s(v_i) = v(w_i)$ for some $w_i \in \mathcal{W}$, and the geodesic (1) lifts to

$$-\overset{v(w_{-1})}{\bullet}\!\!\underset{t_0}{-\!\!-\!\!-}\!\!\overset{v(w_0)}{\bullet}\!\!\underset{t_1}{-\!\!-\!\!-}\!\!\overset{v(w_1)}{\bullet}\!\!\underset{t_2}{-\!\!-\!\!-}\!\!\overset{v(w_2)}{\bullet}\!\!-\!\!-\cdots$$

in $\tilde{X}$. Since (1) is directed, we have

(2)
$$H_{v(w_{-1})} = H_{t_0} \subset H_{v(w_0)} = H_{t_1} \subset \cdots.$$

This chain is non-stationary, since $(\mathcal{H}, Y)$ has no maximal vertex, and the union is H.

0.19. We may assume that $v(w_{-1})$ is of type A. Otherwise, we can remove the first vertex and relabel. We are interested in describing the subgroup H – up to conjugacy in G. Therefore, we are free to replace H by any conjugate H^g, and at the same time apply the automorphism of $\tilde{X}$ which arises from multiplication by g. In other words, we have a commuting diagram

$$\begin{array}{ccc} \tilde{X} \times H & \to & X \\ \downarrow \quad \downarrow & & \downarrow \\ \tilde{X} \times H^g & \to & X \end{array}$$

where $\tilde{X} \to \tilde{X}$ is a multiplication by g, and $H \to H^g$ is conjugation by g. (The notation $\tilde{X} \times H$ is an abuse.)

0.20. The point is that we may replace H by an appropriate conjugate, and assume that $w_{-1}=x_I$ and $w_0=x_J$. This follows from 0.19 and 0.9. It follows, since the vertices $v(w_i)$ lie successively along a geodesic, that there exist sequences $\{a_i\}_{i=1}^{\infty}\subset I$ and $\{e\}_{i=1}^{\infty}\subset J$ such that

$$
\begin{aligned}
&w_1=e_1 && w_2=a_1e_1,\\
(3)\qquad &w_3=e_2a_1e_1, && w_4=a_2e_2a_1e_1,\ldots,\\
&w_{2i-1}=e_ia_{i-1}e_{i-1}\cdots a_1e_1, && w_{2i}=a_ie_i\cdots a_1e_1,\ldots.
\end{aligned}
$$

The vertex $v(w_i)$ is of type A if i is odd and of type E if i is even. According to 0.10, the nested non-stationary chain (2) is

$$
\begin{aligned}
&H\cap A=H\cap B\subset H\cap E=H\cap B^{|w_1|}\\
&\qquad\subset H\cap A^{|w_1|}=H\cap B^{|w_2|}\subset H\cap E^{|w_2|}=H\cap B^{|w_2|}\\
&\qquad\qquad\vdots\\
(4)\qquad&\qquad\subset H\cap A^{|w_{2i-1}|}=H\cap B^{|w_{2i}|}\subset H\cap E^{|w_{2i}|}=H\cap B^{|w_{2i+1}|}\\
&\qquad\qquad\vdots
\end{aligned}
$$

For each integer $i\geq 1$, let

$$
(5)\qquad
\begin{aligned}
H_{i-1}&=H\cap A^{|w_{2i-3}|}=H\cap B^{|w_{2i-2}|},\\
H_i'&=H\cap E^{|w_{2i-2}|}=H\cap B^{|w_{2i-1}|}
\end{aligned}
$$

so that the nested non-stationary chain (4) becomes

$$(6)\qquad H_0\subset H_1'\subset H_1\subset\cdots\subset H_{i-1}\subset H_i'\subset H_i\subset\cdots.$$

For each integer $i\geq 1$, let

$$(7)\qquad S_{i-1}=H_{i-1}^{|w_{2i-2}|^{-1}},\qquad S_i'=H_i'^{|w_{2i-1}|^{-1}}.$$

Since $B^{|w_i|}\subset E^{|w_i|}$ and $B^{|w_i|}\subset A^{|w_i|}$ for each integer $i\geq 0$, we can see from (5) that

$$(8)\qquad H_{i-1}=H_1'\cap B^{|w_{2i-2}|}\quad\text{and}\quad H_i'=H_i\cap B^{|w_{2i-1}|}$$

for each integer $i\geq 1$. Conjugating the groups in the first equation of (8) by $|w_{2i-2}|^{-1}$, referring to (7), we get

$$S_{i-1}=H_i'^{|w_{2i-2}|^{-1}}\cap B=S_i'^{|w_{2i-1}||w_{2i-2}|^{-1}}\cap B=S_i'^{e_i}\cap B.$$

Conjugating the groups in the second equation of (8) by $|w_{2i-1}|^{-1}$, and referring to (7), we get

$$S_i'=H_i^{|w_{2i-1}|^{-1}}\cap B=S_i^{|w_{2i}||w_{2i-1}|^{-1}}\cap B=S_i^{a_i}\cap B.$$

Thus for each integer $i \geq 1$, we have

$$S_{i-1} = S_i'^{e_i} \cap B \quad \text{and} \quad S_i' = S_i^{a_i} \cap B \tag{9}$$

It follows that

$$\begin{aligned} S_i'^{e_i} \subset B &\Leftrightarrow S_{i-1} = S_i'^{e_i} \quad \text{(by (9))} \\ &\Leftrightarrow S_{i-1}^{|w_{2i-2}|} = S_i'^{e_i|w_{2i-2}|} \\ &\Leftrightarrow S_{i-1}^{|w_{2i-2}|} = S_i'^{|w_{2i-1}|} \\ &\Leftrightarrow H_{i-1} = H_i'. \end{aligned} \tag{10}$$

Similarly it follows that

$$S_i^{a_i} \subset B \Leftrightarrow S_i' = S_i^{a_i} \Leftrightarrow H_i' = H_i. \tag{11}$$

Since the nested chain (6) is non-stationary, one concludes from (10) and (11) that there are infinitely many integers $i \geq 1$ for which either

$$S_i'^{e_i} \not\subset B \quad \text{or} \quad S_i^{a_i} \not\subset B \tag{12}$$

The following lemma summarizes the preceding discussion (0.17–0.20).

0.21. LEMMA. *Let H be a subgroup of $G = A *_B E$ such that the graph of groups $(\mathcal{H}, Y)$ (as in 0.12) is a filtering tree of groups which has no maximal vertex. Then there exist sequences $\{a_i\}_{i=1}^{\infty} \subset I$ and $\{e_i\}_{i=1}^{\infty} \subset J$ (I and J are as in 0.1) and subgroups $S_{i-1}, S_i' \subset B$, $i \geq 1$ satisfying:*

(a) *$S_{i-1} = S_i'^{e_i} \cap B$ and $S_i' = S_i^{a_i} \cap B$ for $i \geq 1$.*

(b) *There are infinitely many integers $i \geq 1$ for which either $S_i'^{e_i} \not\subset B$, or $S_i^{a_i} \not\subset B$ such that, upon letting $\{w_j\}_{j=1}^{\infty} \subset \mathcal{W}$ (see 0.7) be defined by $w_{-1} = X_I$, $w_0 = x_J$, $w_{2i-1} = e_i w_{2i-2}$ and $w_{2i} = a_i w_{2i-1}$ for $i \geq 1$, and upon letting $H_{i-1} = S_{i-1}^{|w_{2i-2}|}$, $H_i' = S_i'^{|w_{2i-1}|}$ for $i \geq 1$, then we have a nested, non-stationary chain*

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_{i-1} \subset H_i' \subset H_i \subset \cdots$$

in G, and H is conjugate in G to the union.

Moreover, the argument above can easily be retraced to obtain the following converse to Lemma 0.21.

0.22. LEMMA. *Suppose we are given sequences $\{a_i\}_{i=1}^{\infty} \subset I$ and $\{e_i\}_{i=1}^{\infty} \subset J$ (I and J as in 0.2) and subgroups $S_{i-1}, S_i' \subset B$, $i \geq 1$, satisfying:*

(a) *$S_{i-1} = S_i'^{e_i} \cap B$ and $S_i' = S_i^{a_i} \cap B$ for $i \geq 1$.*

(b) *There are infinitely many integers $i \geq 1$ for which either $S_i'^{e_i} \not\subset B$ or $S_i^{a_i} \not\subset B$. Let $\{w_j\}_{j=-1}^{\infty} \subset \mathcal{W}$ (see 0.7) be defined by $w_{-1} = x_I$, $w_0 = x_J$, $w_{2i-1} = e_i w_{2i-2}$ and $w_{2i} = a_i w_{2i-1}$ for $i \geq 1$, and let $H_{i-1} = S_{i-1}^{|w_{2i-2}|}$, $H_i' = S_i'^{|w_{2i-1}|}$ for $i \geq 1$. Then we have a nested, non-stationary chain.*

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_{i-1} \subset H_i' \subset H_i \subset \cdots$$

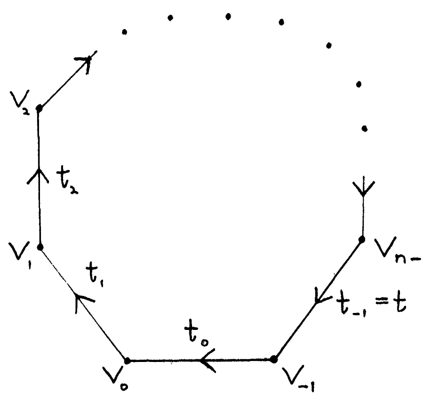
in G, and upon letting H be the union of this chain, the graph of groups $(\mathcal{H}, Y)$ (as in 0.12) is a filtering tree which contains no maximal vertex.

0.23. *Suppose H is a subgroup of $G = A *_B E$, and that $(\mathcal{H}, Y)$, D, and $(\mathcal{H}', Y')$ are as in 0.13. We further assume that the graph Y' consists of one vertex v' and one edge t',*

$$Y' = v' \bullet\!\!\circlearrowleft t',$$

and that t' is a directed edge. The fact that t' is directed means that one of the monomorphisms $\mathcal{H}_{t'} \rightrightarrows \mathcal{H}'_{v'}$ is an isomorphism. In this case, D has one connected component, which is, of course, a filtering tree, and Y has precisely one edge t which is not in D. The fact that t' is directed implies that t is connected to a maximal vertex v_t in D, and that t is directed away from v_t. (This is all immediate from the construction of $(\mathcal{H}', Y')$ in [1].)

0.24. Let v_i be the other edge in Y to which t is connected. Since v_i and v_t are connected in D by a unique geodesic, which is directed toward v_t, we have a circuit



in Y which is directed clockwise (as indicated). One can easily see that when we remove any one of the edges t_i, $0 \le i \le n-2$, the remaining graph of groups D_i is a filtering tree, and v_{i-1} is a maximal vertex in D_i. Thus the reduced graph of groups $(\mathcal{H}', Y')$ could have been constructed using D_i instead of D, and it would be a directed loop. We see then, that after possibly replacing D by some D_i with $0 \le i \le n-2$, and relabeling the vertices in $\mathfrak{G}$ (by rotation), we may assume that $v_i = v_{-1}$ is the image of a vertex in $\tilde{X}$ which is of type A (see 0.7). Once this adjustment is made, we see that v_i comes from a vertex of type A in $\tilde{X}$ if i is odd, and v_i comes from a vertex of type E in $\tilde{X}$ if i is even. Hence n is even; say $n = 2r$. We may proceed from here in a manner similar to that of 0.17–0.20 to obtain the following lemma.

0.25. LEMMA. *Let H be a subgroup of $G = A *_B E$ such that the graph of groups $(\mathcal{H}', Y')$ (as in 0.13) is a directed loop. There exists an integer $r \ge 1$*

and elements $a_1, \dots, a_r \in I$, $e_1, \dots, e_r \in J$, $b \in B$, *and subgroups* $S_0, S_1', S_i, \dots, S_r', S_r$ *of* B *satisfying*

(a) $S_{i-1} = S_i'^{e_i} \cap B$ *and* $S_i' = S_i^{a_i} \cap B$ *for* $i = 1, \dots, r$,

(b) $S_r = S_0^b$,

such that, upon letting $w_{-1}, w_0, w_1, \dots, w_{2r} \in \mathcal{W}$ *(see 0.7) be defined by* $w_{-1} = x_I$, $w_0 = x_J$, $w_{2i-1} = e_i w_{2i-2}$ *and* $w_{2i} = a_i w_{2i-1}$ *for* $i = 1, \dots, r$, *and letting* $H_{i-1} = S_{i-1}^{|w_{2i-2}|}$ *for* $i = 1, \dots, r+1$, $H_i' = S_i'^{|w_{2i-1}|}$ *for* $i = 1, \dots, r$, *and letting* $g = b\,|w_{2r}|$ $(= b a_r e_r \cdots a_1 e_1)$ *we have a nested chain*

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_r' \subset H_r$$

and $H_0^g = H_r$ *and* H *is conjugate in* G *to the subgroup* $\langle H_0, g\rangle$ *of* G. *Since* $H_0^g = H_r \supset H_0$ *we have a nested chain*

$$H_0 \subset H_0^g \subset H_0^{g^2} \subset \cdots \subset H_0^{g^{j-1}} \subset H_0^{g^j} \subset \cdots.$$

Let F *be the union of this chain. Then* $\langle H_0, g\rangle = F \ltimes \langle g\rangle$.

Again, there is a converse to the preceding lemma.

0.26. LEMMA. *Suppose we are given an integer* $r \geq 1$, *and elements* $a_1, \dots, a_r \in I$, $e_1, \dots, e_r \in J$ *(I and J as in 0.1),* $b \in B$, *and subgroups* $S_0, S_1, S_1', \dots, S_r, S_r'$ *of* B *satisfying*

(a) $S_{i-1} = S_i'^{e_i} \cap B$ *and* $S_i' = S_i^{a_i} \cap B$ *for* $i = 1, \dots, r$,

(b) $S_r = S_0^b$.

Let $w_{-1}, w_0, w_1, \dots, w_{2r} \in \mathcal{W}$ *(see 0.7) be defined by* $w_{-1} = x_I$, $w_0 = x_J$, $w_{2i-1} = e_i w_{2i-2}$ *and* $w_{2i} = a_i w_{2i-1}$ *for* $i = 1, \dots, r$, *and let* $H_{i-1} = S_{i-1}^{|w_{2i-2}|}$ *for* $i = 1, \dots, r+1$, $H_i' = S_i'^{|w_{2i-1}|}$ *for* $i = 1, \dots, r$, *and let* $g = b\,|w_{2r}|$ $(= b a_r e_r \cdots a_1 e_1)$. *Then we have a nested chain*

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_r \subset H_r$$

and $H_0^g = H_r (\supset H_0)$.

Let F *be the union of the nested chain*

$$H_0 \subset H_0^g \subset H_0^{g^2} \subset \cdots \subset H_0^{g^{j-1}} \subset H_0^{g^j} \subset \cdots$$

and let $H = \langle F, g\rangle (= \langle H_0, g\rangle)$. *Then* H *is the semidirect product* $F \rtimes \langle g\rangle$, *and* g *is of infinite order. Let* v *be the image in* Y *of the vertex* $v(x_I)$ *and let* t *be the image in* Y *of* $t(x_I, x_J)$. *Then* $D = Y - \{t\}$ *is a filtering tree, in fact it is a maximal filtering forest in* Y, *and* v *is a maximal vertex. The reduced graph of groups* $(\mathcal{H}', Y')$ *with respect to* D *is a directed loop.*

Furthermore, let us record that we can extend the given data to sequences $\{a_i\}_{i=1}^{\infty} \subset I$, $\{e_i\}_{i=1}^{\infty} \subset J$, *and subgroups* $S_{i-1}, S_i' \subset B$, for $i \geq 1$, in such a way that $H_{i-1}^g = H_{i-1+r}$ and $H_i'^g = H_{i+r}'$ for $i \geq 1$. Hence F is the union of

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_{i-1} \subset H_i' \subset H_i \subset \cdots.$$

0.27. I can now give a proof of Theorem 0.3, based on the preceding lemmas, and the main result of [1]. Let H be as in the theorem. Since H is abelian, then according to §6 of [1], $(\mathscr{H}', Y')$ is either a point or a doubly directed loop.

If Y' is a point then Y is a filtering tree. If Y has a maximal vertex, then H is conjugate to a subgroup of A or E (see 0.15), and so (1) holds. If Y has a maximal vertex, then it follows from Lemma 0.21 that (3) holds.

If Y' is a doubly directed loop, we can apply Lemma 0.25. In the notation of 0.25, we see that H is conjugate in G to the group $\langle F, g\rangle$, which is a semidirect product $F \rtimes \langle g\rangle$. But since H is abelian, we have $\langle F, g\rangle = F \times \langle g\rangle$. Now, F is the union of the chain $H_0 \subset H_0^g \subset H_0^{g^2} \subset \cdots$ which, in our situation, is stationary, since H is abelian. Therefore $F = H_0 = S_0 \subset B$. The element g, being cyclicly reduced, is not conjugate to any element of A or E. And so (2) holds.

Clearly (1), (2), and (3) are mutually exclusive, so the theorem is proved.

0.28. Corollary. *Suppose H is an abelian subgroup of $G = A *_B E$, and suppose B is finite. Then H is an abelian subgroup of type* 1 *or type* 3.

Proof. The finiteness of B rules out the possibility of the non-stationary chain of situation (2) in 0.3.

0.29. Corollary. *Suppose B is normal in both A and E, and suppose H is an abelian subgroup of $G = A *_B E$. Then precisely one of the following holds.*

(a) *H is conjugate in G to a subgroup of A or E.*

(b) *$H = F \times \langle g\rangle$, where $F \subset B$, and g is not conjugate to any element of A or E. (Hence g is of infinite order.)*

Proof. Since B is normal in both A and E, B is normal in G. We see, then, that situation (2) of 0.3 cannot occur, because each H_i—and hence H—would be contained in B. Thus we are left with the possibilities (1) and (3). In case (3), F must be contained in B.

The following corollary is a well known fact.

0.30. Corollary. *Suppose $B = \{1\}$, i.e., $G = A * E$, and suppose H is an abelian subgroup of G. Then H is conjugate in G to a subgroup of A or E, or $H = \langle g\rangle$ (infinite cyclic), where g is not conjugate to any element of A or E.*

Proof. This immediate from either of 0.28 and 0.29.

0.31. *Example.* The group $PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\{\pm 1\}$ is isomorphic to $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$ (see [12, Chapter II]). The only non-trivial abelian subgroup of $PSL_2(\mathbf{Z})$ are the conjugates of $\mathbf{Z}/2\mathbf{Z}$, the conjugates of $\mathbf{Z}/3\mathbf{Z}$, and the infinite cyclic subgroups.

0.32. *Example.* The group $SL_2(\mathbf{Z})$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} *_{\mathbf{Z}/4\mathbf{Z}} \mathbf{Z}/6\mathbf{Z}$ where $\mathbf{Z}/4\mathbf{Z}$ is generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\mathbf{Z}/6\mathbf{Z}$ is generated by

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

(ibid.). The non-trivial abelian subgroups of $SL_2(\mathbf{Z})$ are the conjugates of $\mathbf{Z}/4\mathbf{Z}$, the conjugates of $\mathbf{Z}/6\mathbf{Z}$, the conjugates of $\mathbf{Z}/3\mathbf{Z}(\subset \mathbf{Z}/6\mathbf{Z})$, the group $\mathbf{Z}/2\mathbf{Z}$, the infinite cyclic subgroups, and the subgroups of the form $(\mathbf{Z}/2\mathbf{Z})\times \langle g\rangle$, where $\langle g\rangle$ is infinite cyclic. This all follows form Corollary 0.29.

0.33. Definition. Given $G = A *_B E$, and H any subgroup of G, we say that H is a subgroup of *bounded length* if there exists an upper bound for the length of the elements of H (see 0.1). We say that H is of *unbounded length* if no such bound exists.

0.34. One readily verifies that if $H \subset G$ is a subgroup of bounded length, then any subgroup which is conjugate to H is of bounded length. Furthermore, if H contains an element h which is cyclicly reduced, then H is of unbounded degree, since length $(h)^d = d$ length (h). It follows that if H contains an element which is not conjugate to any element of A or E, then H is of unbounded length, because any such element is conjugate to a cyclicly reduced element (see 1.3 of [12]).

0.35. Proposition. *Suppose H is an abelian subgroup of $G = A *_B E$. Then H is of bounded length if and only if H is conjugate to a subgroup of A or E (i.e. H is of type* 1).

(Remark. This proposition is probably true without the assumption that H is abelian.)

Proof. The "if" is clear, in view of the remarks in 0.34. We must show that if H is abelian and of bounded length, H cannot be of type 2 or 3 (see Theorem–Definition 0.3). Any subgroup of type 3 contains an element which is not conjugate to any element of A or E, and so H cannot be of type 3.

Suppose now that H is of type 2. This means that the graph of groups $(\mathcal{H}, Y)$ is a filtering tree with no maximal vertex. By Lemma 0.21, there exists data $\{a_i\}_{i=1}^{\infty} \subset I$, $\{e_i\}_{i=1}^{\infty} \subset J$, and $S_{i-1}, S_i \subset B$, for $i \geq 1$, satisfying (a) and (b) of 0.21, such that upon letting H_{i-1}, H_i' be defined as in 0.21, H is conjugate to the union of

$$H_0 \subset H_1' \subset H_1 \subset \cdots \subset H_{i-1} \subset H_i' \subset H_i \subset \cdots.$$

In fact, we may assume that H is the union, since replacing H by a conjugate doesn't alter the bounded length condition. Condition (b) of 0.21 guarantees that for any integer m, there exists $n > m$ such that either $S_n'^{e_n} \not\subset B$ or $S_n^{a_n} \not\subset B$. Assume the latter. Let $s \in S_n$ such that $s^{a_n} \notin B$. Since $H_n = S_n^{a_n e_n \cdots a_1 e_1}$, then $s^{a_n e_n \cdots a_1 e_1} = t$ is in H. The length of t is $2n-1$, since

$s^{a_n} \notin B$. We can do a similar thing in the case $S_n'^{e_n} \not\subset B$. This shows that H is of unbounded length—a contradiction—and so the proposition is proved.

0.36. *Example.* Let k be a field, and let $k[T]$ be the polynomial ring in one variable over k. The group $GL_2(k[T])$ is a free product with amalgamation as follows: $GL_2(k[T]) = GL_2(k) *_{B_2(k)} B_2(k[T])$ where B_2 denotes the lower triangular subgroup (see [12], Chapter II).

The set

$$I = \left\{ \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \middle| x \in k \right\}$$

forms a system of non-trivial right coset representatives of $GL_2(k)$ modulo $B_2(k)$. This fact is easily seen, and will be demonstrated in 1.6. The set

$$J = \left\{ \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \middle| f \in k[T], \quad f(0) = 0, \quad f \neq 0 \right\}$$

forms a system of non-trivial coset representatives of $B_2(k[T])$ modulo $B_2(k)$.

Let

$$t = \begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in B_2(k).$$

A direct computation shows that if, for some $a \in I$, we have $t^a \in B_2(k)$, then $w = 0$. On the other hand, if there exists $e \in J$ such that $t^e \in B_2(k)$, then $u = v$. Obviously, if both of these conditions hold, then t is a scalar matrix. Let us denote by $C_2(k)$ the scalar matrices in $GL_2(k)$. $C_2(k)$ is the center of $GL_2(k[T])$.

We claim that there are no abelian subgroups of type 2 in $GL_2(k[T])$. In light of Lemma 0.21, suppose there exist sequences $\{a_i\}_{i=1}^{\infty} \subset I$, $\{e_i\}_{i=1}^{\infty} \subset J$ and subgroups $S_{i-1}, S_i' \subset B_2(k)$, $i \geq 1$ satisfying (a) of the theorem. We will show that (b) cannot possibly be satisfied, which, according to the theorem, will prove the claim.

Let $t \in S_i'$ for some $i \geq 1$. According to condition (a), there exists $s \in S_i$ such that $t = s^{a_i}$. Therefore s is of the form

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

with $u, v \in k^*$. There also exists $r \in S_{i+1}'$ such that $s = r^{e_{i+1}}$. If

$$e_{i+1} = \begin{pmatrix} 1 & 0 \\ f & 0 \end{pmatrix},$$

then

$$e_{i+1}^{-1} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \in J,$$

and since $s^{(e_{i+1}^{-1})}=r\in B$, we must have $u=v$. Therefore $s\in C_2(k)$. Since $t=s^{a_i}$, $t=s$. This shows that $S_i'\subset C_2(k)$. It follows from (a) that $S_{i-1}\subset C_2(k)$. Since S_{i-1}, $S_i'\subset C_2(k)$ for $i\geq 1$, we see (b) does not hold.

Now suppose $H\subset GL_2(k[T])$ is an abelian subgroup of type 3. Then, by Lemmas 0.25 and 0.26, we have the same data $\{a_i\}, \{e_i\}, S_{i-1}, S_i$ as above, with $S_{i-1}=S_i'^{e_i}\cap B$ and $S_i'=S_i^{a_i}\cap B$, and an element $b\in B_2(k)$, $g=ba_re_r\cdots a_1e_1$ such that H is conjugate in $GL_2(k[T])$ to $F\times\langle g\rangle$ where F is the union of

$$H_0\subset H_0^g\subset H_0^{g^2}\subset\cdots\subset H_0^{g^{j-1}}\subset H_0^{g^j}\subset\cdots. \tag{13}$$

Just as in the last paragraph, we can argue that $S_{i-1}, S_i'\subset C_2(k)$. It follows from (13) that $F=H_0=S_0$, and that H is actually equal to $F\times\langle g'\rangle$ where g' is conjugate to g.

We have proved the following proposition.

0.37. PROPOSITION. *Let k be a field. Suppose H is an abelian subgroup of $GL_2(k[T])$. Then precisely one of the following situations holds.*

(a) *H is conjugate in $GL_2(k[T])$ to a subgroup of $GL_2(k)$ or $B_2(k[T])$.*

(b) *$H=F\times\langle g\rangle$ where $F\subset C_2(k)$, and g is not conjugate to any element of $GL_2(k)$ or $B_2(k[T])$ (hence g is of infinite order).*

0.38. *Remark.* We define the *degree* of an element $\gamma\in GL_2(k[T])$ (or $GL_n(k[T])$ to be the maximum of the degrees of its entries—i.e. the degree of γ as an element of the graded ring of 2×2 matrices with coefficients in $k[T]$. One can prove by an easy induction argument that if $a_1,\ldots,a_r\in I$, $e_1,\ldots,e_r\in J$ (I and J as in 0.36), and if

$$e_i=\begin{pmatrix}1 & 0\\ f_i & 1\end{pmatrix},$$

with $d_i=\deg(f_i)$, then

$$\deg(a_1e_1,\ldots,a_re_r)=\deg(e_1a_1,\ldots,e_ra_r)$$
$$=\deg(e_1a_1,\ldots,e_{r-1}a_{r-1}e_r)=d_1+\cdots+d_r.$$

(In fact, this is one way to see that $GL_2(k)*_{B_2(k)}B_2(k[T]\to GL_2(k[t])$ is an isomorphism.) Thus if $\gamma=a_re_r\cdots a_1e_1$, and if $d=d_1+\cdots+d_r$, then $\deg(\gamma^n)=nd$. In particular, we can see that for any $\gamma\in GL_2(k[T])$,

$$\deg(\gamma)>(\text{length }\gamma-1)/2.$$

Let $H\subset GL_2(k[T])$. We say that H is a *subgroup of bounded degree* if there is an upper bound for the degrees of the elements of H. It follows from the above remarks that if H is of bounded degree, then H is of bounded length. Therefore, we can apply Proposition 0.35 to get:

0.39. PROPOSITION. *Any abelian subgroup of $GL_2(k[T])$ which is of bounded degree is conjugate either to a subgroup of $GL_2(k)$ or to a subgroup of $B_2(k[T])$.*

1. The group $GA_2(k)$

1.1. Suppose k is a ring. We denote by $GA_n(k)$, or just GA_n, the group of k-automorphisms of the polynomial ring $k[X_1, \ldots, X_n]$. An element $\varphi \in GA_n$ is determined by a vector $(F_1, \ldots, F_n)$, when $F_1, \ldots, F_n \in k[X_1, \ldots, X_n]$, and so we write $\varphi = (F_1, \ldots, F_n)$. If $\varphi = (F_1, \ldots, F_n)$ and $\gamma = (G_1, \ldots, G_n)$, then

$$\gamma\varphi = (F_1(G_1, \ldots, G_n), \ldots, F_n(G_1, \ldots, G_n)).$$

1.2. The group $GL_n(k)$ is identified as a subgroup of $GA_n(k)$ by the monomorphism which sends the invertible matrix (c_{ij}) to the vector

$$\left(\sum_i c_{ij}X_i\right)_{j=1}^{n} \in GA_n(k).$$

The additive group k^n is identified as subgroup of GA_n via the monomorphism $l: k^n \to GA_n$ defined by $l(c_1, \ldots, c_n) = (X_1 + c_1, \ldots, X_n + c_n)$, and we write $\mathcal{L}_n$ for the image of l. The additive group $k[X_2, \ldots, X_n]$ is identified as a subgroup of $GA_n(k)$ via the monomorphism $e: k[X_2, \ldots, X_n] \to GA_n$ defined by

$$e(f) = (X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n);$$

we write $\mathscr{E}_n$ for the image of e. We call the elements of $\mathscr{E}_n$ *elementary* automorphisms.

1.3. The group GL_n normalizes $\mathcal{L}_n$ (in GA_2), and $\mathcal{L}_n GL_n$ is a semidirect product, the action of conjugation of GL_n on $\mathcal{L}_n$ being given by $l(c)^g = c \cdot g$, where $c \in k^n$, $g \in GL_n$ ($c \cdot g$ denotes matrix multiplication). We denote by Af_n the subgroup $\mathcal{L}_n \rtimes GL_n \subset GA_n$. Elements of Af_n are called *linear* automorphisms. One easily verifies that Af_n is isomorphic to the subgroup of $GL_{n+1}(k)$ consisting of matrices of the form

$$\begin{pmatrix} g & 0 \\ c & 1 \end{pmatrix}$$

where $g \in GL_n(k)$, $c \in k^n$. (This matrix gets identified with $g \cdot l(c) \in Af_n(k)$.) *We will often write elements of Af_n as matrices, rather than as vectors.*

1.4. The diagonal subgroup D_n of GL_n normalizes $\mathscr{E}_n$, and the action is given by

$$e(f(X_2, \ldots, X_n))^r = e(r_1 f(r_2^{-1}X_2, \ldots, r_n^{-1}X_n)) \quad \text{where } r = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \tag{14}$$

The subgroup of GA_n generated by $\mathscr{E}_n$ and D_n is a semidirect product. The subgroup of GA_n generated by $\mathscr{E}_n \rtimes D_n$ and $\mathscr{L}_n$ is a product $(\mathscr{E}_n \rtimes D_n) \cdot \mathscr{L}_n$, although neither of these groups normalizes the other. We will denote this group by E_n. Each element of E_n has the form

$$(r_1X_1 + f(X_2, \ldots, X_n) + c_1, r_2X_2 + c_2, \ldots, r_nX_n + c_n) \tag{15}$$

where $r_1, \ldots, r_n \in k^*$, $c_1, \ldots, c_n \in k$.

1.5. For the case $n = 2$, and *k a field*, the linear and elementary automorphisms generate the entire automorphism group. This was proved by Jung in [6] for the case when k has characteristic zero, and was generalized to arbitrary characteristics by van der Kulk in [13]. In fact, the group $GA_2(k)$ is the free product of the groups Af_2 and E_2, amalgamated along their intersection, which is the semidirect product $\mathscr{L}_2 \rtimes B_2$, where B_2 denotes the lower triangular subgroup of GL_2. Upon letting $A = Af_2$, $E = E_2$, and $B = \mathscr{L}_2 \rtimes B_2$, we have

$$GA_2 = A *_B E. \tag{16}$$

(See [10, p. 31] or [12, §5] for a proof of this. Both these references furnish a more complete description of the group $GA_2(k)$.)

For the rest of §1 *we will be assuming that k is a field.*

1.6. We will choose a system of non-trivial left coset representatives of A modulo B. Since $A = \mathscr{L}_2 \rtimes GL_2$, and $B = \mathscr{L} \rtimes B_2$, we can do this by choosing representatives of GL_2 modulo B_2. We claim that the set

$$I = \left\{ \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \,\middle|\, x \in k \right\} \tag{17}$$

is such a system. Given

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2$$

with $b \neq 0$, α is represented (modulo B_2) by

$$\begin{pmatrix} a/b & 1 \\ 1 & 0 \end{pmatrix},$$

since, upon letting $u = \det(\alpha)$, we have

$$\begin{pmatrix} 1/b & 0 \\ d/u & -b/u \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a/b & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, if $x \neq y$, then

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -y \end{pmatrix} = \begin{pmatrix} 1 & x-y \\ 0 & 1 \end{pmatrix} \notin B_2$$

which shows that

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$$

represent distinct left cosets modulo B_2. Hence upon identifying A with the subgroup

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}$$

of GL_3 (see 1.3), we see that the set

$$I=\left\{\begin{pmatrix} x & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x\in k\right\} \tag{18}$$

is our system of left coset representatives of A modulo B. We will let

$$a(x)=\begin{pmatrix} x & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\in I. \tag{19}$$

1.7. We now choose a system of non-trivial left coset representatives of E modulo B. Let $\overline{k[Y]}=Yk[Y]$. We claim that

$$J=\{e(f)\mid f\in \overline{k[Y]}^2, \qquad f\neq 0\} \tag{20}$$

is such a system. (See 6.2 for notation. Here we consider GA_2 to be the automorphism group of $k[X, Y]$.) This is seen as follows. Let

$$\varphi=(uX+g(Y)+r,\, vY+s)\in E$$

(see (15)), where u, $v\in k^*$, r, $s\in k$, and $g\in \overline{k[Y]}$. Let

$$\beta=\begin{pmatrix} \dfrac{1}{u} & 0 & 0 \\ 0 & \dfrac{1}{v} & 0 \\ \dfrac{-r-g\left(-\dfrac{s}{v}\right)}{u} & -\dfrac{s}{v} & 1 \end{pmatrix}\in B.$$

Writing β in vector notation, we have

$$\beta=\left(\frac{1}{u}X-\frac{r+g\left(-\dfrac{s}{v}\right)}{u},\ \frac{1}{v}Y-\frac{s}{v}\right).$$

Direct computation shows that

$$\beta\varphi=e(f')\in \mathscr{E}_2 \quad \text{where} \quad f'=g\left(\frac{1}{v}Y=\frac{s}{v}\right)-g\left(-\frac{s}{v}\right)\in k[Y].$$

Write $f' = a_1Y + \cdots + a_dY^d$. Since $\varphi \notin B$, then $d > 1$. Now $e(-a_1Y) \in B$, and

$$e(-a_1Y)\beta\varphi = e(f) \quad \text{where} \quad f = a_2X^2 + \cdots + a_dX^d \in \overline{k[Y]}^2.$$

Thus we see that φ is represented by $e(f)$ modulo B. Furthermore, if $f_1, f_2 \in \overline{k[Y]}^2 - \{0\}$, and if $f_1 \neq f_2$, then

$$e(f_1)e(f_2)^{-1} = e(f_1)e(-f_2) = e(f_1 - f_2) \notin B,$$

since $f_1 - f_2$ is not linear. This shows that the elements of J represent distinct left cosets modulo B.

1.8. DEFINITION. Given $\varphi = (F, G) \in GA_2(k)$ we define the degree of φ ($\deg(\varphi)$) to be maximum of the total degrees of the polynomials F and G. Given a subgroup $H \subset GA_2(k)$, we say that H is of *bounded degree* if there is an upper bound for the degrees of the elements of H; we say that H is of *unbounded degree* if no such bound exists.

1.9. PROPOSITION. *Suppose we are given $a_1, \ldots, a_r \in I$, $e_1, \ldots, e_r \in J$ (as defined in* (62) *and* (64)), *with $e_1 = e(f_i)$, and $d_i = \deg(f_i)$ for $i = 1, \ldots, r$. Then*

$$\deg(a_1e_1 \cdots a_re_r) = \deg(e_1a_1 \cdots e_ra_r) = \deg(e_1a_1 \cdots e_{r-1}a_{r-1}e_r) = \prod_{i=1}^{r} d_i.$$

Proof. It is evident that multiplication of an automorphism on the right or left by a linear automorphism does not alter its degree, and so we only need to prove

$$\deg(a_1e_1 \cdots a_re_r) = \prod_{i=1}^{r} d_i.$$

Let $\varphi = a_1e_1 \cdots a_re_r$. We will prove the following statement, by induction: If $\varphi = (F, G)$, then $\deg(\varphi) = \deg(F) = \prod_{i=1}^{r} d_i$. For $r = 0$ this makes sense, and is obviously true. Suppose it is true for $r - 1$. Let

$$\varphi' = a_1e_1 \cdots a_{r-1}e_{r-1} = (F', G').$$

Then by induction, $\deg(\varphi') = \deg(F') = \prod_{i=1}^{r-1} d_i$. Let

$$a_r = a(x_r) = \begin{pmatrix} x_r & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\varphi = \varphi' a_re_r$ is given by $(x_rF' + G' + f_i(F'), F')$; clearly the degree of $x_rF' + G' + f_i(F')$ is $d_i \deg(F')$, since $\deg(F') \geq \deg(G')$. Thus we have $\varphi = (F, G)$ and $\deg(\varphi) = \deg(F) = \prod_{i=1}^{r} d_i$ as required.

1.10. It follows from Proposition 6.9 that for any $\varphi \in GA_2$,

$$\deg(\varphi) \geq 2^{(\mathrm{length}(\varphi)-1)/2} \tag{21}$$

(see 0.1). Hence, if H is a subgroup of GA_2 of bounded degree, then H is of bounded length (see Definition 0.33). The following proposition is then immediate from Proposition 0.35.

1.11 PROPOSITION. *Let k be a field. Suppose H is an abelian subgroup of bounded degree in $GA_2(k)$. Then H is conjugate either to a subgroup of $Af_2(k)$, or to a subgroup of $E_2(k)$.*

1.12. I wonder if some analogue of Proposition 1.11 is true for $GA_n(k)$, $n>2$. If so it could be useful in treating the polynomial cancellation problem:

Suppose R is a k-algebra such that $R[Y]\cong_k k[X_i, \ldots, X_n]$. Then is

$$R \cong_k k[X_1, \ldots, X_{n-1}]?$$

(We refer the reader to [14] for a further discussion of this problem.) In the situation $R[Y]\cong k[X_1, \ldots X_d]$, we get two abelian subgroups of $GA_n(k)$ of bounded degree in the following ways. Given $c\in k^*$, let $\varphi_c: R[Y]\to R[Y]$ be the R-automorphism defined by $Y\mapsto cY$. Since $R[Y]=k[X_1, \ldots X_d]$, $c\mapsto \varphi_c$ defines a monomorphism $k^*\to GA_n(k)$. In addition, for any $a\in k^*$, we can define $\gamma_a: R[Y]\to R[Y]$ by $Y\mapsto Y+a$. Then $a\mapsto \gamma_a$ defines an additive monomorphism $k^+\to GA_n(k)$. It is not hard to see that these inclusions yield subgroups of bounded degree. It would be extremely helpful if one knew that such subgroups were conjugate in $GA_2(k)$ to some more managable subgroup (e.g. Af_n).

1.13. In order to be able to apply the theorems of §0 to GA_2, we will study the effects of conjugating elements of B by elements of I and J.

Let $x\in k$, and let

$$s=\begin{pmatrix} u & 0 & 0\\ w & v & 0\\ r_1 & r_2 & 1\end{pmatrix}\in B.$$

Computing, we see that

$$s^{a(x)}=\begin{pmatrix} 0 & 1 & 0\\ 1 & -x & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} u & 0 & 0\\ w & v & 0\\ r_1 & r_2 & 1\end{pmatrix}\begin{pmatrix} x & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix} xw+v & w & 0\\ xu-x^2w-xv & u-xw & 0\\ xr_1+r_2 & r_1 & 1\end{pmatrix}$$

(22) We see that $s^{a(x)}\in B$ *if and only if* $w=0$, *and in this case we have*

$$s^{a(x)}=\begin{pmatrix} v & 0 & 0\\ x(u-v) & u & 0\\ xr_1+r_2 & r_1 & 1\end{pmatrix}.$$

Now let $f=c_2Y^2+\cdots+c_dY^d\in\overline{k[Y]}^2$, $c\neq 0$. Writing s and $e(f)$ in vector notation, we see that

$$\begin{aligned}s^{e(f)}&=(X-f(Y),\,Y)(uX+wY+r_1,\,vY+r_2)(X+f(Y),\,Y)\\&=(uX+wY-uf(Y)+f(vY+r_2)+r_1,\,vY+r_2)\end{aligned}$$

In particular, we see that

(23) $s^{e(f)}\in B$ *if and only if* $f(vY+r_2)-uf(Y)$ *is linear or constant.*

Now, for any v, $r\in k$,

$$f(vY+r)=\sum_{i=2}^{d}v^i\left(\sum_{t=i}^{d}\binom{t}{i}c_tr^{t-i}\right)Y^i+vD_1f(r)Y+f(r).$$

(D_i denotes ith derivative.) If the characteristic of k is zero, the above can be written

$$f(vY+r)=\sum_{i=0}^{d}\frac{v^iD_if(r_2)}{i!}Y^i.$$

Clearly, then, $f(vY-r_2)-uf(Y)$ *is linear or constant if and only if*

$$v^i\sum_{t=i}^{d}\binom{t}{i}c_tr_2^{t-1}=uc_i \tag{24}$$

for $i=2,\ldots,d$. *In this case we have*

$$v^d=u \tag{25}$$

(putting $i=d$ in (24)), and

$$s^{e(f)}=\begin{pmatrix}u&0&0\\w+vD_1f(r_2)&v&0\\r_1+f(r_q)&r_2&1\end{pmatrix}. \tag{26}$$

1.14. We will study how abelian subgroups of types 2 and 3 occur as subgroups of $GA_2(k)$ (see Theorem–Definition 0.3). In light of Lemmas 0.21, 0.22, 0.25, and 0.26, the following assumptions are in order.

Suppose we are given sequences $\{a(x_i)\}_{i=1}^{\infty}\subset I$ *and* $\{e(f_i)\}_{i=1}^{\infty}\subset J$, *and abelian subgroups* $S_{i-1}, S_i'\subset B$, $i\geq 1$, *such that, upon letting* $a_i=a(x_i)$ *and* $e_i=e(f_i)$, *we have*

$$S_{i-1}=S_i'^{e_i}\cap B,\quad S_i'=S_i^{a_i}\cap B\quad \textit{for } i\geq 1. \tag{27}$$

These will be the standing assumptions in 1.14–1.19.

As in 0.24 and 0.26, we let

$$H_{i-1}=s_{i-1}^{a_{i-1}e_{i-1}\cdots a_1e_1},\qquad H_i'=S_i'^{e_ia_{i-1}\cdots a_1e_1}. \tag{28}$$

It follows from (27) that

$$H_0\subset H_1'\subset H_1\subset\cdots\subset H_{i-1}\subset H_i'\subset H_i\subset\cdots. \tag{29}$$

Let F be the union of this chain. For each of the polynomials $\{f_i\}_{i=1}^{\infty}$, write

$$f_i = c_{i,2}Y^2 + \cdots + c_{i,d_i}Y^{d_i}, \quad c_{i,d_i} \neq 0. \tag{30}$$

The following statements are immediate from (22), (23), and (26).

$$S_i' = \left\{ \begin{pmatrix} v & 0 & 0 \\ x_i(u-v) & u & 0 \\ r_2 - x_i r_i & r_1 & 1 \end{pmatrix} \middle| \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in S_i \text{ and } w = 0 \right\}, \tag{31}$$

$$S_{i-1} = \left\{ \begin{pmatrix} u & 0 & 0 \\ w + D_1 f_i(r_2) & 0 & 0 \\ r_1 + f_i(r_2) & r_2 & 1 \end{pmatrix} \middle| \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in S_i' \text{ and } f_i(vY + r_2) - uf_i(Y) \text{ is linear} \right\}. \tag{32}$$

Since $B = \mathscr{L}_2 \rtimes B_2$, we have the exact sequence $1 \to \mathscr{L}_2 \to B \to B_2 \to 1$ and so for S_{i-1} and S_i' we have the exact sequences

$$1 \to K_{i-1} \to S_{i-1} \to T_{i-1} \to 1, \qquad 1 \to K_i' \to S_i' \to T_i' \to 1 \tag{33}$$

of abelian groups, where $K_{i-1} = S_{i-1} \cap \mathscr{L}_2$, *$T_{i-1}$ is the image of S_{i-1} in B_2; and similarly for K_i' and T_i'.*

We will write $U(n)$ for the group of n-th roots of unity in k.

1.15. PROPOSITION. (*a*) *Suppose*

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T_{i-1}.$$

Then $u = v^{d_i}$ and $v = u^{d_{i+1}}$. Hence u, $v \in U(d_{i+1}d_{i+1} - 1)$.

(b) *Suppose*

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T_i'.$$

Then $v = u^{d_{i+1}}$ and $u = v^{d_{i+2}}$. Hence u, $v \in U(d_{i+1}d_{i+2} - 1)$.

(c) *The homomorphism $T_i' \to U(d_{i+1}d_{i+2} - 1)$ defined by*

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \mapsto u$$

is injective, hence carries $T_i' \to$ onto a subgroup $U(n)$.

(*d*) *If $K_i = 1$, then S_i' is conjugate in A to $T_i \cap D_2 (\subset B_2 \subset B)$.*

Proof. Suppose

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T_{i-1}.$$

By (22) we see that, for some $r \in k$, $f_i(vY + r) - uf_i(Y)$ is linear, and therefore

$u = v^{d_i}$ (see (25)). Suppose

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T'_{i-1}.$$

It follows from (31), and what has just been said, that $v = u^{d_{i+1}}$.

Again, suppose

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T_{i-1}:$$

It follows from (22) that, for some $w' \in k$,

$$\begin{pmatrix} u & 0 \\ w' & v \end{pmatrix} \in T'_i,$$

and so $v = u^{d_{i+1}}$. Now suppose

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T'_i.$$

We conclude from (31), and the above, that $u = v^{d_{i+2}}$. This proves (a) and (b).

To see that the homomorphism of (c) is injective, we observe that, if

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T'_i,$$

then $v = u^{d_{i+1}}$, by (b), and $w = x_i(v-u)$, by (31).

We will now prove (d). It follows from (31) that

$$s_i'^{a(x_i)^{-1}} = \left\{ \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in S_i \,\middle|\, w = 0 \right\}.$$

Call this group S. The projection of S on B_2 is clearly $T_i \cap D_2$. Suppose s, $s' \in S$, $\neq 1$. Write

$$s = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ r & t & 1 \end{pmatrix}, \qquad s' = \begin{pmatrix} u' & 0 & 0 \\ 0 & v' & 0 \\ r' & t' & 1 \end{pmatrix}.$$

Now

$$ss' = \begin{pmatrix} uu' & 0 & 0 \\ 0 & vv' & 0 \\ u'r + r' & v't + t' & 1 \end{pmatrix} \text{ and } s's = \begin{pmatrix} u'u & 0 & 0 \\ 0 & vv' & 0 \\ ur' + r & vt' + t & 1 \end{pmatrix}.$$

Since S is abelian, $ss' = s's$, and so

$$u'r + r' = ur' + r \quad \text{and} \quad v't + t' = vt' + t.$$

Therefore

$$(u'-1)r=(u-1)r' \quad\text{and}\quad (v'-1)t=(v-1)t'.$$

Since $S \subset S_i$, and since $K_i = 1$, it follows from (a) that $u, v, u', v' \neq 1$ (because $s, s' \neq 1$). Therefore we can write

$$\frac{r}{u-1}=\frac{r'}{u'-1} \quad\text{and}\quad \frac{t}{v-1}=\frac{t'}{v'-1}.$$

Thus we have $c, d \in k$ such that for each $s \in S$, $s \neq 1$, with

$$s=\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ r & t & 1 \end{pmatrix},$$

then $c = r/(u-1)$, $d = v/(v-1)$. Upon letting

$$q=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix}$$

one easily verifies that

$$s^q=\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore S^q is the projection of S onto B_2, which is $T_i \cap D_2$. We have showed that $S'^{a(x_i)^{-1}q} = S^q = T_i \cap D_2$, which proves (d).

1.16. Proposition. *Either $T_i' = 1$ for all $i \geq 1$, or $K_i' = 1$ for all $i \geq 1$. In the later case, $K_{i-1} = 1$, for all $i \geq 1$.*

Proof. We make the following claims, for each integer $i \geq 1$.

Claim 1. If $K_i = 1$, then $K_i' = 1$.
This is apparent from (31). For if

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in K_i', \neq 1,$$

then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_2 & r_1 + x_i r_2 & 1 \end{pmatrix} \in K_i, \neq 1.$$

Claim 2. If $K_i' = 1$, then $K_{i-1} = 1$.

Suppose

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in K_{i-1}, \neq 1.$$

Then, by (76), we have

$$\begin{pmatrix} 1 & 0 & 0 \\ w & 1 & 0 \\ r_1 - f_i(r_2) & r_2 & 1 \end{pmatrix} \in S_i', \neq 1,$$

for some $w \in k$. But, by (31), $w = x_i(1-1) = 0$. Hence $K_i' \neq 1$.

Claim 3. If $K_i = 1$, or if $K_i' = 1$, then $K_j', K_{j-1} = 1$, for $j \leq i$.
This is immediate from Claims 1 and 2.
Claim 4. If $T_i = 1$, then $T_i' = 1$.
Assume $T_i = 1$. Let

$$\begin{pmatrix} u & 0 \\ w & r \end{pmatrix} \in T_i'.$$

By (31), we see that

$$\begin{pmatrix} v & 0 \\ w' & u \end{pmatrix} \in T_i$$

for some $w' \in k$, and $w = x_i(u-v)$. Since $T_i = 1$, then $w' = 0$, $u = v = 1$, and so $w = 0$. Therefore $T_i' = 1$.
Claim 5. If $T_{i+1}' = 1$, then $T_i' = 1$.
Assume $T_{i+1}' = 1$. Let

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in T_i'.$$

By (31) we see that $w = x_i(u-v)$. By (31) and (32) we see that

$$\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} \in T_{i+1}'.$$

Therefore $u = v = 1$ and $w = 0$. Therefore $T_i' = 1$.
Claim 6. One of the following holds.
(a) $T_i = 1$
(b) $K_1 = 1$
(c) Each element of K_i is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & 0 & 1 \end{pmatrix}$$

and each element of T_i is of the form

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}.$$

Assume neither of situations (a) and (b) holds. We can choose

$$s = \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ c_1 & c_2 & 1 \end{pmatrix} \in S$$

such that

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \neq 1,$$

and

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in K_i, \neq 1.$$

Write $s = s't'$, where

$$s' = \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \in B_2, \qquad t' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c_1 & c_2 & 1 \end{pmatrix} \in \mathscr{L}_2.$$

(s' and r' may not be in S_i.) Since S_i is abelian, we have $t^s = t$; and since $\mathscr{L}_2$ is abelian we have $t^{s'} = t$. The equation $ts' = s't$ tells us that

$$\begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1u + r_2w & r_2v & 1 \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix},$$

i.e., that $r_1u + r_2w = r_1$ and $r_2v = r_2$. Thus if $r_2 \neq 0$, then $v = 1$. By (a) of Lemma 1.15, $u = 1$, and hence, by the first equation above, $w = 0$. But this contradicts the fact that

$$\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \neq 1.$$

Therefore $r_2 = 0$. It follows from the first equation, above, that $u = 1$. Therefore $v = 1$ ((a) of 1.15) and

$$s = \begin{pmatrix} 1 & 0 & 0 \\ w & 1 & 0 \\ c_1 & c_2 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & 0 & 1 \end{pmatrix}$$

and the claim follows easily.

Claim 7. Either $T_i' = 1$ or $K_i' = 1$.

We examine S_i in the light of Claim 6. If $T_i = 1$, then $T_i' = 1$ by Claim 4. If $K_i = 1$, then $K_i' = 1$ by Claim 1. Now suppose (c) of Claim 6 holds. Let

$$\begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in S_i'$$

It follows from (31) that $w = x_i(v-u)$, and

$$\begin{pmatrix} v & 0 & 0 \\ 0 & u & 0 \\ r_2 & r_1 + xr_2 & 1 \end{pmatrix} \in S_i.$$

Then $u = v = 1$, by (c), and so $w = 0$. This proves that $S_i = K_i$, i.e., that $T_i = 1$. Claim 7 is proved.

Now, Proposition 1.16 follows from Claims 3, 5, and 7.

1.17. Definition. Given a polynomial $f \in k[Y]$, with $f = \sum c_i Y^i$, and $p = \operatorname{char}(k)$, we say that f is an *additive* polynomial if $c_i = 0$ whenever i is not a power of p.

(Note: For $p = 0$, this just says that $f = cY$.)

1.18. Proposition. If all but a finite number of the polynomials $\{f_i\}_{i=1}^{\infty}$ are additive, and if $T_i' = 1$ for $i \geq 1$, then the chain (29) is stationary.

Proof. We will show that $S_i = S_{i+1}'^{e_{i+1}}$ and $S_i' = S_i^{a_i}$, if $T_{i+1}' = 1$ and f_{i+1} is additive. This implies that $H_i' = H_i = H_{i+1}'$; whence the proposition.

To show that $S_i = S_{i+1}'^{e_{i+1}}$, we must show that if $s \in S_{i+1}'$, then $s^{e_{i+1}} \in B$ (since $S_i = S_{i+1}'^{e_{i+1}} \cap B$). Since $T_{i+1}' = 1$, s is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_1 & r_2 & 1 \end{pmatrix}.$$

Since f_{i+1} is additive, $f_{i+1}(Y+r) = f_{i+1}(Y) + f_{i+1}(r)$ for any $r \in K$. In particular $f_{i+1}(Y+r_2) - f_{i+1}(Y)$ is constant, and so by (23) we see that $s^{e_{i+1}} = s^{e(f_{i+1})} \in B$.

Now, by (32) we see that each element of S_i is of the form

$$\begin{pmatrix} u & 0 & 0 \\ w + vD_1 f_{i+1}(r_2) & v & 0 \\ r_1 + f_{i+1}(r_2) & r_2 & 1 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} u & 0 & 0 \\ w & v & 0 \\ r_1 & r_2 & 1 \end{pmatrix} \in S_{i+1}'.$$

But since $T_{i+1}' = 1$, $u = v = 1$, and $w = 0$. And since f_{i+1} is additive, and $f_{i+1} \in \overline{k[Y]}^2$, $D_1 f = 0$. It follows that $S_i \subset \mathscr{L}_2$, and so, by (22), $S_i^{a_i} = S_i^{a(x_i)} \subset B$. This proves that $S_i' = S_i^{a_i}$, as required.

1.19. Proposition. *Suppose $T_i' = 1$ for all $i \geq 1$, and suppose infinitely* *many of the polynomials $\{f_i\}_{i=1}^{\infty}$ are not additive (e.g. if the chain (29) is non-stationary—by Proposition 1.18). Then the groups S_{i-1}, S_i' are finite for $i \geq 1$.*

Proof. We need this lemma.

LEMMA. *Suppose $f \in \overline{k[Y]}$ is not additive. Then there are only finitely many constants $r \in k$ for which $f(Y+r)-f(Y)$ is a constant.*

Proof of lemma. Write $f = g + \not{h}$ where $\not{h}$ is additive and g has no additive terms, i.e., $g = \sum_{i=1}^{d} b_i Y^i$, and $b_i = 0$ whenever i is a power of $p = \operatorname{char}(k)$. We may assume $b_d \neq 0$. Clearly it suffices to prove the lemma for g. For $r \in k$, we have

$$g(Y+r)-g(Y) = \sum_{i=1}^{d} \left(\sum_{t=i+1}^{d} \binom{t}{i} b_t r^{t-i} \right) Y^i + g(r)$$

Hence if $g(Y+r)-g(Y) \in k$, we must have $\sum_{t=i+1}^{d} \binom{t}{i} b_t r^{t-i} = 0$, $i = 2, \ldots, d-1$. Since d is not a power of p, at least one of the coefficients $\binom{d}{1}$, $\binom{d}{2}, \ldots, \binom{d}{d-1}$ is non-zero in k. If $\binom{d}{d-j} \neq 0$, we have

$$\sum_{t=d-j+1}^{d} \binom{t}{d-j} b_t r^{t-d+j} = 0$$

Since $b_d \binom{d}{d-j} \neq 0$, the above algebraic constraint on r is non-trivial, and so only finitely many constants r satisfy it. This proves the lemma.

Proof of the Proposition. We assume each $T_i' = 1$ and that infinitely many f_i's are not additive. Recalling the notation of 1.2, we write $l(r_1, r_2)$ for the corresponding element of $\mathscr{L}_2$. Since $T_i' = 1$, each element of S_i' is of this form.

Claim 1.

$$S_i' = \{l(r_2 + x_i r_1 - x_i f_{i+1}(r_2), r_1 + f_{i+1}(r_2)) \mid l(r_1, r_2) \in S_{i+1}' \text{ and } f_{i+1}(Y - r_2) - f_{i+1}(Y) \in k\}.$$

By (32), we deduce that the elements of S_i are the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ D_1 f_{i+1}(r_2) & 1 & 0 \\ r_1 + f_{i+1}(r_2) & r_2 & 1 \end{pmatrix}$$

where $l(r_1, r_2) \in S_{i+1}'$ and $f_{i+1}(Y + r_2) - f_{i+1}(r_2)$ is linear or constant. By (31) we see that the elements of S_i' are the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r_2 - x_i r_1 - x_i f_{i+1}(r_2) & r_1 + f_{i+1}(r_2) & 1 \end{pmatrix}$$

where $l(r_1, r_2) \in S_{i+1}'$, $f_{i+1}(Y + r_2) - f_{i+1}(Y)$ is linear or constant and $D_1 f_{i+1}(r_2) = 0$. These last two conditions say that $f_{i+1}(Y + r_2) - f_{i+1}(Y)$ is constant, since $f_{i+1} \in \overline{k[Y]}^2$. This proves Claim 1.

Claim 2. Suppose $1 \le i < j$. There exist polynomials $F, G \in k[T_1, T_2]$ with $\deg_{T_1}(G) \ge \deg_{T_1}(F)$ such that the leading (highest power) T_1-coefficient in G is a constant, and such that each element of S'_i is the form $l(F(r_1, r_2), G(r_1, r_2))$ where $l(r_1, r_2) \in S'_j$, and $f_j(Y+r_2)-f_j(Y)$ is a constant. If $i = j-1$, we take

$$F = T_2 - x_1T_1 - x_1f_j(T_2), \qquad G = T_1 + f_{i+1}(T_2)$$

and refer to Claim 1. Now we perform induction on $j-i$. By induction we have F and G satisfying the requirements in the claim, such that each element of S'_{i+1} is of the form $l(F(r_1, r_2), G(r_1, r_2))$ where $l(r_1, r_2) \in S'_j$ and $f_j(Y-r_2)-f_j(Y) \in k$. By Claim 1, we see that the polynomials

$$F' = G - x_iF - x_if_{i+1}(G), \qquad G' = F + f_{i+1}(G)$$

satisfy the requirements for S'_i.

Now, to prove the proposition, it is enough to show that S'_i is finite whenever f_{i+1} is not additive. (This follows from (27).) There are infinitely many f_j's which are not additive so suppose f_{i+1} and f_{i+j} are not additive, with $j > 1$. By Claim 2, we can choose polynomials $F, G \in k[T_1, T_2]$ such that each element of S'_{i+1} is of the form $l(F(r_1, r_2), G(r_1, r_2))$, where $l(r_1, r_2) \in S'_{i+j}$ and $f_{i+j}(Y+r_2)-f_{i+j}(r_2) \in k$. Since f_{i+j} is not additive, the lemma tells us that only finitely many constants r_2 satisfy this condition. By Claim 1 each element of S'_i is of the form

$$l(G(r_1, r_2) - x_iF(r_1, r_2) - x_if_{i+1}(G(r_1, r_2)), G(r_1, r_2) + f_{i+1}(F(r_1, r_2))) \tag{34}$$

where $l(r_1, r_2) \in S'_{i+j}$, $f_{i+j}(Y+r_2)-f_{i+j}(Y) \in k$, and

$$f_{i+1}(Y + G(r_1, r_2)) - f_{i+1}(Y) \in k. \tag{35}$$

Now f_{i+1} is not additive, and so $G(r_1, r_2)$ can take on only finitely many values in order for the above condition to be satisfied. Since the coefficient of the highest power of T_1 in $G(T_1, T_2)$ is a constant, we see that for each value of r_2, there are only finitely many values of r_1 such that $G(r_1, r_2)$ satisfies (35). Therefore, each element of S'_i is of the form (34), where r_1 and r_2 each take on only finitely many values; and so S'_i is finite.

1.20. Now we drop our assumptions, made in 1.14, and assume that H is an abelian subgroup of $GA_2(k)$ of type 2, with respect to the decomposition $GA_2 = A *_B E$ of (16) (see Theorem–Definition 0.3). This means that the graph of groups $Y = T/H$, where T is the tree on which GA_2 acts (see §0) is a filtering tree, with no maximal vertex. Thus we are in the situation of Lemma 0.21. We apply that theorem to get the data $\{a_i\}_{i=1}^{\infty} \subset I$, $\{e_i\}_{i=1}^{\infty} \subset J$, S_{i+1}, $S'_i \subset B$ as specified, and upon letting H_{i-1}, H_i be defined as in the theorem H is conjugate to the union of

$$H_0 \subset H'_1 \subset H_1 \subset \cdots \subset H_{i-1} \subset H'_i \subset H_1 \subset \cdots$$

which is non-stationary. We can now apply Propositions 1.15, 1.16, 1.18, and 1.19. We let K_{i-1}, T_{i-1}, K_i', T_i' be as in (33). According to Proposition 1.16, either each $T_i'=1$ or each $K_i'=1$. In the first case, we apply Propositions 1.18 and 1.19 to get that each S_i' $(=K_i')$ is a finite subgroup of $\mathscr{L}_2$. Thus we can say that each H_i' (and hence each H_i) is conjugate in GA_2 to a finite subgroup of $\mathscr{L}_2$. Obviously this case can't be realized if $\operatorname{char}(k)=0$, since then each element of $\mathscr{L}_2$ has infinite order. In the case $K_i'=1$ for all $i\geq 1$, we have each $K_i=1$ (by 1.16), and so by (d) and (a) of Proposition 1.15, each S_i' is conjugate in A to a subgroup of D_2 of the form

$$\left\{\begin{pmatrix} u & 0 \\ 0 & u^d \end{pmatrix} \middle|\, u\in U(n)\right\}$$

for some integers d, n, with $(d, n)=1$ (since u is a power of u^d). Whence the following theorem.

1.21. THEOREM. *If H is an abelian subgroup of type* 2 *in GA_2 (with respect to the decomposition $GA_2=A *_B E$ of* (16)), *then there exists a (non-stationary) chain*

$$H_0\subset H_1\subset\cdots\subset H_{i-1}\subset H_i\subset\cdots$$

such that $H=\bigcup_{i=0}^{\infty} H_i$, and such that (a) *each H_i is conjugate in GA_2 to a finite subgroup of $\mathscr{L}_2$, or* (b) *each H_i is conjugate to a subgroup of D_2 of the form*

$$\left\{\begin{pmatrix} u & 0 \\ 0 & u^d \end{pmatrix} \middle|\, u\in U(n)\right\}$$

where d and n are integers, depending on i, with $(d, n)=1$. If $\operatorname{char}(k)=0$, *case* (b) *holds.*

Note. We will see in Examples 2.2 and 2.5 that *both the possibilities* (a) *and* (b) *in the conclusion of Theorem* 1.21 *can be realized.*

1.22. COROLLARY. *$GA_2(k)$ has no abelian subgroup of type two in each of the following cases:*
(1) *k is finite*
(2) $\operatorname{char}(k)=0$ *and k contains only finitely many roots of* 1.

1.23. Now suppose that H is an abelian subgroup of type 3 in $GA_2=A *_B E$. This means that the reduced graph of groups Y' (see §0) is a directed loop (see Theorem–Definition 0.3). According to Lemma 0.25, and Lemma 0.26, there exists $\{a_i\}_{i=1}^{\infty}\subset I$, $\{b_i\}_{i=1}^{\infty}\subset J$, S_{i-1}, $S_i'\subset B$, $g=ba_re_r\cdots a_1e_1$, such that, upon letting H_{i-1}, H_i' be defined as in 0.25, we have

$$F=H_0=H_1'=H_1=\cdots=H_{i-1}=H_i'=H_i=\cdots$$

(everywhere stationary, since H is abelian) and H is conjugate in GA_2 to $F\times(g)\subset GA_2$. Again we can apply Propositions 1.16 and 1.15, which tell us that either F is a (possibly infinite) subgroup of $\mathscr{L}_2$, or else G is conjugate to a subgroup of D_2 of the form

$$\left\{\begin{pmatrix} u & 0 \\ 0 & u^d \end{pmatrix} \middle| u \in U(n)\right\}$$

where $(n, d) = 1$. Proposition 1.19, guarantees that, if $\operatorname{char}(k) = 0$, and if $F \subset \mathscr{L}_2$, then $F = 1$, since none of the polynomials f_i are additive, and there are no non-trivial finite subgroups of $\mathscr{L}_2$. We have proved the following theorem.

1.24. THEOREM. *If H is an abelian subgroup of type 3 in $GA_2(k)$ $(= A *_B E)$, then there exist a subgroup $F \subset H$ and an element $g \in H$ which is not conjugate to any element of A or E (and therefore has infinite order such that $H = F \times (g)$, and such that* (a) *F is conjugate in $GA_2(k)$ to a subgroup of $\mathscr{L}_2$, or* (b) *F is conjugate to a subgroup of D_2 of the form*

$$\left\{\begin{pmatrix} u & 0 \\ 0 & u^d \end{pmatrix} \middle| u \in U(n)\right\}$$

where n and d are integers and $(n, d) = 1$. If $\operatorname{char}(k) = 0$, *case* (b) *holds.*

Note. We will see in Examples 2.9 and 2.10 that *both the possibilities* (a) *and* (b) *in the conclusion of Theorem* 1.24 *can be realized.*

2. Examples of abelian subgroups in $GA_2(k)$

2.1. In this section we will display some examples of abelian subgroups of type 2 and 3 in $GA_2(k)$, for various kinds of fields k. We do this by employing the technical apparatus furnished by Lemmas 0.22 and 0.26. We will often be referring to the systems I and J of coset representatives of A and E, respectively, modulo B (see 1.6 and 1.7).

2.2. *Example.* Let k be a field which contains all the roots of 1. Let

$$a_i = a(0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in J$$

for each integer $i \geq 1$. Let $\{d_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of integers ≥ 2 such that $d_i d_{i+1} - 1$ divides $d_{i+1} d_{i+2} - 1$, and such that $p = \operatorname{char}(k)$ does not divide $d_i d_{i+1} - 1$ (If $p > 0$, we can form such a sequence by taking $d_1 = p$, $d_2 = p^2$, $d_{i+2} = p d_i d_{i+1} + d_i - p$. Then clearly

$$(d_i d_{i+1} - 1)(p d_{i+1} + 1) = d_{i+1} d_{i+2} - 1,$$

and one sees by induction that p does not divide $d_{i+1} d_{i+2} - 1$. If $p = 0$, let $d_1 = 2$, $d_2 = 3$, $d_{i+2} = d_i d_{i+1} + d_i + 1$.) Let $f_i = Y^{d_i} \in \overline{k[Y]^2}$, and let $e_i = e(f_i) \in J$.

Let

$$S_{i-1}=\left\{\begin{pmatrix} v^{d_i} & 0 & 0\\ 0 & v & 0\\ 0 & 0 & 1\end{pmatrix}\middle| v\in U(d_id_{i+1}-1)\right\},$$

$$S_i'=\left\{\begin{pmatrix} v & 0 & 0\\ 0 & v^{d_{i+1}} & 0\\ 0 & 0 & 1\end{pmatrix}\middle| v\in U(d_{i+1}d_{i+2}-1)\right\}$$

for $i\geq 1$. We observe from (22) of 1.13 that $S_i'=S_i^{a_i}=S_i^{a_i}\cap B$. From (23), (24), and (25), we conclude that

$$S_i'^{e^i}\cap B=S_i'^{e(f_i)}\cap B$$

$$=\left\{\begin{pmatrix} v & 0 & 0\\ 0 & v^{d_{i+1}} & 0\\ 0 & 0 & 1\end{pmatrix}\middle| \begin{array}{l} v\in U(d_{i+1}d_{i+2}-1),\text{ and } f_i(v^{d_{i+1}}Y)-vf_i(Y)\\ \text{is linear or constant}\end{array}\right\}.$$

The condition $f_i(v^{d_{i+1}}Y)-vf_i(Y)$ is linear or constant says that $v^{d_id_{i+1}}Y^{d_i}=vY^{d_i}$, i.e., that $v^{d_id_{i+1}-1}=1$. Since $d_id_{i+1}-1$ divides $d_{i+1}d_{i+2}-1$, the conditions $v\in U(d_{i+1}d_{i+2}-1)$ and $v^{d_id_{i+1}-1}=1$ may be replaced by the condition $v\in U(d_id_{i+1}-1)$. If $v\in U(d_id_{i+1}-1)$, and if $u=v^{d_{i+1}}$, then $v=u^{d_i}$. Thus we see that from (36) that

$$s_{i-1}=S_i'^{e_i}\cap B\subsetneq S_i'^{e_i}. \tag{37}$$

The proper containment results from the fact that $U(d_id_{i+1}-1)\subsetneq U(d_{i+1}d_{i+2}-1)$, which follows from the fact that $p\nmid d_{i+1}d_{i+2}-1$. Therefore the hypothesis of Lemma 0.21 are satisfied by these data, and so, upon letting H_{i-1}, H_i be defined as in this lemma, we take H to be the union of the non-stationary chain

$$H_0\subset H_1'\subset H_1\subset\cdots\subset H_{i-1}\subset H_i\subset H_i'\subset\cdots.$$

Then, according to the lemma, *H is an abelian subgroup of type 2 in GA_2.*

Viewing this example in the light of Theorem 1.21, we observe that case (b) of the conclusion of the theorem holds in this example, since each S_i is of the required form.

2.3. A further analysis of the preceding example shows that if we define the isomorphisms $\varphi_i\colon S_i'\to U(d_{i+1}d_{i+2}-1)$ by

$$\varphi_i\begin{pmatrix} u & 0 & 0\\ 0 & v & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{cases} u & \text{if } i \text{ is odd,}\\ v & \text{if } i \text{ is even}\end{cases}$$

then following diagram commutes:

$$\begin{array}{ccc} H'_{i-1} & \subset & H'_i \\ \downarrow & & \downarrow \\ U(d_id_{i+1}-1) & \subset & U(d_{i+1}d_{i+2}-1) \end{array}$$

This shows that H is isomorphic to the union of the groups $U(d_id_{i+1}-1)$.

Note that the requirement $p \nmid (d_id_{i+1}-1)$ was only used to insure that the containment of (37) is proper for each $i \geq 1$. This is actually more than one needs to apply Lemma 0.21. The hypothesis of this theorem is met if only the proper containment of (37) holds for infinitely many integers $i \geq 1$. Since the proper, containment for i holds precisely when $U(d_id_{i+1}-1) \subset U(d_{i+1}d_{i+2}-1)$ is proper, we can replace the condition $p \nmid d_id_{i+1}-1$ by the condition

$$U(d_1d_2-1) \subset U(d_2d_3-1) \subset \cdots \subset U(d_id_{i+1}-1) \subset U(d_{i+1}d_{i+2}-1) \subset \cdots$$

is non-stationary.

It follows that if k has infinitely many roots of 1, we can find an abelian subgroup H of type 2 in $GA_2(k)$ which is isomorphic to the group U of all roots of 1 in k, provided we can choose the sequence $\{d_i\}_{i=1}^{\infty}$ in such a way that for each integer n, n divides $d_id_{i+1}-1$ for i sufficiently large. For, if the d_i's can be so chosen, the union of the groups $U(d_id_{i+1}-1)$ is all the roots of 1. In fact, this can be done, as follows. List the prime numbers $\{p_i\}_{i=1}^{\infty}$ in order. We want to arrange that (1) $d_id_{i+1}-1$ divides $d_{i+1}d_{i+2}-1$, and that (2) $(p_1 \cdots p_i)^i$ divides $d_id_{i+1}-1$. For the sake of choosing the d_i's inductively, we will also arrange that (3) $p_i \nmid d_i$. Now, the conditions are satisfied (where they make sense) if we let $d_1=3$, $d_2=5$. Suppose we have defined $d_1, \ldots, d_{t+1}$ such that (3) is satisfied for $i=1, \ldots, t+1$, (2) is satisfied for $i=1, \ldots, t$, and (1) is satisfied for $i=1, \ldots, t-1$. Let n be a positive integer. If we were to let $d_{t+2}=nd_td_{t+1}+d_t-n$, we would have

$$(nd_{t+1}+1)(d_td_{t+1}-1)=d_{t+1}d_{t+2}-1,$$

and so (1) would be satisfied for $i=t$. By (2) and (3) for $i=t+1$ we see that none of the primes $p_1, \ldots, p_{t+1}$ divides d_{t+1}, and so the integer n can be chosen so that $nd_{t+1} \equiv -1 \pmod{(p_1 \cdots p_{t+1})^{t+1}}$, by the Chinese Remainder Theorem. With n so chosen, we see that (2) is satisfied for $i=t+1$ (since $nd_{t+1}+1$ divides $d_{t+1}d_{t+2}-1$). To be able to continue the induction, we must insure that (3) is satisfied for $i=t+2$, i.e., that $p_{t+2} \nmid d_{t+2}$. Suppose that, with n chosen as above, we have

$$p_{t+2} \mid d_{t+2}=nd_td_{t+1}+d_t-n.$$

Then $p_{t+2} \nmid d_td_{t+1}-1$, since $d_td_{t+1}-1$ divides $d_{t+1}d_{t+2}-1$. Therefore upon replacing n by $n+(p_1 \cdots p_{t+1})^{t+1}$, we see that condition (3) is satisfied, for $i=t+2$, and conditions (1) and (2) remain intact.

Thus we have proved:

2.4. THEOREM. *Let k be a field containing infinitely many roots of 1. There exists a subgroup H of $GA_2(k)$ which is isomorphic to the group U of roots of 1 in k, but which is not conjugate to any subgroup of Af_2 or E_2. Any such subgroup H is of type 2.*

Proof. All is proved, except the last statement. But any subgroup of type 3 contains an infinite cyclic subgroup (see Theorem–Definition 5.1), and therefore is not isomorphic to U.

2.5. *Example.* The following lemma enables us to exhibit a different kind of type 2 abelian subgroup in $GA_2(k)$, when $\operatorname{char}(k)>2$.

LEMMA. *Let k be a field of characteristic $p>0$, such that k contains the algebraic closure of $\mathbf{F}_p$. Let n be a positive integer, and let $f\in k[Y]$ be defined by $f=(Y-Y^{p^n})^d$ where d is any integer >1 such that $p\nmid d$. Then if $r\in\mathbf{F}_{(p^n)}$, we have $f(Y+r)=f(Y)$. The elements of $\mathbf{F}_{(p^n)}$ are precisely those elements $r\in k$ for which $\deg(f(Y+r)-f(Y))\le 1$.*

Proof. This follows readily from the fact that the elements of $\mathbf{F}_{(p^n)}$ are precisely the roots of $Y-Y^{p^n}$.

Now, to give the example, we let $\mathbf{F}_p$ be the algebraic closure of $\mathbf{F}_p$, and assume k is a field containing $\bar{\mathbf{F}}_p$. As in example 2.2 we let

$$a_i=a(0)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\in I,$$

for each integer $i>1$. Let $\{p_i\}_{i=1}^{\infty}$ be a list of all the primes, and let $n(i)=(p_1\cdots p_i)^i$. Let $e_{2i-1}=e_{2i}=e(f_i)\in J$, where $f_i=(Y-Y^{p^{n(i)}})^{d_i}$ where d_i is an integer >1 such that $p_i\nmid d_i$. For each integer $i\ge 1$, let

$$\begin{aligned}
S'_{2i}&=\mathscr{L}_2(\mathbf{F}_{(p^{n(i+1)})})=\left\{\begin{pmatrix}1&0&0\\0&1&0\\r&t&1\end{pmatrix}\,\middle|\, r,t\in\mathbf{F}_{(p^{n(i+1)})}\right\},\\
S_{2i-1}&=\left\{\begin{pmatrix}1&0&0\\0&1&0\\r&t&1\end{pmatrix}\,\middle|\, r\in\mathbf{F}_{(p^{n(i+1)})},\ t\in\mathbf{F}_{(p^{n(i)})}\right\},\\
S'_{2i-1}&=\left\{\begin{pmatrix}1&0&0\\0&1&0\\r&t&1\end{pmatrix}\,\middle|\, r\in\mathbf{F}_{(p^{n(i)})},\ t\in\mathbf{F}_{(p^{n(i+1)})}\right\},\\
S_{2i-2}&=\mathscr{L}_2(\mathbf{F}_{(p^{n(i)})})=\left\{\begin{pmatrix}1&0&0\\0&1&0\\r&t&1\end{pmatrix}\,\middle|\, r,t\in\mathbf{F}_{(p^{n(i)})}\right\}.
\end{aligned}\tag{38}$$

Now, given

$$s=\begin{pmatrix}1&0&0\\0&1&0\\r&t&1\end{pmatrix}\in\mathscr{L}_2(k),$$

it follows from the lemma, and from (23) and (26) of 6.13, that $s^{e_{2i}}\in B$ if and only if $t\in\mathbf{F}_{(p^{n(i)})}$, and in this case, $s^{e_{2i}}=s$, since both $f_i(t)$ and $D_1f_i(t)$ are zero. Also, we see from (22) of 1.13 that $s_i^{a_i}\in B$, and

$$s_i^{a_i}=\begin{pmatrix}1&0&0\\0&1&0\\t&r&1\end{pmatrix}.$$

It follows that

$$S_{2i-1}=s_{2i}^{\prime e_{2i}}\cap B\subsetneqq S_{2i}^{\prime e_{2i}},\qquad S'_{2i-1}=S_{2i-1}^{a_{2i-1}}\cap B=S^{a_{2i-1}}_{2i-1}$$
$$S_{2i-2}=S_{2i-1}^{\prime e_{2i-1}}\cap B\subsetneqq S_{2i-1}^{\prime e_{2i-1}},\qquad S'_{2i-2}=S_{2i-1}^{a_{2i-1}}\cap B=S^{a_{2i-1}}_{2i-1}$$

We have satisfied the hypothesis of Lemma 0.22, and so, upon letting H_{j-1}, H'_j be defined as in the theorem, we have a non-stationary chain

$$H_0\subset H'_1\subset H_1\subset\cdots\subset H_{j-1}\subset H'_j\subset H_j\subset\cdots\tag{39}$$

and the union H is an abelian subgroup of type 2 in $GA_2(k)$. Since H_j is conjugate to S_j, a finite subgroup of $\mathscr{L}_2(k)$, we see that case (a) of the conclusion of Theorem 1.21 is realized in this example.

2.6 In fact, in this example, H'_{2i} is conjugate to $\mathscr{L}_2(\mathbf{F}_{(p^{n(i+1)})})$ (by (38)), and the induced isomorphism $H'_{2i}\to\mathscr{L}_2(\mathbf{F}_{(p^{n(i+1)})})$ is such that the diagram

$$\begin{array}{ccc}H'_{2i}&\subset&H'_{2i+2}\\ \downarrow\wr&&\downarrow\wr\\ \mathscr{L}_2(\mathbf{F}_{(p^{n(i+1)})})&\subset&\mathscr{L}_2(\mathbf{F}_{(p^{n(i+2)})})\end{array}$$

commutes. By the choice of the n_i's, we have $\bigcup_{i=1}^{\infty}\mathbf{F}_{(p^{n(i)})}=\bar{\mathbf{F}}_p$, and so

$$\bigcup_{i=1}^{\infty}\mathscr{L}_2(\mathbf{F}_{(p^{n(i)})})=\mathscr{L}_2(\bar{\mathbf{F}}_p).$$

It follows that $H\cong\mathscr{L}_2(\bar{\mathbf{F}}_p)$.

One really doesn't need to assume $k\supset\bar{\mathbf{F}}_p$ to construct an example. The only thing one needs to know is that condition (b) of Lemma 0.22 is satisfied. (This has the effect of insuring that the chain (39) is non-stationary.) This can be accomplished by assuming that $k\cap\bar{\mathbf{F}}_p$, the algebraic closure of $\mathbf{F}_p$ in k, is infinite. With this assumption, let $F=k\cap\bar{\mathbf{F}}_p$, $F_i=k\bigcap\mathbf{F}_{(p^{n(i)})}$ ($n(i)$ defined as in 2.5). Then the chain $F_1\subset F_2\subset F_3\subset\cdots$ is non-stationary, the union being F. Upon replacing $\mathbf{F}_{(p^{n(i)})}$ by F_i in (38), the hypothesis of Lemma 0.22 is again satisfied, and we get a non-stationary chain

$$H_0\subset H_1\subset H'_1\subset\cdots\subset H_{i-1}\subset H'_i\subset H_i\subset\cdots$$

such that the union H is an abelian subgroup of type 2. In this case, H'_{2i} is conjugate in GA_2 to $\mathscr{L}_2(F_{i+1})$, and the diagram (39) commutes, replacing $\mathbf{F}_{(p^{n(i+1)})}$ and $\boldsymbol{F}_{(p^{n(i+2)})}$ by F_{i+1} and F_{i+2} respectively. Therefore $H \cong \mathscr{L}_2(F)$, and we have the following theorem.

2.1. THEOREM. *Let k be a field of characteristic $p \neq 0$. Let F be the algebraic closure of $\mathbf{F}_p$ in k, and assume F is infinite. There exists a subgroup H of $GA_2(k)$ which is isomorphic to the additive group F^2 which is not conjugate to any subgroup of Af_2 or E_2. Any such subgroup H is of type 2.*

Proof. We need to prove the last statement. Any subgroup of type 3 has an element of infinite order, and hence is not isomorphic to F^2.

(Note. One can also produce an abelian subgroup $H \subset GA_2$ of type 2 such that H is isomorphic to the additive group of $F = k \cap \dot{\mathbf{F}}_p$.)

2.8. To produce examples of type 3, we need only to give an abelian subgroup $F \subset B$ and an element $g \in GA_2$ which is not conjugate to any element of A or E, and which commutes with every element of F. In this case, one sees from Theorem–Definition 0.3 that the subgroup $\langle F, g\rangle = F \cdot \langle g\rangle$ is an abelian subgroup of type 3.

2.9. *Example.* Let d, $d' > 1$ be integers. Let

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = a(0) \in I.$$

Let $f, f' \in \overline{k[Y]^2}$ be defined by $f = Y^d$, $f' = Y^{d'}$ and let $e = e(f)$, $e' = e(f') \in J$. Let k be a field containing all the $(dd'$-1)th roots of 1, and let

$$F = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & u^{d'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, u \in U(dd'-1) \right\} \subset B.$$

Let $g = eae'a$. We claim that g commutes with every element of F. For any diagonal element

$$s = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \in D_2,$$

we have

$$s^a = \begin{pmatrix} v & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

by (22) of 1.13. By (23), (24), and (26) of 1.13, we see that $s^e \in B$ if and only if $u = v^e$, and in this case $s^e = s$; and $s^{e'} \in B$ if and only if $u = v^{d'}$, and in this case $s^{e'} = s$. If $s \in F$, then $v = u^{d'}$ and $u = v^d$, and it follows that

$$s^g = s^{eae'a} = s^{ae'a} = \begin{pmatrix} v & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}^{e'a} = \begin{pmatrix} v & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}^a = s.$$

Hence g commutes with every element of F. Clearly g is not conjugate to any subgroup of A or E, since g is cyclicly reduced (see 0.1). It follows from 2.8 that $H = \langle F, g \rangle$ is an abelian subgroup of type 3 in $GA_2(k)$. In fact, $H = F \times \langle g \rangle$.

In light of Theorem 1.24, note that case (b) of the conclusion holds for this example.

2.10. *Example.* Let k be a field of characteristic $p \neq 0$, and let $n, m \geq 1$ be integers. Assume $k \subset \mathbf{F}_{(p^n)}, \mathbf{F}_{(p^m)}$. Let $f = (Y - Y^{p^n})^d$, $f' = (Y - Y^{p^m})^d$ where d is an integer >1 such that $p \nmid d$ and let $e = e(f)$, $e' = e(f') \in J$. Let

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = a(0) \in I.$$

Let

$$F = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & t & 1 \end{pmatrix} \middle| \, r \in \mathbf{F}_{(p^m)}, \ t \in \mathbf{F}_{(p^n)} \right\},$$

and let $g = eae'a'$. For any element

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & t & 1 \end{pmatrix} \in \mathscr{L}(k)$$

we have

$$s^a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & r & 1 \end{pmatrix}$$

by (22) of 1.13. By (23) and (26) of 1.13, and by the lemma of 2.5, we see that $s^e \in B$ if and only if $t \in \mathbf{F}_{(p^n)}$, and in this case $s^e = s$; also $s^{e'} \in B$ if and only if $t \in \mathbf{F}_{(p^m)}$, and in this case $s^{e'} = s$. Now it follows easily that if $s \in F$, $s^g = s$. Therefore g commutes with every element of F. Also, g is cyclicly reduced, hence not conjugate to any element of A or E. It follows, using 2.8, that $H = \langle F, g \rangle$ is an abelian subgroup of type 3 in $GA_2(k)$. In fact, $H = \langle F \times \langle g \rangle$.

Note that case (a) of Theorem 1.24 holds in this example. The author does not know if an example exists where the group F is an infinite subgroup of $\mathscr{L}_2(k)$.

3. Actions of commutative k-group schemes on the affine plane

3.1. For any field (or ring) k a k-group scheme $\mathscr{G}$ is a functor from the category of k-schemes to the category of groups which is represented by a map $\alpha: \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ of k-schemes. An action of $\mathscr{G}$ on a k- scheme $\mathscr{S}$ is a functor from the category of k-schemes to the category of groups acting on sets, represented by a map $\gamma: \mathscr{G} \times \mathscr{S} \to \mathscr{S}$.

3.2. Given an action $\gamma: \mathscr{G} \times \mathscr{S} \to \mathscr{S}$ of $\mathscr{G}$ on $\mathscr{S}$, one can easily see that the map

$$id \times \gamma: \mathscr{G} \times \mathscr{S} \to \mathscr{G} \times \mathscr{S},$$

(where id is the projection onto $\mathscr{G}$) is an isomorphism.

3.3. Letting $\mathscr{T} = \operatorname{spec}(k)$, we have the group homomorphism from the group $\operatorname{Hom}(\mathscr{T}, \mathscr{G})$ of points in $\mathscr{T}$ (k-rational points) into $\operatorname{Aut}(\mathscr{S})$ which takes $\varphi \in \operatorname{Hom}(\mathscr{T}, \mathscr{G})$ to the element of $\operatorname{Aut}(\mathscr{S})$ given by the composite

$$\mathscr{S} = \mathscr{T} \times \mathscr{S} \xrightarrow{\varphi \times \mathrm{id}} \mathscr{G} \times \mathscr{S} \xrightarrow{\gamma} \mathscr{S}.$$

3.4. Two actions γ and γ' of $\mathscr{G}$ on $\mathscr{S}$ are said to be *equivalent* if there exists an automorphism ρ of $\mathscr{S}$ such that γ' is given by the composite

$$\mathscr{G} + \mathscr{S} \xrightarrow{\mathrm{id} \times \rho} \mathscr{G} \times \mathscr{S} \overset{\gamma}{-} \mathscr{S} \xrightarrow{\rho^{-1}} \mathscr{S}.$$

This is equivalent to saying that $\mathrm{id} \times \gamma': \mathscr{G} \times \mathscr{S} \to \mathscr{G} \times \mathscr{S}$ is given by the composite

$$\mathscr{G} \times \mathscr{S} \xrightarrow{\mathrm{id} \times \rho} \mathscr{G} \times \mathscr{S} \xrightarrow{\mathrm{id} \times \gamma} \mathscr{G} \times \mathscr{S} \xrightarrow{\mathrm{id} \times \rho^{-1}} \mathscr{G} \times \mathscr{S}.$$

3.5. Now suppose we take $\mathscr{S} = \mathbf{A}^2(k) = \operatorname{spec}(k[X, Y])$, the affine plane; and let $\mathscr{G}$ be some affine k-group scheme $\operatorname{spec}(R)$. Suppose we have an action $\gamma: \mathscr{G} \times \mathbf{A}^2 \to \mathbf{A}^2$ on the affine plane. Then γ corresponds to a k algebra homomorphism $k[X, Y] \to R[X, Y]$, which is determined by a vector (P, Q), with $P, Q \in R[X, Y]$. According to 3.2, the vector (P, Q) also determines an automorphism of $R[X, Y]$, i.e., (P, Q) represents an element of $GA_2(R)$ (see 1.1). By 3.4 two actions, determined by vectors (P, Q) and (P', Q'), which in turn represent $\psi, \psi' \in GA_2(R)$, are equivalent if and only if there exists $\varphi \in GA_2(k)$ such that $\psi' = \psi^{(\varphi \otimes_k R)}$.

3.6. It follows from 3.3 that an action γ of $\mathscr{G}$ on $\mathbf{A}^2$, given by (P, Q), gives us a homomorphism from the (abstract) group $\operatorname{Hom}_k(R, k)$ into $\operatorname{Aut}_k(k[X, Y]) = GA_2(k)$. This homomorphim carries $\varphi \in \operatorname{Hom}(R, k)$ onto $(\varphi(P), \bar{\varphi}(Q))$, where $\bar{\varphi}$ is the extended homomorphism $\varphi \otimes_k k[X, Y]$.

3.7. Now *suppose k is a field.* Let $\mathscr{G} = \operatorname{spec}(R)$ be a commutative k-group scheme, and γ an action of $\mathscr{G}$ on $\mathbf{A}^2$. We have, as in 3.6, an induced homomorphism $\operatorname{Hom}(R, k) \to GA_2(k)$. It is clear from 3.6 that the image H

of this homomorphism is an abelian subgroup of $GA_2(k)$ which is of bounded degree (in the sense of 1.8), since the degree of any element of H is at most max (deg P, deg Q). Therefore, by Proposition 1.11, H is conjugate in $GA_2(k)$ to a subgroup of Af_2 or E_2. It follows from 3.5, that after replacing γ by an equivalent action, we may assume $H \subset Af_2$ or $H \subset E_2$.

3.8. In order to obtain the results in this section, we must appeal to the following proposition, which arises from the theory of linear algebraic groups. We denote by U_2 the lower triangular unipotent subgroup of GL_2, and by D_2 the diagonal subgroup.

PROPOSITION. *If k is an algebraically closed field, then any abelian subgroup of $GL_n(k)$ whose Zariski closure is connected is conjugate to a subgroup of the lower triangular subgroup.*

For the proof we refer the reader to [3, Theorem 15.4]. Note that such an abelian subgroup of GL_n may be replaced by its Zariski closure, which is again abelian [3, Chapter I].

3.9. PROPOSITION. *Suppose k is a field, and suppose k' is some field extension of k. Suppose H is a subgroup of $GL_2(k)$ which is conjugate in $GL_2(k')$ to a subgroup of $U_2(k')$. Then H is conjugate in $GL_2(k)$ to a subgroup of $U_2(k)$.*

Proof. Let

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k')$$

be such that $H^\alpha \subset U_2(k')$. Clearly α can be chosen to be in $SL_2(k')$.

Let

$$\begin{pmatrix} 1 & 0 \\ h & 0 \end{pmatrix} \in H^\alpha.$$

Then

$$\alpha\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\alpha^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1+bdh & -b^2h \\ d^2h & 1-bdh \end{pmatrix}$$

is in H. Each element of H has the form (40), where h varies. Now, if $b=0$, $H \subset U_2(k)$ and we are done. If $d=0$, H is contained in the upper triangular unipotent group, and so H conjugated by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is contained in $U_2(k)$. Otherwise, $b, d \neq 0$. By (40) we see that $1-bdh, -b^2h \in k$ whenever

$$\begin{pmatrix} 1 & 0 \\ h & 0 \end{pmatrix} \in H^\alpha.$$

If H is non-trivial, choose h to be non-zero, and it follows that $bd^{-1} \in k$. Let

$$\gamma = \begin{pmatrix} d & 0 \\ -c & d^{-1} \end{pmatrix} \in B_2(k'),$$

and let

$$\beta = \alpha\gamma = \begin{pmatrix} 1 & bd \\ 0 & 1 \end{pmatrix}^{-1}.$$

Then $\beta \in GL_2(k)$, and $H^\beta = \gamma^{-1}(H^\alpha)\gamma$. Since $H^\alpha \in U^2(k')$, and since $U_2(k')$ is normal in $B_2(k')$, we see that $H^\beta \subset U_2(k') \cap GL_2(k) = U_2(k)$.

Now we are prepared to prove the following proposition.

3.10. PROPOSITION. *Suppose k is a field, and $\mathcal{G} = \operatorname{spec}(R)$ is an affine commutative k-group scheme which is reduced, and assume that the k-rational points form a dense set in $\mathcal{G}$. Any action of $\mathcal{G}$ on $\mathbf{A}^2(k)$ is equivalent either to an action given by a vector of the form*

$$(uX + f(Y), vY + s) \tag{41}$$

where $u, v \in R^*$, $s \in R$, $f \in R[Y]$, or to an action given by a vector of the form

$$(aX + bY + r, cX + dY + t) \quad \text{where } \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(R), \text{ and } r, t \in R. \tag{42}$$

Proof. As in 3.7 we may replace the action of $\mathcal{G}$ *on* $\mathbf{A}^2$ by an equivalent action and assume that the image H of the induced homomorphism $\operatorname{Hom}(R, k) \to GA_2(k)$ is contained in Af_2 or E_2. Since $\mathcal{G}$ is reduced, and since the k-rational points are dense in $\mathcal{G}$, it follows that if $x \in R$ vanishes at each k-rational point, then $x = 0$. Now, the action of $\mathcal{G}$ on $\mathbf{A}^2$ is represented by a vector (P, Q), with $P, Q \in R[X, Y]$ (see 3.5). If $H \subset E_2$, then when we evaluate (P, Q) at any k-rational point, we get a vector of the form (41) with $u, v \in k^*$, $s \in k$, $f \in k[Y]$. Therefore, (P, Q) itself is of the form (41) with u, v, $s \in R$, $f \in R[Y]$, since no non-zero coefficients vanish throughout $\operatorname{Hom}(R, k)$. Since (P, Q) determines an element of $GA_2(R)$ (see 3.5), we must have $u, v \in R^*$.

On the other hand, if $H \subset Af_2$, then when we evaluate (P, Q) at any k-rational point we get a vector of the form (42) with

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(k), \quad r, t \in k.$$

It follows that (P, Q) is of the form (42), and that

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(R),$$

since (P, Q) determines an element of $GA_2(R)$.

3.11. THEOREM. *Suppose k is an algebraically closed field, and suppose $\mathcal{G} = \operatorname{spec}(R)$ is a connected, reduced, affine, commutative k-group scheme. Then any action of $\mathcal{G}$ on $\mathbf{A}^2(k)$ is equivalent to an action given by a vector of* *the form*

$$(43) \qquad (uX+f(Y),\ vY+s)$$

where $u,\ v \in R^*,\ s \in R,\ f \in R[Y]$.

Proof. According to Proposition 3.10, the action is equivalent either to one of the required form, or else to an action given by

$$(44) \quad (P, Q) = (aX+bY+r,\ cX+dY+t) \quad \text{with } \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(R),\ r, t \in R.$$

In the latter case, (P, Q) determines an element of $Af_2(R)$. Since

$$Af_2(R) = \mathcal{L}_2(R) \rtimes GL_2(R),$$

the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(R)$$

determines an algebraic group homomorphism $\mathcal{G} \to GL_2(k)$. The image H of $\mathrm{Hom}\,(R, k) \to GA_2(k)$ is contained in $Af_2(k)$, and the projection T of H onto $GL_2(k)$ is the (closed point) image of the algebraic group homomorphism. Since $\mathcal{G}$ is connected, T is connected. We appeal to the proposition of 3.8 to conclude that T is conjugate in $GL_2(k)$ to some subgroup of $B_2(k)$. It follows that H in conjugate in $Af_2(k)$ to a subgroup $\mathcal{L}_2 B_2 = B$. Thus after conjugating $(P, Q) \in Af_2(R)$ by an appropriate element of $GL_2(k)$ (this amounts to replacing the action of $\mathcal{G}$ on $\mathbf{A}^2$ by an equivalent action) we may assume that $H \subset B$. Hence the coefficient c of (44) vanishes everywhere, and therefore $c = 0$, since $\mathcal{G}$ is reduced. Thus we see that (P, Q) is of the form (43), with f a linear polynomial.

4. Actions of vector groups on the affine plane

4.1. For any ring k, the k-group G_a^n (n dimensional vector group) is the affine k-scheme $\mathrm{spec}\,(k[T_1, \ldots, T_n])$, together with the map $G_a^n \times G_a^n \to G_a^n$ defined by the homomorphism

$$k[T_1, \ldots, T_n] \to k[T_1, \ldots, T_n] \otimes_k k[T_1, \ldots, T_n] = k[T_1, \ldots, T_n, T_1', \ldots, T_n']$$

which sends T_i to $T_i + T_i'$, for $i = 1, \ldots, n$. Or, in other words,

$$G_a^n = G_a \times \cdots \times G_a \ (n \text{ times})$$

where $G_a = G_a^1$. We will simply write T and T' for $T_1, \ldots, T_n$ and $T_1', \ldots, T_n'$, respectively.

4.2. An action of G_a^n on $\mathbf{A}^2(k)$ is given by a vector (P, Q) with P,

$Q \in k[T, X, Y]$ (see 3.5) satisfying the conditions

(a) $(P(T+T', X, Y), Q(T+T', X, Y))$
$$= (P(T, X, Y), Q(T, X, Y)) \cdot (P(T', X, Y), Q(T', X, Y)),$$

(b) $$(P(0, X, Y), Q(0, X, Y)) = (X, Y).$$

(The vector multiplication of (a) is performed as if we were composing elements of $GA_2(k[T, T'])$ (see 1.1).) Such a vector (P, Q) necessarily determines an element of $GA_2(k[T])$ (see 3.5).

4.3. The (abstract) group of k-rational points $\mathrm{Hom}(k[T], k)$ can be identified with the group $(k^+)^n$. (We write k^+ for the additive group in k.) An action (P, Q) of G_a^n on A^2 gives a homomorphism $(k^+)^n \to GA_n(k)$ which takes

$$(c) = (c_1, \ldots, c_n) \in (k^+)^n \quad \text{to} \quad (P(c, X, Y), Q(c, X, Y)) \in GA_2(k).$$

4.4. Recall from Theorem 2.7 that if k is a field of characteristic $p \neq 0$ such that k is an infinite algebraic extension of $\mathbf{F}_p$, then there exists a subgroup of $GA_2(k)$ which is isomorphic to $(k^+)^2$ which is not conjugate to any subgroup of Af_2 or E_2. Thus there exist faithful (non-algebraic) actions of $(k^+)^2$—and in fact k^+—on $\mathbf{A}^2$ which are not "linear" or "elementary", up to equivalence.

4.5. Of course, if $H \subset GA_2(k)$ (k a field) is the image of a homomorphism $(k^+)^n \to GA_n(k)$ arising from an (algebraic) action of G_a^n on $\mathbf{A}^2$, then H is of bounded degree, and hence is conjugate to a subgroup of Af_2 or E_2 (Proposition 1.11). This is the principle which has already been exploited, in a more general setting, to obtain Proposition 3.10 and Theorem 3.11. Now we will show, in proving Theorem 4.9, how this fact allows us to describe explicitly all actions of G_a^n on $\mathbf{A}^2$ (and in §5, all actions of tori on $\mathbf{A}^2$), up to equivalence. First, some preliminaries.

4.6. Definition. Let $f \in k[T_1, \ldots, T_n]$. We say that f is an *additive* polynomial if

(45) $$f(T_1 + T_1', \ldots, T_n + T_n') = f(T_1, \ldots, T_n) + f(T_1', \ldots, T_n')$$

(Note that if k is an infinite field, this is equivalent to saying that f defines an endomorphism of $(k^+)^n$.)

4.7. Proposition. *Suppose k is a domain and $f \in k[T_1, \ldots, T_n]$. If* $\mathrm{char}(k) = 0$, *then f is additive if and only if f is linear and homogeneous. If* $\mathrm{char}(k) = p > 0$, *then f is additive if and only if f is a k-linear combination of monomials of the form $T_i^{p^r}$.*

Proof. The sufficiency in each statement is obvious. The necessity, for $n = 1$, follows from the fact that if a positive integer d is not a power of p, then at least one of the coefficients $\binom{d}{1}, \ldots, \binom{d}{d-1}$ is non-zero in k. For $n > 1$, we reason as follows. Write

$$f = f_0 + f_1 T_n + \cdots + f_d T_n^d,$$

where $f_0, \ldots, d_d \in k[T_1, \ldots, T_{n-1}]$. Putting $T_n = T'_n = 0$ in (45) we see that f_0 is additive, and hence by induction is of the required form. Now we substitute only $T'_n = 0$ in (45) to get $f_i(T_1 + T'_1, \ldots, T_{n-1} + T'_{n-1}) = f_i(T_1, \ldots, T_n)$ for $i = 1, \ldots, d$. This clearly implies that $f_1, \ldots, f_d \in k$. Finally we set $T_1 = \cdots = T_{n-1} = 0 = T'_1 = \cdots T'_{n-1}$ in (45) and appeal to the case $n = 1$ to see that, for $i = 1, \ldots, d$, $f_i = 0$ if i is not a power of p. (When $p = 0$, this says that $f_2 = \cdots f_d = 0$.) This proves that f is of the required form.

(Note. Proposition 4.7 shows that, for $n = 1$, Definition 4.6 agrees with the definition of additive polynomial given in 1.17.)

4.8. *Remark.* Suppose $f \in k[T_1, \ldots, T_n]$. If f is an additive polynomial, then f defines an endomorphism of $(k^+)^n$. We have already remarked that the converse is true when k is an infinite field. However, if $k = \mathbf{F}_{(p^r)}$, the polynomial $(T^{p^r} - T)^d$ defines the zero endomorphism on k^+, but is not additive, if d is not a power of p.

4.9. THEOREM. *Suppose k is an infinite field. Any action of the n-dimensional vector group G_a^n on the affine plane $\mathbf{A}^2(k)$ is equivalent either to an action of the form*

$$(46) \qquad (X + g_0(T) + g_1(T)Y + \cdots + g_d(T)Y^d, Y)$$

where $g_0, \ldots, g_d \in k[T_1, \ldots, T_n]$ are additive polynomials; or to an action of the form

$$(47) \qquad (X + g(T), Y + h(T))$$

where $g, h \in k[T_1, \ldots, T_n]$ are additive polynomials.

The proof of this theorem is done in 4.10–4.27.

4.10. Let γ be an action of G_a^n on $\mathbf{A}^2$. γ is given by a vector (P, Q), with $P, Q \in k[T, X, Y]$, satisfying (a) and (b) of 4.2. Let H be the image of the induced homomorphism $(k^+)^n \to GA_2(k)$ (see 4.3). Since k is infinite, no polynomial in $k[T]$ vanishes on k^n (this says that the k-rational points of G_a^n form a dense set), and so according to Proposition 3.10, we can replace γ by an equivalent action, and assume that either (P, Q) is of the form

$$(48) \qquad (uX + f(Y), vY + s)$$

where $u, v \in k[T]^* = k^*$, $s \in k[T]$, $f \in k[T, Y]$ (i.e. $H \subset E_2$); or (P, Q) is of the form

$$(49) \quad (aX + bY + r, cX + dY + t) \quad \text{where} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(k[T]),\ r, t \in k[T],$$

(i.e. $H \subset Af_2$). We will dispense with the second possibility by again replacing γ by an equivalent action. Note that, if k were assumed to be algebraically closed, we could appeal to Theorem 3.11.

3.11. In any case, we can make the base change to $\bar{k}$, the algebraic closure of k. The action γ extends to an action γ of $G_a(\bar{k})$ on $\mathbf{A}^2(\bar{k})$, and $\bar{\gamma}$ is given by the vector (P, Q). Let $\bar{H}$ be the image of $(\bar{k}^+)^n \to GA_2(\bar{k})$. If (P, Q) is of the form (49), we have

$$\begin{array}{ccc} H & \subset & \bar{H} \\ \cap & & \cap \\ Af_2(k) & \subset & Af_2(\bar{k}) \end{array}.$$

Recall that $Af_2 = \mathscr{L}_2 \times GL_2$. Let $T, \bar{T}$ be the respective images of $H, \bar{H}$ in $GL_2(k)$, $GL_2(\bar{k})$. Then $\bar{T}$ is the (closed point) image of the algebraic group homomorphism $G_a^n(\bar{k}) \to GL_2(\bar{k})$ defined by

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Since $G_a^n(\bar{k})$ is connected, $\bar{T}$ is connected. Therefore, by the Proposition of 3.8, $\bar{T}$ is conjugate in $GL_2(\bar{k})$ to a subgroup of $B_2(\bar{k})$. But there are no non-trivial algebraic homomorphisms from a vector group into a torus (one can convince himself of this by an easy direct argument), and so we must have $\bar{T}$, and hence T, conjugate in $GL_2(\bar{k})$ to a subgroup of $U_2(\bar{k})$, the lower triangular unipotent group. We apply Proposition 3.9 to conclude that T is conjugate in $GL_2(k)$ to a subgroup of $U_2(k)$.

It follows that H is conjugate in Af_2 so a subgroup of $\mathscr{L}_2 U_2 \subset Af_2$. Therefore, if we replace γ by the appropriate equivalent action, γ is given by a vector (P, Q) of the form $(X + bY + r, Y + t)$. In particular, γ is of the form (48) of 4.10.

4.12. Thus we have shown that any action γ of G_a^n on A^2 is equivalent to one defined by a vector (P, Q) of the form (48). Note that the argument of 4.11 was necessary to show that γ is equivalent *over* k to such an action—not merely equivalent after going to $\bar{k}$.

4.13. So now we may assume γ is given by (P, Q) of the form $(uX + f(Y), vY + s)$ with $u, v \in k^*$, $s \in k[T]$, $f \in k[T, Y]$. It follows from (b) of 4.2 that $u = v = 1$. It follows from (a) of 4.2 that s is an additive polynomial in $k[T_1, \ldots, T_n]$. Write

$$f(Y) = g_0 + g_1 Y + \cdots + g_d Y^d,$$

with $g_0, g_1, \ldots, g_d \in k[T]$. If $s = 0$, we can deduce from (b) of 4.2 that each of $g_0, \ldots, g_d$ is an additive polynomial. Hence, if $s = 0$, the action is of the form (46) of 4.9, as required.

4.14. Otherwise, (P, Q) is of the form

$$(X + f(Y), Y + s) \tag{50}$$

where $f \in k[T, Y]$, and $s \in k[T]$ is a non-zero additive polynomial. As above, write

$$f(Y) = g_0 + g_1 Y + \cdots + g_d Y^d \tag{51}$$

where $g_0, \ldots, g_d \in k[T]$, $g_d \neq 0$. If $d = 0$, we again appeal to (b) of 4.2 to see that g_0 is additive, and so (P, Q) has the form (47) specified in the theorem. In the case $d > 0$ we will show that we can conjugate $(P, Q) \in GA_2(k[T])$ by a well chosen elementary automorphism in $GA_2(k)$ which has the effect of leaving the form (50) of (P, Q) intact, but lowering the degree d of f. This will prove the theorem, since conjugating (P, Q) by an element of $GA_2(k)$ is tantamount to replacing γ by an equivalent action (see 3.5).

4.15. Since (P, Q) is of the form (50), condition (b) of 4.2 says that

$$\begin{aligned} &(X + f(T + T', Y), Y + s(T + T')) \\ &= (X + f(T, Y), Y + s(T))(X + f(T', Y), Y + s(T')) \\ &= (X + f(T, Y) + f(T', Y + s(T)), Y + s(T) + s(T')). \end{aligned} \tag{52}$$

One sees, then, that

$$f(T + T', Y) = f(T, Y) + f(T', Y + s(T)). \tag{53}$$

Now,

$$f(T', Y + s(T)) = \sum_{i=0}^{d} \left(\sum_{t=i}^{d} \binom{t}{i} g_t(T') s(T)^{t-i} \right) Y^i) \tag{54}$$

and so (53) and (54) imply that

$$g_i(T + T') = g_i(T) + g_i(T') + \sum_{t=i+1}^{d} \binom{t}{i} g_t(T') s(T)^{t-i}, \quad i = 0, 1, \ldots, d.) \tag{55}$$

In particular, we see that the polynomial

$$A_i(T, T') = \sum_{t=i+1}^{d} \binom{t}{i} g_t(T') s(T)^{t-1}$$

has the property that $A_i(T, T') = A_i(T', T)$, since $A_i(T, T') = g_i(T + T') - g_i(T) - g_i(T')$.

For any polynomial $A \in k[T, T']$ we will say that A is T, T'-symmetric if $A(T, T') = A(T', T)$. We will be repeatedly exploiting the fact that the polynomials $A_i(T, T')$ are T, T'-symmetric.

4.16. Given $h \in k[Y]$, we have $e(h) \in GA_2(k)$ given by $(X + h(Y), Y)$. Direct computation shows that

$$\begin{aligned} (P, Q)^{e(h)} &= (X + f(T, Y), Y + s(T))^{e(h)} \\ &= (X + f(T, Y) - h(Y) + h(Y + s(T)), Y + s(T)). \end{aligned} \tag{56}$$

In particular, upon letting $h=cY^m$, where $c\in k$, (56) becomes

$$(57)\qquad \begin{aligned}(P,Q)^{e(cY^m)} &= (X+f(T,Y),\ Y+s(T))^{e(cY^m)}\\ &= (X+f(T,Y)-cY^m+c(Y+s(T))^m,\ Y+s(T)).\end{aligned}$$

Note that the effect of (57) on $(P,Q)=(X+f(T,Y),Y+s(T))$ is to replace $f(T,Y)$ by $f(T,Y)-cY^m+c(Y+s(T))^m$. We wish to choose c and m so that the conjugation (57) lowers the degree d (in Y) of f. This will prove the theorem, as was explained in 4.14. We begin by proving the following technical lemma.

4.17. Lemma. *Suppose the action (P,Q) is of the form $(X+f(T,Y),Y+s(T))$ with*

$$f=g_0+g_qY^q+\cdots+g_{d'q}Y^{d'q},$$

where $g_0,\ldots,g_{d'q}\in k[T]$ and q is a power of $p=\operatorname{char}(k)$. (If $p=0$, then $q=1$). Let u be an integer such that $1\le u\le d'$ and $p\nmid u+1$; and let $a\in k$. There exists an elementary automorphism $\varphi\in GA_2(k)$ such that

$$(P,Q)^\varphi=(X+f'(T,Y),\ Y+s(T))$$

where

$$f'=g_0'+g_q'Y^q+\cdots+g_{d'q}'Y^{d'q},$$

$g_0',\ldots,g_{d'q}'\in k[T]$, $g_{jq}'=g_{jq}$ for $j>u$, and $g_{uq}'=g_{uq}-as(T)^q$.

Proof. In (57) of 4.16, take $m=(u+1)q$, $c=-a/(u+1)$, and we see that $\varphi=e(cY^m)$. works.

4.18. Now we assume that $d\ (=\deg_Y f)>0$. Let q be the highest power of $p=\operatorname{char}(k)$ which divides every integer j for which $g_j\neq 0$, and write

$$(58)\qquad f=g_0+g_qY^q+\cdots+g_{d'q}Y^{d'q}$$

where $d=d'q$. (If $p=0$, then $q=1$.) Note that, with f in this form, equation (55) becomes

$$(59)\qquad G_{iq}(T+T')=g_{iq}(T)+g_{iq}(T')+\sum_{t=i+1}^{d'}\binom{t}{i}g_{tq}(T')s(T)^{(t-i)q},\qquad i=0,1,\ldots,d'.$$

This uses the fact that $\binom{tq}{iq}=\binom{t}{i}$ in k.

4.19. We first assume that $p\nmid d'$ and $p\nmid d'+1$. Equation (59) with $i=(d'-1)$, says that

$$g_{(d'-1)q}(T+T')=g_{(d'-1)q}(T)+g_{(d'-1)q}(T')+d'g_d(T')s(T)^q.$$

Thus we see that $d'g_d(T')s(T)^q$ must be T,T'-symmetric. Since d' and $s(T)$ are non-zero (in $k[T]$), this easily implies that $g_d(T)=cs(T)^q$ for some non-zero $c\in k$. Since $p\nmid d'+1$, we see by Lemma 4.17 that we can perform a conjugation of (P,Q) by an element of $GA_2(k)$ which has the effect of cancelling $g_d(T) = cs(T)^q$, and thereby lowering the degree (in Y) of f. Thus we can lower the degree (in Y) of f by an appropriate conjugation, as long as $d', d'+1 \neq 0$ in k. This completes the proof of the theorem for the case char $(k) = 0$, and so we will proceed under the assumption $p > 0$.

4.20. We will dispense with the case $p \mid d'+1$ by showing it cannot occur. Suppose $p \mid d'+1$. We again appeal to the equation (59), with $i = d'-1$ to see that $g_d(T) = cs(T)^q$ for some non-zero $c \in k$. This can be done just as in 4.19, since $d' \neq 0$ in k.

Let us first assume $d'+1 \neq p$, i.e. $d' > p-1$. We write (59) with $i = p-1$ to get

$$(60) \quad g_{(p-1)q}(T+T') = g_{(p-1)q}(T) + g_{(p-1)q}(T') + \sum_{t=p}^{d'} \binom{t}{p-1} g_{tq}(T')s(T)^{(t-p+1)q}.$$

Since $\binom{p}{p-1} = p = 0$ in k, we see from (60) that the polynomial

$$A(T, T') = \sum_{t=p+1}^{d'} \binom{t}{p-1} g_{tq}(T')s(T)^{(t-p+1)q}$$

is T, T'-symmetric. (Note that since $d' > p-1$, we also have $d' > p$, and so the above sum is non-empty.) However, this is impossible because $s(T)^{2q}$ divides $A(T, T')$, but since $g_d(T')$ $(= g_{d'q}(T')) = cs(T')^q$, and since $\binom{d'}{p-1} \neq 0$ in k (this follows from the fact that $p \mid (d'+1)$. p does not divide

$$\frac{d'!}{(d'-p+1)!\,(p-1)!} = \binom{d'}{p-1}$$

in $\mathbf{Z}$), one sees that $s(T')^{2q}$ does not divide $A(T, T')$. This violates the T, T'-symmetry of $A(T, T')$, and gives a contradiction, when $d'+1 \neq p$.

4.21. If $d'+1 = p$ we must make a special argument to get the contradiction. As we have seen in 4.20, $g_d(T) = g_{(p-1)q}(T) = cs(T)^q$, with $c \in k, \neq 0$.

4.22. We claim that, by conjugating (P, Q) by a well-chosen element of $GA_2(k)$, we can arrange that, for $j = 1, \ldots, p-1$, there exists $c_j \in k, \neq 0$ such that $g_{(p-j)q}(T) = c_j s(T)^{jq}$.

4.23. We already have this for $j = 1$. Assuming this has been arranged for $j = 1, \ldots, m-1$, with $1 < m \leq p-1$, we study the polynomial $g_{(p-m)q}(G)$ by writing the equation (59) for $i = p-m-1$. We get

$$\begin{aligned} & g_{(p-m-1)q}(T+T') - g_{(p-m-1)q}(T) - g_{(p-m-1)q}(T') \\ (61) \quad & = \sum_{t=p-m}^{p-1} \binom{t}{p-m-1} g_{tq}(T')s(T)^{(t-p+m+1)q} \\ & = \sum_{r=1}^{m} \binom{p-r}{p-m-1} g_{(p-r)q}(T')s(T)^{(m-r+1)q} \\ & = \sum_{r=1}^{m-1} \binom{p-r}{p-m-1} g_{(p-r)q}(T')s(T)^{(m-r+1)q} - mg_{(p-m)q}(T')s(T)^q. \end{aligned}$$

Call this polynomial $B(T, T')$. Note that all the binomial coefficients

$$\binom{p-r}{p-m-1}, \quad 1 \le r \le m,$$

are non-zero in k (since $0<p-r<p$). Therefore we have

$$(62) \quad B(T, T') = b_1 s(T')^q s(T)^{mq} + b_2 s(T')^{2q} s(T)^{(m-1)q} + \cdots + b_{m-1} s(T')^{(m-1)q} s(T)^{2q} - m g_{(p-m)q}(T') s(T)^q$$

with $b_1, \ldots, b_{m-1} \in k, \neq 0$ (also $m \neq 0$ in k). Now $B(T, T')$ is T, T'-symmetric, by (61) and $S(T)^q$ divides $B(T, T')$. Therefore $s(T')^q$ divides $B(T, T')$. Let

$$D(T, T') = \frac{B(T, T')}{s(T)^q s(T')^q}.$$

Then $D(T, T')$ is a polynomial, and is T, T'-symmetric. It follows from (62) that $s(T')^q$ divides $g_{(p-m)q}(T')$, and

$$(63) \quad D(T, T') = b_1 s(T)^{(m-1)q} + b_2 s(T')^q s(T)^{(m-2)q} + \cdots + b_{m-1} s(T')^{(m-2)q} s(T)^q - m \frac{g_{(p-m)q}(T')}{s(T')^q}.$$

Since s is additive, we have $s(0)=0$. Since $D(T, T')$ is T, T'-symmetric, we have $D(T, 0) = D(0, T)$, which implies, according to (63) that

$$b_1 s(T)^{(m-1)q} - m \frac{g_{(p-m)q}}{s^q}(0) = -m \frac{g_{(p-m)q}}{s^q}(T).$$

Hence, letting

$$a = \frac{g_{(p-m)q}}{s^q}(0) \in k,$$

we have

$$(64) \qquad -\frac{b_1}{m} s(T)^{mq} + a s(T)^q = g_{(p-m)q}(T).$$

Since p does not divide $p-m+1$, Lemma 4.17 tells us that we can conjugate (P, Q) by an appropriate element of $GA_2(K)$, and this will have the effect of subtracting $as(T)^q$ from $g_{(p-m)q}(T)$ without altering $g_{(p-m+1)q}, \ldots, g_{(p-1)q} = g_d$, and without introducing power of Y in $f(T, Y)$ which are not divisible by q. And so, after this conjugation, we have $g_{(p-m)q}(T) = c_m s(T)^{mq}$, where $c_m = -b_1/m \in k, \neq 0$ (see (63)). Thus the claim of 4.22 is established.

4.24. We write the equation (59) with $i=0$, and we get (in view of the claim)

$$(65) \qquad g_0(T+T') - g_0(T) - g_0(T') = \sum_{j=1}^{p-1} c_j s(T')^{jq} s(T)^{(p-j)q}.$$

Let $w(T)$ be the non-zero homogeneous form of minimal degree in $s(T)$. Since $s(T)$ is an additive polynomial, $w(T)$ is of the form $a_1T_1^{p^e}+\cdots+a_nT_n^{p^e}$ for some integer $e\geq 0$. It follows from (65) that the non-zero form of minimal degree in $g_0(T+T')-g_0(T)-g_0(T')$ is $\sum_{j=1}^{p-1} c_jw(T')^{jq}w(T)^{(p-j)q}$, and the degree of this form is $p^{e+1}q$. In fact, if we let $h(T)$ be the homogeneous form of degree $p^{e+1}q$ in $g_0(T)$, we must have

$$h(T+T')-h(T)-h(T')=\sum_{j=1}^{p-1} c_j(T')^{jq}w(T)^{(p-j)q}. \tag{66}$$

However, this equation gives a contradiction, as follows. Since $w(T)\neq 0$, we may assume $a_1\neq 0$. Note that each of the terms

$$c_1a_1^{pq}T_1'^{p^ejq}T_1^{p^e(p-j)q}, \qquad j=1,\ldots,p-1 \tag{67}$$

appears in the right hand side of (66). On the other hand, if we substitute $T_2=\cdots=T_n=0=T_2'=\cdots=T_n'$ into $h(T+T')-h(T)-h(T')$, we get zero, since h is homogeneous and $\deg h$ is a power of p, and so a term like (67) cannot possibly appear in (66).

4.25. Thus we have shown that the situation $p\mid(d'+1)$ cannot occur. The only situation we have left to deal with is the case $p\mid d'$ (see 4.19).

4.26. Assume $p\mid d'$. Recall that $f=g_0+g_qY^q\cdots+g_{d'q}Y^{d'q}$, where $d=d'q$, and q is the highest power of p which divides each of the integers j for which $g_j\neq 0$. We want to show that we can conjugate (P,Q) by a carefully chosen element of $GA_2(k)$ which "cancels out" those terms $g_{iq}Y^{iq}$ in f for which $p\nmid i$. With this accomplished, f is of the form

$$f=g_0+g_{q'}Y^{q'}+\cdots+g_{d''q'}Y^{d''q'}$$

where $q'=pq$ and $d''=d'/p$. We can repeat this process until we have $p\nmid d''$, and then go to the first case treated to lower the degree (in Y) of f.

We will cancel the unwanted g_{iq}'s (i.e., those for which $p\nmid i$) starting from the top.

4.27. For each integer u such that $1\leq u\leq d'$, let C_u be the following statement.

(68) C_u: Let i be an integer such that $u\leq i\leq d'$. If $p\nmid i$, then $g_{iq}(T)=0$. If $p\mid i$, and if $s(T)^q$ divides $g_{iq}(T)$, then $(g_{iq}/s^q)(0)=0$.

4.28. The statement $C_{d'}$ is just the statement that if $s(T)^q$ divides $g_d(T)=g_{d'q}(T)$, then $(g_d/s^q)(0)=0$. Since $p\nmid d'+1$, we see by Lemma 4.17 that this can be arranged by conjugating (P,Q) by an appropriate automorphism in $GA_2(k)$, which has the effect of adding $cs(T)^q$ to $g_d(T)$ (and probably disturbing some of the lower g_j's) without raising the degree of f.

We will prove that if $1\leq u\leq d'$, and if C_{u+1} holds, then we can perform a conjugation of (P,Q) as in Lemma 4.17, which disturbs only those g_{jq}'s for which $j\leqq u$, to arrange that C_u holds. This will prove the theorem, because

C_1 says that $g_{jq}=0$ whenever $p \nmid j$, and so we can replace q by pq, as was explained in 4.26.

So *assume* C_{u+1} *holds*. In order to make arrangements for C_u, we must consider five cases.

4.29. *Case I.* $p \mid u$. In this case, we need only to arrange that if $s(T)^q$ divides $g_{uq}(T)$, then $(g_{uq}/s^q)(0)=0$. Since $p \nmid (u+1)$, this can be accomplished using Lemma 4.17, just as we did in 4.28 for the case $u=d'$.

4.30. *Case II.* $p \mid u-1$. Now we must arrange that $g_{uq}=0$. We write equation (59) with $i=u-1$. Since $g_{jq}=0$ if $j>u$ and $p \nmid j$ (*by* C_{u+1}), we get

$$(69)\quad g_{(u-1)q}(T+T')-g_{(u-1)q}(T)-g_{(u-1)q}(T')$$
$$=\left(\sum_{\substack{u<j\le d'\\ p\mid j}}\binom{j}{u-1}g_{jq}(T')s(T)^{(j-u+1)q}\right)+ug_{uq}(T')s(T)^q$$

Call the above polynomial $A(T,T')$. Then $A(T,T)$ is T,T'-symmetric. Obviously $s(T)^q$ divides $A(T,T')$, and therefore $s(T')^q$ must divide $A(T,T')$. Upon studying the right hand side of (69) we see that $s(T')^q$ must divide $g_{jq}(T')$ whenever $p \mid j$ and $\binom{j}{u-1}\neq 0$ in k; and also $s(T')^q$ divides $g_{uq}(T')$. Consider the polynomials

$$B(T,T')=\frac{A(T,T')}{s(T)^q}\quad\text{and}\quad B'(T,T')=\frac{A(T,T')}{s(T')^q}.$$

Obviously $B(0,T)=B'(T,0)$. Now, $B(0,T)=ug_{uq}(T)$, since $s(0)=0$. Since $(g_{jq}/s^q)(0)=0$ whenever $j>u$ and $p \mid j$ we see from (69) that

$$B'(T,0)=\frac{ug_{uq}}{s^q}(0)s(T)^q.$$

Hence, upon letting $a=(g_{ua}/s^q)(0)\in k$, we have $g_{uq}(T)=as(T)^q$. Since $pp \nmid u+1$ (unless[1] $p=2$), we can appeal to Lemma 4.17, and conjugate (P,Q) by an appropriate element of $GA_2(k)$ which cancels $g_{uq}(T)$, and leaves $g_{jq}(T)$, $u<j\le d'$, intact. And so now C_u holds.

4.31. *Case III.* $p \mid u+1$, $p\neq u+1$. Again, we write equation (59) for $i=u-1$. For each $j>u$ such that $g_{jq}\neq 0$, we have $p \mid j$, and therefore $\binom{j}{u-1}=0$ in k, since $p \nmid u-1$ (unless $p=2$—see the footnote in Case II). Thus equation (59) is

$$g_{(u-1)q}(T+T')-g_{(u-1)q}(T)-g_{(u-1)q}(T')=ug_{uq}(T')s(T)^q.$$

Therefore $ug_{uq}(T')s(T)^q$ is T,T'-symmetric, and since $u, s(T)^q\neq 0$ (in $k[T]$), we must have $g_{uq}(T)=as(T)^q$ for some $a\in k$. We claim $a=0$. (Note that we

[1] The case $p=2$ requires a special argument here. If $p=2$, then case II is covered by cases III and IV. The proof in case IV holds if $p=2$. *However, in case III, we encounter a difficulty in getting* $g_{uq}(T)$ to be of the form as $(T)^q$, $a\in k$. But since this much is accomplished in case II, for $p=2$, we can patch cases II and III together to cover the case $2\mid u-1$, $2\neq u+1$.

cannot employ Lemma 4.17 to cancel $g_{uq}(T)$ as we did in Cases I and II, since $p \mid u+1$.) We write equation (59) with $i=p-1$. For each $j>u$ such that $g_{jq} \neq 0$, we have $p \mid j$, and so $\binom{j}{p-1}=0$ in k. Also $\binom{p}{p-1}=p=0$ in k, and so the equation is

$$g_{(p-1)q}(T+T')-g_{(p-1)q}(T)-g_{(p-1)q}(T')=\sum_{t=p+1}^{u}\binom{t}{p-1} g_{tq}(T')s(T)^{(t-p+1)q}.$$

This is a non-empty sum, since $u>p$. Note that $s(T)^{2q}$ divides this polynomial. Since the polynomial is T, T'-symmetric, $s(T')^{2q}$ must divide it also. It follows that $s(T')^{2q}$ divides $g_{tq}(T')$ whenever $\binom{t}{p-1}\neq 0$ in k. Now, $\binom{u}{p-1}\neq 0$ in k, since $p \mid u+1$ (this was explained in 4.20, for d') and so $s(T')^{2q}$ divides $g_{uq}(T')=as(T')^q$, which implies that $a=0$. Hence C_u holds.

4.32. *Case IV.* $p=u+1$. In this case we can argue as in Case III (Case II if $p=2$) that $g_{uq}(T)$ $(=g_{(p-1)q}(T))=as(T)^q$ for some $a \in k$. We claim $a=0$.

Assume $a \neq 0$. We will arrive at a contradiction by a similar procedure to that of 4.21–4.24 where we proved the impossibility of $p=d'+1$.

(70) We claim, as in 4.22, that, by conjugating (P, Q) by a well-chosen element of $GA_2(k)$ which leaves g_{jq} intact for $j>u$ we can arrange that, for $j=1,\dots,p-1$, there exists $c_j \in k, \neq 0$ such that $g_{(p-j)q}(T)=c_j s(T)^{jq}$.

This is already the case for $j=1$, letting $c_1=a$. As in 4.23, we assume that this has been arranged for $j=1,\dots,m-1$, where $1<m\leq p-1$, and we write equation (59) for $i=p-m-1$. Now, as long as $m<p-1$, all the binomial coefficients $\binom{j}{p-m-1}$ are zero in k whenever $p \mid j$ since $p \nmid p-m-1$. Therefore, if $m<p-1$ this equation is equation (61) of 4.23, and we can argue just as in 9.23 that after suitable conjugation, $g_{(p-m)q}(T)$ is of the form $c_m s(T)^{mq}$ with $c_m \in k, \neq 0$; and the higher g_{jq}'s remain undisturbed.

When $p>2$ we must make a special argument for the last step when $m=p-1$. In this case the coefficients $\binom{j}{p-m-1}$ are all l, and so when we write (59) with $i=0$, we get

$$\begin{aligned} &g_0(T+T')-g_0(T)-g_0(T')\\ &=\left(\sum_{\substack{j\geq p\\ p\mid j}} g_{jq}(T')s(T)^{jq}\right)+\left(\sum_{r=1}^{p-1} g_{(p-r)q}(T')s(T)^{(p-r)q}\right)\\ &=\left(\sum_{\substack{j\geq p\\ p\mid j}} g_{jq}(T')s(T)^{jq}\right)+\left(\sum_{r=1}^{p-2} c_r s(T')^{rq}s(T)^{(p-r)q}\right)+g_q(T')s(T)^q. \end{aligned}$$

Call this polynomial $B(T, T')$. Obviously $B(T, T')$ is T, T'-symmetric, and $s(T)^q$ divides $B(T, T')$, and therefore $s(T')^q$ divides $B(T, T')$. It follows that $s(T')_q$ divides each of the polynomials $g_{jq}(T')$ where $j\geq p$, and also $s(T')^q$ divides $g_q(T')$. Since the assertion C_p (68) holds, we have $(g_{iq}/s^q)(0)=0$, for $j\geq p$. Let $D(T, T')$ be the polynomial $B(T, T')/s(T')^q s(T)^q$. Then $D(T, T')$ is

T, T'-symmetric, and we have $g_q(T)/s(T)^q = D(0, T)$, since $s(0)=0$. Also

$$D(T,0) = c_1 s(T)^{(p-2)q} + \frac{g_q}{s^q}(0)$$

since $(g_{jq}/s^q)(0)=0$ for $j \geq p$. By the T, T'-symmetry of $D(T, T')$, we have $D(T,0)=D(0,T)$, and therefore $g_q(T) = c_1 s(T)^{(p-2)q} s(T)^q + bs(T)^q$ where $b = (g_q/s^q)(0) \in k$. Since $p > 2$, we know that $p \nmid 2$, and so, by Lemma 4.17 we can conjugate (P, Q) by an element of $GA_2(k)$ to subtract $bs(T)^q$ from $g_q(T)$. After doing so, we have $g_q(T) = c_1 s(T)^{(p-2)q} s(T)^q = c_1 s(T)^{(p-1)q}$ as required in the claim (70). This validates the claim.

And so we arrange that $g_{(p-j)q}(T) = c_j s(T)^{jq}$, $c_j \in k, \neq 0$, for $j = 1, \ldots, p-1$. As in 4.24, we study the equation (59) with $i=0$, which is

(71)
$$g_0(T+T') - g_0(T) - g_0(T') = \left(\sum_{\substack{j \geq p \\ p \mid j}} g_{jq}(T')s(T)^{jq} \right) + \left(\sum_{r=1}^{p-1} c_r s(T')^{rq} s(T)^{(p-r)q} \right).$$

Upon letting $w(T)$ be the non-zero homogeneous form of minimal degree in $s(T)$ we see from (71) that the non-zero form of minimal degree in $g_0(T+T') - g_0(T) - g_0(T')$ is

$$\sum_{j=1}^{p-1} c_j w(T')^{jq} w(T)^{(p-j)q}. \tag{72}$$

(Note. The fact that $g_{jq}(0)=0$ shows that the first summation of (71) does not contribute to the form of minimal degree in $g_0(T+T') - g_0(T) - g_0(T')$.) Since $w(T)$ is an additive form, its degree is a power of p, say p^e, and $w = a_1 T_1^{p^e} + \cdots + a_n T_n^{p^e}$. The degree of the form (72) is $p^{e+1}q$. If we let $h(T)$ be the form in $g_0(T)$ of degree $p^{e+1}q$, we must have, by (71),

$$h(T+T') - h(T) - h(T') = \sum_{j=1}^{p-1} c_j w(T')^{jq} w(T)^{(p-j)q},$$

which is the same equation as (66) in 4.24, and leads to the same contradiction.

Thus we have shown, by contradiction, that $g_{uq}(T) = g_{(p-1)q}(T) = 0$, which implies that C_u holds.

4.26. *Case V.* $p \nmid u+1$, $p \nmid u$, $p \nmid u-1$. Write equation (59) with $i = u-1$. Since $p \mid j$ whenever $j > u$ and $g_{jq} \neq 0$ (by C_{u+1}) we have $\binom{j}{u-1} = 0$ in k, and so the equation is

$$g_{(u-1)q}(T+T') - g_{(u-1)q}(T) - g_{(u-1)q}(T') = u g_{uq}(T')s(T)^q.$$

Since $u \neq 0$ in k, and since $ug_{uq}(T')s(T)^q$ is T, T'-symmetric, it follows that $g_{uq}(T) = bs(T)^q$ for some $b \in k$. Now we can employ Lemma 4.17 to perform a conjugation of (P, Q) by an element of $GA_2(k)$ which cancels $g_{uq}(T)$, so that C_u holds.

4.27. This completes the proof of Theorem 4.9. Note that the crux of the argument comes in 4.10, where it depends on the fact that the abelian subgroups of bounded degree in $GA_2(k)$ are all conjugate to linear and elementary subgroups. The rest of the proof, although lengthy and notationally difficult, is conceptually fairly straightforward.

In the case $n=1$, the theorem can be refined as follows.

4.28. COROLLARY. *Suppose k is an infinite field. Any action of G_a on the affine plane $A_2(k)$ is equivalent to an action given by a vector of the form*

$$(X+g_0(T)+g_1(T)Y+\cdots+g_d(T)Y^d,\ Y)$$

where $g_0,\ldots,g_d$ are additive polynomials in $k[T]$ (one variable).

Proof. (We present the proof assuming $\operatorname{char}(k)=p>0$. Basically the same proof works if $\operatorname{char}(k)=0$, but things are simpler.) Given an action γ of G_a on A^2, Theorem 4.9 tells us that γ is equivalent either to an action given by such a vector, or else to an action given by (P, Q) of the form $(X+g(T), X+h(T))$, where $g, h\in k[T]$ are additive polynomials. Suppose $g, h\neq 0$, and let aT^{p^u}, bT^{p^v} be the leading (highest degree) terms of g and h, respectively. Suppose $u\geq v$. We conjugate (P, Q) by

$$e\left(-\frac{b}{a}Y^{p^{(u-v)}}\right)=\left(X-\frac{b}{a}Y^{p^{(u-v)}},\ Y\right)\in GA_2(k).$$

(This gives us an action equivalent to γ.) Conjugating (P, Q) by $e(-(b/a)Y^{p^{(u-v)}})$ we get

$$\left(X+\frac{a}{b}Y^{p^{(u-v)}},\ Y\right)(X+g(T),\ Y+h(T))\left(X-\frac{a}{b}Y^{p^{(u-v)}},\ Y\right)$$
$$=\left(X+g(T)-\frac{a}{b}h(T)^{p^{(u-v)}},\ Y+h(T)\right)$$

The leading term of $-(a/b)h(T)^{p^{(u-v)}}$ is $-aT^p$, which cancels the leading term of $g(T)$. Thus we have replaced $g(T)$ by an additive polynomial of lower degree. Of course, we can do a similar thing if $v\geq u$ (conjugate by $(X, Y-(a/b)Y^{p^{(v-u)}})$ instead) to lower the degree of $h(T)$. We can continue this until we have either $g=0$ or $h=0$. If $h=0$, we are done, since (P, Q) is of the form specified in the corollary, with $d=0$. If $g=0$, we conjugate (P, Q) by $(Y, X)\in GA_2(k)$ to get $(X+h(T), Y)$, which is of the required form. This proves the corollary.

4.29. If k is algebraically closed, so that G_a^n and A^2 are algebraic groups in the classical sense, we restate Theorem 4.9 and Corollary 4.28 (in reverse order) in terms of the action of closed points.

Restatement for the case where k is algebraically closed. (1) (Rentschler, Miyanishi). Any action of G_a on the affine plane $\mathbf{A}^2$ is equivalent to an action of the form

$$t \cdot (x, y) = (x + g_0(t) + g_1(t)y + \cdots + g_d(t)y^d, y)$$

where $g_0, \ldots, g_d$ are additive polynomials.

(2) Any action of the n-dimensional vector group G_a^n on $\mathbf{A}^2$ is equivalent *either* to an action of the form

$$(t_1, \ldots, t_n) \cdot (x, y) = (x + g_0(t) + g_1(t)y + \cdots + g_d(t)y^d, y)$$

where $g_0, \ldots, g_d$ are additive polynomials in $t_1, \ldots, t_n$; *or* to an action of the form

$$(t_1, \ldots, t_n)(x, y) = (x + g(t), y + h(t))$$

where g and h are additive polynomials in $t_1, \ldots, t_n$.

5. Actions of tori on the affine plane (Gutwirth's theorem)

5.1. For any ring k, the k-group G_m^n (the n-dimensional torus) is the affine k-scheme spec(A) where $A = k[T_i, T_i^{-1}]_{i=1}^n$, the map $G_m^n \times G_m^n \to G_m^n$ being given by the homomorphism

$$A \to A \otimes_k A \cong k[T_i, T_i', T_i^{-1}, T_i'^{-1}]_{i=1}^n$$

which sends T_i to T_iT_i', for $i = 1, \ldots, n$. We will write just T for $T_1, \ldots, T_n$, and T^{-1} for $T_1^{-1}, \ldots, T_n^{-1}$, so that $A = k[T, T^{-1}]$, $A \otimes_k A \cong k[T, T', T^{-1}, T'^{-1}]$.

5.2. An action of G_m^n on $\mathbf{A}^2$ is given by a vector $(P, Q) \in k[T, T^{-1}, X, Y]$ (see 3.5) satisfying the following conditions:

(a) $(P(TT', X, Y), Q(TT', X, Y))$
$$= (P(T, X, Y), Q(T, X, Y)) \cdot (P(T', X, Y), Q(T', X, Y))$$

(b) $(P(1, X, Y), Q(1, X, Y)) = (X, Y)$.

(The vector multiplication of (a) is performed as if we were composing elements of $GA_2(k[T, T', T^{-1}, T'^{-1}])$ (see 1.1).) Such a vector (P, Q) necessarily determines an element of $GA_2(k[T, T^{-1}])$ (see 3.5).

5.3. For G_m^n, the (abstract) group of k-rational points Hom$(k[T, T^{-1}], k)$ is identified with the multiplicative group $(k^*)^n$. An action (P, Q) of G_m^n on $\mathbf{A}^2$ gives rise to the homomorphism $(k^*)^n \to GA_2(k)$ which takes

$$(u) = (u_1, \ldots, u_n) \in (k^*)^n \quad \text{to} \quad (P(u, X, Y), Q(u, X, Y)) \in GA_2(k).$$

5.4. One sees from Theorem 2.4 that if k is an infinite field extension of $\mathbf{F}_p$, there exist subgroups of $GA_2(k)$ isomorphic to k^* which are not conjugate in GA_2 to any subgroup of Af_2 or E_2. Thus there are faithful (non-algebraic) actions of k^* on $\mathbf{A}^2$ which are not "linear" or "elementary", up to equivalence.

5.5. However, if k is a field and $H \subset GA_2(k)$ is the image of a homomorphism $(k^*)^n \to GA_2$ induced by an (algebraic) action of G_m^n on $\mathbf{A}^2$, then H is of bounded degree, and so H is conjugate to a subgroup of Af_2 or E_2 (Proposition 1.11). As with actions of vector groups on $\mathbf{A}^2$ (§4), we will exploit this fact to prove Theorem 5.9 which explicitly describes actions of tori on $\mathbf{A}^2$, up to conjugacy.

5.6. Definition. Let $f \in k[T_i, T_i^{-1}]_{i=1}^n$, $f \neq 0$. We say that f is a multiplicative (Laurant) polynomial if

$$f(T_1, T_1', \dots, T_n T_n') = f(T_1, \dots, T_n) f(T_1', \dots, T_n'). \tag{73}$$

(Note that if k is an infinite field, this is equivalent to saying that f defines an endomorphism of the group $(k^*)^n$).

5.7. Proposition. *Suppose k is a domain and $f \in k[T_i, T_i^{-1}]_{i=1}^n$, $f \neq 0$. Then f is a multiplicative polynomial if and only if f is a Laurant monomial, i.e., f is of the form $\prod_{i=1}^n T_i^{\alpha_i}$, where $\alpha_1, \dots, \alpha_n \in \mathbf{Z}$.*

Proof. The *if* is obvious. Conversely, suppose f is multiplicative. Write $f = (\prod_{i=1}^n T_i^{\alpha_i}) \cdot g$ in such a way that $g \in k[T_1, \dots, T_n]$ and g is not divisible in $k[T_1, \dots, T_n]$ by any of the variables. Then g is also multiplicative. We claim $g = 1$. In equation (73) with g instead of f, set $T_1' = \cdots = T_{n-1}' = 1$, $T_n' = 0$ to get

$$g(T_1, \dots, T_{n-1}, 0) = g(T_1, \dots, T_n) \cdot g(1, \dots, 1, 0).$$

Since T_n doesn't divide g, $g(T_1, \dots, T_{n-1}, 0) \neq 0$, and so $g(1, \dots, 1, 0) \neq 0, \in k$. It follows that $g \in k[T_1, \dots, T_{n-1}]$. We continue this to get $g \in k$. Since g is multiplicative, $g = 1$.

5.8. *Remark.* One easily verifies that if k is a domain, the units of $k[T_i, T_i^{-1}]_{i=1}^n$ are precisely the elements uf where f is multiplicative and $u \in k^*$.

5.9. Theorem (Gutwirth). *Suppose k is an infinite field. Any action of the n-dimensional torus G_m^n on the affine plane $\mathbf{A}^2(k)$ is equivalent to an action of the form $(u(T)X, v(T)Y)$ where $u, v \in k[T_i, T_i^{-1}]_{i=1}^n$ are Laurant monomials (i.e., multiplicative).*

The proof of Theorem 5.9 is done in 5.10–5.15, and it is like the proof of Theorem 4.9 in that we first use the fact that abelian subgroups of $GA_2(k)$ of bounded degree are conjugate to linear or elementary type subgroups, and then complete the proof by making observations about the polynomials.

5.10. Let γ be an action of G_m^n on $\mathbf{A}^2$ given by (P, Q) satisfying (a) and (b) of 5.2. Let H be the image of the induced homomorphism $(k^*)^n \to GA_2(k)$ (see 3.6). The field k is infinite, and so no polynomial in $k[T, T^{-1}]$ vanishes on $(k^*)^n$, i.e., the k-rational points of G_m^n form a dense set. By Proposition 3.10 we can replace γ by an equivalent action and assume that either (P, Q) is of the form

$$(74)\qquad (uX+f(Y), vY+s)$$

with $u, v \in k[T, T^{-1}]^*$, $s \in k[T, T^{-1}]$, $f \in k[T, T^{-1}, Y]$ (i.e. $H \subset E_2$); or (P, Q) if of the form

$$(75)\qquad (aX+bY+r, cX+dY+t)$$

where

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL_2(k[T, T^{-1}]),$$

$r, t \in k[T, T^{-1}]$ (i.e. $H \subset Af_2$). We will deal with both possibilities.

5.11. First assume (P, Q) is of the form (44). Since u and v are units in $k[T, T^{-1}]$, and since $u(1)=v(1)=1$ by (b) of 5.2, we see by 5.8 that u and v are Laurant monomials. Condition (a) of 5.2 says that

$$\begin{aligned}(u(TT')X+f(TT', Y), v(TT')Y+s(TT'))\\ =(u(T)u(T')X+u(T')f(T, Y)+f(T', v(T)Y+s(T)), v(T')v(T)Y\\ +v(T')s(T)+s(T')).\end{aligned}$$

In particular,

$$(76)\qquad f(TT', Y)=u(T')f(T, Y)+f(T', v(T)Y+s(T)).$$

If $f \neq 0$, write $f(T, Y)=g_0+g_1Y+\cdots+g_dY^d$ with $g_o, \ldots, g_d \in k[T, T^{-1}]$, $g_d \neq 0$. Then equation (76) implies that

$$(77)\qquad g_d(TT')=u(T')g_d(T)+v(T)^dg_d(T').$$

A close look at (77) tells us exactly what g_d is, up to constant multiple. The Laurant monomials form a k-basis for $k[T, T^{-1}]$. All the monomials appearing in $g_d(TT')$ are T, T'-symmetric. It follows from (77) that $u(T)$ and $v(T)^d$ are the only monomials which can appear in $g_d(T)$. For if a monomial $w(T)$ appears, with $w \neq u, v^d$, then $u(T')w(T)$ is not symmetric, and it appears in $u(T')g_d(T)$. However, $u(T')w(T)$ is clearly not cancelled by any term of $v(T)^dg_d(T')$, since $w \neq v^d$, and so $u(T')w(T)$ appears in $g_d(TT')$—a contradiction. Therefore $g_d(T)=au(T)+bv(T)^d$ for some $a, b \in k$. The fact that $g_d(1)=0$ (see (b) of 5.2) implies that $b=-a$, and so $g_d(T)=au(T)-av(T)^d$ and $a \neq 0$.

5.12. Now, if we conjugate (P, Q) by an element of $GA_2(k)$, the resulting vector gives an action equivalent to γ. (This was explained in 3.5 and used extensively in §4. Upon performing the computation, one sees that conjugating (P, Q) by $(X+aY^d, Y) \in GA_2(k)$ yields

$$(78)\qquad (uX+f(Y)-auY^d+a(vY+s)^d, vY+s).$$

Note that the leading (highest degree) term (in Y) of $-auY^d+a(vY+s)^d$ is $(-au+av^d)Y$. This cancels the leading term of $f(Y)$, which is g_dY^d.

Hence the vector (78) is of the form $(uX+f'(Y), vY+s)$ with $\deg_Y f'<d$. Observe that this argument works to cancel f entirely if $d=0$. We can continue to conjugate until we get $f=0$, i.e., γ is given by $(uX, vY+s)$. We conjugate this vector by $(Y, X)\in GA_2(k)$ to get $(vX+s, uY)$ and employ the same reasoning as above (with $d=0$) to eliminate s. The resulting action is given by (vX, uY), which is of the form required.

5.13. Now assume (P, Q) is of the form (75) of 5.10, so that $H\subset Af_2$. The projection T of H onto $GL_2(k)$ is the image of the algebraic group homomorphism $G_m^n\to GL_2$ defined by $\left(\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right)$. Therefore T is connected, since G_m^n is connected. It follows from [3, Prop. 8.4, p. 203] that the image of T is conjugate in $GL_n(k)$ to a subgroup of D_2.

5.14. It follows that we can conjugate $(P, Q)=(aX+bY+r, cX+dY+t)$ by an element of $GL_2(k)$ $(\subset GA_2(k))$ to get $(uX+r', vY+t')$ where $r', t'\in k[T, T^{-1}]$. Now the vector is of the form (74), a situation which we have already treated. This concludes the proof of Theorem 5.9.

5.15. If k is algebraically closed, so that G_m^n and A^2 are algebraic groups, we restate Theorem 5.9 in terms of the action of closed points.

Restatement for the case when k is algebraically closed. Any action of the n-dimensional torus G_m^n on the affine plane $A^2(k)$ is equivalent to an action of the form

$$(t_1,\ldots,t_n)\cdot(x, y)=(t_1^{\alpha_1}\cdots t_n^{\alpha_n}x, t_1^{\beta_1}\cdots t_n^{\beta_n}y)$$

where $\alpha_1,\ldots,\alpha_n, \beta_1,\ldots,\beta_n\in\mathbf{Z}$.

This is the assertion proved by Gutwirth in [5].

REFERENCES

1. H. BASS, *Some remarks on group actions on trees*, Comm. Alg. vol. 4 (1976), pp. 1091–1126.
2. A. BIALYNICKI-BIRULA, *Remarks on the action of an algebraic torus k^n*, Bull. Acad. Polon. Sci., vol. 14 (1966), pp. 177–181.
3. A. BOREL, *Linear algebra groups*, Benjamin, New York, 1969.
4. M. DEMAZURE, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. Ecole Norm. Sup., 4ᵉ série, t. 3, pp. 507–588.
5. A. GUTWIRTH, *The action of an algebraic torus on the affine plane*, Trans. Amer. Math. Soc., vol. 105 (1962), pp. 407–414.
6. H. W. E. JUNG, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math., vol. 184 (1942), pp. 161–172.
7. A. KARRASS and D. SOLITAR, *The subgroups of a free product of two groups with an amalgamated subgroup*, Trans. Amer. Math. Soc., vol. 150 (1970) pp. 227–255.
8. M. MIYANISHI, *G_a-action of the affine plane*, Nagoya Math. J., vol. 41 (1971), pp. 97–100.
9. D. I. MOLDAVANSKI, *Certain subgroups of groups with one defining relation*, Sibirsk. Mat. Ž., vol. 8 (1967), pp. 1370–1384.
10. M. NAGATA, *On the automorphism group of k[X, Y]*, Lectures in Math., Kinokuniya-Tokyo.
11. R. RENTSCHLER, *Opérations du groupe additif sur le plan affine*, C. R. Acad. Soc. Paris, vol. 267 (1968), pp. 384–387.

12. J.-P. SERRE, *Arbes, amalgames, et SL_2*, Astérisque, vol. 46 (1977).
13. W. VAN DER KULK, *On polynomial rings in two variables*, Nieuw Arch. Wisk. vol. I (1953), pp. 33–41.
14. D WRIGHT, *Algebras which resemble symmetric algebras*, thesis, Columbia University.

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