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UNIVALENCE CRITERIA DEPENDING ON THE SCHWARZIAN DERIVATIVE[1]

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It is known [6, 7] that an analytic function f in $\Delta=\{z:|z|<1\}$ will be univalent if it satisfies certain conditions of the type

$$|\{f, z\}| \le R(|z|), \tag{1}$$

where $\{f, z\}$ denotes the Schwarzian derivative

$$\{f, z\} \equiv \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2}$$

The most important such condition [6] is

$$|\{f, z\}| \le \frac{2}{(1-|z|^2)^2} \tag{3}$$

This sufficient univalence criterion has two notable features: (a) If the constant 2 is replaced by 6, (3) is a necessary criterion for univalence [5], [6]. (b) If 2 is replaced by a smaller positive number, condition (3) guarantees, in addition, that f has a quasi-conformal extension to the entire complex plane [2], [3], [1].

Two other univalence criteria of the type (1) which, like (3), have the merit of using functions R of simple character are

$$|\{f, z\}| \le \frac{\pi^2}{2} \qquad [6] \tag{4}$$

and

$$|\{f, z\}| \le \frac{4}{1-|z|^2} \qquad [8] \tag{5}$$

In both cases the constants are the best possible. (For the sharpness of (3), cf. [4].)

The general procedure, described in [7], for generating univalence criteria of the type (1) relied heavily on the theory of linear second-order differential equations. In particular, the determination of the function R required the explicit knowledge of a solution of an associated differential equation. In the present paper we develop an alternative procedure which circumvents these difficulties. The only use of ideas properly belonging to the theory of differential equations will appear in the proof of the following lemma.

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LEMMA I. *Let $g(s)$ be a non-constant complex-valued function with three continuous derivatives on a real interval I, and let $g'(s) \neq 0$ on I. If*

$$\operatorname{Re}[\{g, s\}] \leq 0, \quad s \in I, \tag{6}$$

and a, b are two distinct points of I, then $g(a) \neq g(b)$.

We now state our main result.

THEOREM I. *Let f be a non-constant analytic function in Δ, and let $F(t)$ be a real-valued function on $[0, 1)$ with the following properties: F has two continuous derivatives; $\{F, t\}$ is continuous; $F' > 0$; $F''(0) = 0$; the expression*

$$\{F, t\}(1-t^2)^2 \tag{7}$$

is non-increasing. If

$$|\{f, z\}| \leq \{F, |z|\}, \quad z \in \Delta, \tag{8}$$

then f is univalent in Δ.

It will be shown later that the conclusion of Theorem I remains valid if F, F', and F'' are permitted to have simple discontinuities at a finite number of points and $F''_- < F''_+$ at these points. This more general result will also make it possible to prove Theorem I under the assumption $F''(0) \geq 0$ rather than the more restrictive condition $F''(0) = 0$.

In analogy to the three special cases mentioned before, we may say that condition (8) is sharp if $\{F, |z|\}$ cannot be replaced by $A\{F, |z|\}$, $A > 1$. While this is not true in all cases, there is a rather general class of functions F for which condition (8) is sharp in an even stronger sense.

THEOREM II. *Let $F(z)$ be an analytic function in Δ for which $\{F, z\} > 0$ for $z \in [0, 1)$ and $|\{F, z\}| \leq \{F, |z|\}$. Further, let $F(|z|) \to \infty$ for $|z| \to 1$, and let $F(|z|)$ satisfy all the hypotheses of Theorem I. If $G(z)$ is analytic in Δ and positive on $(-1, 1)$, and if ε is an arbitrarily small positive number, then the condition*

$$|\{f, z\}| \leq \{F, |z|\} + \varepsilon G(|z|) \tag{9}$$

is not sufficient to guarantee the univalence of f in Δ.

We list here three examples of a function F which satisfies Theorems I and II.

If

$$F(t) = \int_0^t \frac{ds}{(1-s^2)^{\mu+1}}, \quad 0 \leq \mu \leq 1,$$

then

$$\{F, t\} = \frac{2(1+\mu)(1-\mu t^2)}{(1-t^2)^2},$$

and this has all the required properties. Hence, f is univalent in Δ if

$$|\{f, z\}| \le \frac{2(1+\mu)(1-\mu|z|^2)}{(1-|z|^2)^2}, \quad 0 \le \mu \le 1,$$

and this result is sharp in the sense of Theorem II. For $\mu=0$ and $\mu=1$, this yields criteria (3) and (5), respectively. Another function which may be used in Theorems I and II is

$$F(t) = \int_0^t \frac{(1+s^2)^\mu}{(1-s^2)^{\mu+1}}\, ds, \quad 0 \le \mu \le 1.$$

This function leads to the univalence criterion

$$|\{f, z\}| \le \frac{2(1-\mu^2)}{(1-|z|^2)^2} + \frac{2\mu(2+\mu)}{(1+|z|^2)^2}, \quad 0 \le \mu \le 1.$$

Criterion (4) is obtained by using the function $F(t) = \tan \frac{1}{2}\pi t$, for which $\{F, t\} = \pi^2/2$.

Turning now to the proof of Theorem I, we note that under a conformal mapping $z \to \zeta$ the Schwarzian derivative transforms according to the formula

$$\{f, z\} = \left(\frac{d\zeta}{dz}\right)^2 \{f, \zeta\} + \{\zeta, z\}. \tag{10}$$

$\{f, z\}$ remains unchanged if f is subjected to a Moebius transformation. Thus, if $\zeta(z)$ is a Moebius transformation, $\{\zeta, z\} = \{z, z\} = 0$, and (10) simplifies to

$$\{f, z\} = \left(\frac{d\zeta}{dz}\right)^2 \{f, \zeta\}. \tag{11}$$

Suppose now that f is not univalent in Δ and that, as a result, there exist two distinct points z_1, z_2 in Δ such that $f(z_1) = f(z_2)$. If C is the intersection of Δ and the circle orthogonal to $|z|=1$ which passes through z_1 and z_2, it is easy to see that there exist constants κ, ρ ($|\kappa|=1$, $0 \le \rho < 1$) such that the transformation

$$z = \kappa\left(\frac{\zeta + i\rho}{1 - i\rho\zeta}\right) \tag{12}$$

(which maps Δ onto itself) carries C into the diameter $-1 < \zeta < 1$. Since

$$|dz|\,(1-|z|^2)^{-1} = |d\zeta|\,(1-|\zeta|^2)^{-1},$$

it follows from (11) that $|\{f, z\}|\,(1-|z|^2)^2 = |\{f, \zeta\}|\,(1-|\zeta|^2)^2$. For $z \in C$ we have $\zeta = \operatorname{Re}\{\zeta\} = t$ and condition (8) is thus seen to be equivalent to

$$|\{f, t\}| \le \left(\frac{1-|z|^2}{1-t^2}\right)^2 \{F, |z|\}, \quad z \in C, \tag{13}$$

where $-1 < t < 1$.

(12) shows that, for $z \in C$ (i.e., $\zeta = t$), $|z|^2 - t^2 = \rho^2(1+\rho^2 t^2)^{-1}(1-t^4)$ and thus $|z| > |t|$. Since the expression (7) is nonincreasing on $[0, 1)$, it follows that

$$\{F, |z|^2\}(1-|z|^2)^2 \leq \{F, |t|\}(1-t^2)^2,$$

and (13) is found to imply the inequality $|\{f, t\}| \leq \{F, |t|\}$ for $-1 < t < 1$. This may also be written in the form

$$|\{f, t\}| \leq \{F, t\}, \quad t \in (-1, 1), \tag{14}$$

if we require $F(0) = 0$ (which we may do without changing the value of $\{F, t\}$) and define F for negative t by $F(-t) = -F(t)$. With this definition, F' and $\{F, t\}$ become even functions and, because of $F''(0) = 0$, F has all the requisite derivatives at $t = 0$.

Let now $s(t)$ be a real-valued function with three continuous derivatives (and with $s'(t) > 0$) which maps $-1 < t < 1$ onto some interval $(-\alpha, \alpha)$, $\alpha > 0$ (where α may be $+\infty$). By (10) (applied to real variables s, t) we have

$$\{F, t\} = \left(\frac{ds}{dt}\right)^2 \{F, s\} + \{s, t\} \tag{15}$$

and a similar formula for $\{f, t\}$. Since $s'(t) > 0$, $\{F, s\}$ will be continuous on $(-\alpha, \alpha)$. By (14), $\operatorname{Re}[\{f, t\} - \{F, t\}] \leq 0$ and thus, by (15) and the corresponding formula for f,

$$s'^2(t) \operatorname{Re}[\{f, s\} - \{F, s\}] \leq 0. \tag{16}$$

The function $F(t)$ has three continuous derivatives, and we may therefore set $s(t) = F(t)$. Since $\{F, s\} = \{F, F\} = 0$, we obtain $\operatorname{Re}[\{f, s\}] \leq 0$, $s = F(t)$, or, if we write $f[z(s)] = g(s)$ $(z \in C)$,

$$\operatorname{Re}[\{g, s\}] \leq 0, \quad -\alpha < s < \alpha. \tag{17}$$

The existence of two distinct points z_1, z_2 on C such that $f(z_1) = f(z_2)$ is equivalent to the existence of a, b such that $-\alpha < a < b < \alpha$ and $g(a) = g(b)$. Because of (17) and Lemma I we therefore conclude that g is constant on (a, b). Hence, f is constant on the section of C between z_1 and z_2, i.e., f must be a constant. The proof of Theorem I will therefore be complete if we establish the truth of Lemma I.

To do so, we define two functions u and v on the real interval I by

$$g' = 1/v^2, \qquad u/v = g - g(a). \tag{18}$$

Since $g' \neq 0$, v and u have two continuous derivatives on I. By the second formula (18), $g' = (vu' - uv')v^{-2}$ and thus $vu' - uv' = 1$, i.e., $vu'' - uv'' = 0$, or $u''/u = v''/v$. Since, by the first formula (18),

$$\tfrac{1}{2}\{g, s\} = -v''/v, \tag{19}$$

this shows that u is a solution of the linear differential equation

$$u''+\tfrac{1}{2}\{g, s\}u=0. \tag{20}$$

If $g(a)=g(b)$, it follows from (18) that $u(a)=u(b)=0$. Multiplying (20) by $\bar{u}$ and integrating from a to b, we obtain

$$[\bar{u}\cdot u']_a^b-\int_a^b |u'|^2\,ds+\tfrac{1}{2}\int_a^b \{g, s\}\,|u|^2\,ds=0.$$

Taking real parts, and noting that $u(a)=u(b)=0$, we arrive at the identity

$$\int_a^b |u'|^2\,ds=\tfrac{1}{2}\int_a^b \operatorname{Re}\,([\{g, s\}]\,|u|^2\,ds.$$

But, by (6), this is absurd unless $u'(s)\equiv 0$ on $[a, b]$. Because $u(a)=0$, this implies $u\equiv 0$. Thus, by (18), g is constant on $[a, b]$, as asserted. This completes the proof of Theorem I.

To prove Theorem II, it is sufficient to show that a function f for which

$$\{f, z\}=\{F, z\}+\varepsilon G(z) \tag{21}$$

(where F and G are as described in Theorem II) cannot be univalent on $(-1, 1)$.

By Theorem I, a function $F(z)$ subject to the hypotheses of Theorem II is univalent in Δ. Since $F(|z|)\to\infty$ for $|z|\to 1$, the mapping $s=F(t)$ transforms the linear segment $(-1, 1)$ into the real line $(-\infty, \infty)$. Using the transformation formula (15) we thus find, by (21), that

$$\varepsilon G(t)=\{f, t\}-\{F, t\}=s'^2(t)[\{f, s\}-\{F, s\}]=s'^2\{f, s\},$$

i.e., $\{g, s\}=2H(s)$, where $g(s)=f[t(s)]$ and $H(s)$ is positive on $(-\infty, \infty)$. Our result will be proved if we can show that for such a function g we cannot have $g(a)\neq g(b)$ for all a, b such that $-\infty<a<b<\infty$.

We choose an arbitrary real value $g(a)$ and define the functions u, v by (18). As shown before, u and v will then be two independent solutions of the differential equation (20). Since $\{g, s\}$ is real, u and v can be chosen to be real functions, and g is thus real on $(-\infty, \infty)$. By (18), $g(a)=g(b)$ is equivalent to the existence of a solution of (20) for which $u(a)=u(b)=0$. Because of $\{g, s\}=2H(s)$, our task thus reduces to showing that a real solution of

$$u''+H(s)u=0, \quad H(s)>0, \tag{22}$$

must have at least two zeros on the real line.

This is accomplished by constructing a function $R(s)$ such that $R(s)\leq H(s)$, and showing that the equation $w''+R(s)w=0$ has a solution which vanishes twice on $(-\infty, \infty)$. By the Sturm comparison theorem it then follows that (22) must likewise have a solution with at least two zeros on $(-\infty, \infty)$. If α is a positive number, we denote by δ the (positive) minimum of $H(s)$ on

$[\alpha, -\alpha]$ and set $R(s)=\delta$ on $[-\alpha, \alpha]$ and $R(s)=0$ elsewhere (the discontinuities of R do not affect the Sturm comparison theorem). Clearly, $R(s)\le H(s)$ on $(-\infty, \infty)$. If $\alpha\sqrt{\delta}\ge \pi/2$, the solution $w_0=\cos\sqrt{\delta}\, s$ of $w''+R(s)w=0$ on $[-\alpha, \alpha]$ has at least two zeros on this interval, and we are done. If $\alpha\sqrt{\delta}<\pi/2$, we extend this solution to the interval $(-\infty, \infty)$ by setting $w=w_0(\alpha)+w_0'(\alpha)(t-\alpha)$ on (α, ∞) and $w=w_0(-\alpha)+w_0'(-\alpha)(t+\alpha)$ on $(-\infty, -\alpha)$. Since $w_0'(\alpha)<0$ and $w_0'(-\alpha)>0$, both these half-lines must intersect the s-axis. Thus, on a sufficiently large interval the solution w has two zeros, and our proof is complete.

Finally, we have to substantiate the remark—made after the statement of Theorem I—concerning the possible relaxation of the regularity conditions imposed on F. Since, for given $\{F, t\}$, F is determined only up to an arbitrary Moebius transformation, the three independent constants entering the latter may be used to compensate for possible simple discontinuities of the originally chosen function F and two of its derivatives at a given point. To explore these possibilities, it is sufficient to consider the case of one such point, say t_0. Suppose, then, that $F=F_0$ on $[0, t_0]$ and $F=F_1$ on $[t_0, 1)$, and that these functions are non-negative and increasing and have three continuous derivatives on their respective intervals. Clearly, we may normalize F_1 by the condition $F_0'(t_0)=F_1'(t_0)$. We now replace F_1 by the function

$$F_2(t)=\frac{F_0(t_0)+[1+\mu F_0(t_0)][F_1(t)-F_1(t_0)]}{1+\mu[F_1(t)-F_1(t_0)]}, \tag{23}$$

where μ is a constant. We then have $F_2(t_0)=F_0(t_0)$, $F_2'(t_0)=F_0'(t_0)$, and $\{F_2, t\}=\{F_1, t\}$. To prevent the denominator from vanishing on $(t_0, 1)$, we restrict μ to non-negative values. Since

$$\frac{F_2''(t)}{F_2'(t)}=\frac{F_1(t)}{F_1'(t)}-\frac{2\mu F_1'(t)}{1+\mu[F_1(t)-F_1(t_0)]}$$

we shall have $F_0''(t_0)=F_2''(t_0)$ if μ is so chosen that $2\mu F_1'(t_0)=F_1''(t_0)-F_0''(t_0)$. In view of the condition $\mu\ge 0$ and $F'>0$, this is possible if $F_1''(t_0)\ge F_0''(t_0)$. Thus, if we set $F=F_0$ on $[0, t_0]$ and $F=F_2$ on $[t_0, 1)$, F, F', F'' and—because of the assumed continuity of $\{F, t\}$—also F''' will be continuous on $[0, 1)$, as required in the proof of Theorem I.

This flexibility in choosing the function F for given $\{F, t\}$ can also be used to show that the assumption $F''(0)=0$ made in Theorem I may be replaced by $F''(0)\ge 0$. The condition $F''(0)=0$ was imposed to ensure the continuity of F'' at 0 if F is defined for negative t by $F(-t)=-F(t)$. By the procedure just described, we may replace $-F(t)$ by a Moebius transformation of the type (23), provided $-F''(0)\le F''(0)$, i.e., provided $F''(0)\ge 0$. The function F obtained in this way will then have three continuous derivatives on $(-1, 1)$. It is easily verified that the lack of symmetry of this function has no effect on the proof of Theorem I.

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