TOEPLITZ OPERATORS ON THE BALL WITH PIECEWISE CONTINUOUS SYMBOL

BY

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1. Introduction

Let B denote the open unit ball in $\mathbf{C}^n$ and let $A^2(B)$ denote the Bergman space of square-integrable holomorphic functions on B. The Toeplitz operator with symbol φ, T_φ, is defined by $T_\varphi f = P(\varphi f)$, $f \in A^2(B)$, where φ is in $L^\infty(B)$, and P is the orthogonal projection of $L^2(B)$ onto $A^2(B)$. Let E be a $(2n-1)$-dimensional real hyperplane in $\mathbf{C}^n$ intersecting B. The set $B \setminus E$ then consists of two components, which we will label B_+ and B_-. We define

$$HC(B) = \{\varphi \in L^\infty(B): \varphi \,|B_+, \varphi| \, B_- \text{ are uniformly continuous}\}.$$

The main purpose of this paper is to compute the essential spectrum of T_φ for φ in $HC(B)$, and to show, in particular, that it is connected.

Note that $HC(B)$ is a closed subalgebra of $L^\infty(B)$, and that we can write

$$HC(B) = \{\langle f, g \rangle: f \in C(\bar{B}_+), g \in C(\bar{B}_-)\}.$$

Our interest in $HC(B)$ stems from the fact that in many ways it seems like a reasonable analogue of the algebra PC on the unit circle T. Recall that PC is the closed subalgebra of $L^\infty(T)$ consisting of piecewise continuous functions on T, and that $\varphi^\#$ is the curve obtained by joining left- and right-hand limits of φ by a line segment at points of discontinuity. For details, and a proof of the following, see [3, pp. 20–23].

PROPOSITION 1.1.   *If φ and ψ are in PC, then:*
   (i) $T_\varphi T_\psi - T_\psi T_\varphi$ *is compact.*
   (ii) T_φ *is Fredholm if and only if $\varphi^\#$ does not pass through the origin.*

Here T_φ denotes the Toeplitz operator acting on the Hardy space H^2 on the circle.

We will show that this proposition remains essentially true for φ and ψ in $HC(B)$. We point out that 1.1(i) depends on the fact that we can approach a point of discontinuity of a function from only two directions on T. This property is retained by the functions in $HC(B)$. For suppose λ is in E and $\varphi = \langle f, g \rangle$ is in $HC(B)$. If we approach λ through B_+, then $\lim \varphi(z) = f(\lambda)$, and if we approach λ through B_-, then $\lim \varphi(z) = g(\lambda)$.

Although everything we do is true for any hyperplane, we will assume

$$E = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 \text{ is real}\}$$

to simplify notation and some of the constructions. We will also assume $n \geq 2$, the case $n = 1$ being much like the situation for PC on the circle. Let

$$B_+ = \{(z_1, \ldots, z_n) \in B : \text{Im } z_1 > 0\}, \quad B_- = \{(z_1, \ldots, z_n) \in B : \text{Im } z_1 < 0\}.$$

Denote $E \cap \partial B$ by ∂E.

Let $\mathfrak{L}(A^2)$ denote the set of bounded linear operators on $A^2(B)$, and $\mathfrak{R}$ denote the ideal of compact operators in $\mathfrak{L}(A^2)$. Recall that the operator S in $\mathfrak{L}(A^2)$ is said to be Fredholm if the element $S + \mathfrak{R}$ is invertible in the quotient algebra $\mathfrak{L}(A^2)/\mathfrak{R}$. If S is Fredholm then dim ker S and dim ker S^* are finite, and the index of S, $j(S)$, is defined by

$$j(S) = \dim \ker S - \dim \ker S^*.$$

The essential spectrum of S, ess $\sigma(S)$, is the set of all λ in $\mathbb{C}$ such that $S - \lambda I$ is not Fredholm. The norm of $S + \mathfrak{R}$ in $\mathfrak{L}(A^2)/\mathfrak{R}$ will be denoted by $\|S + \mathfrak{R}\|$.

The following proposition summarizes the basic properties of Toeplitz operators which we will need in this paper.

PROPOSITION 1.2.   *If $\varphi, \psi \in L^\infty(B)$ and $\alpha \in \mathbb{C}$, then:*
   (i) $T_{\varphi + \psi} = T_\varphi + T_\psi$, $T_{\bar{\varphi}} = (T_\varphi)^*$, $\alpha T_\varphi = T_{\alpha\varphi}$.
   (ii) T_1 *is the identity operator.*
   (iii) $\|T_\varphi\| \leq \|\varphi\|_\infty$.
   (iv) $\|T_\varphi + \mathfrak{R}\| \leq \lim_{\varepsilon \to 1} \sup_{1 > |z| > \varepsilon} |\varphi(z)|$.
   (v)   *For ψ in $C(\bar{B})$, $T_\varphi T_\psi - T_{\varphi\psi} \in \mathfrak{R}$.*
   (vi) *If $\varphi, \psi \in C(\bar{B})$, $T_\varphi T_\psi - T_\psi T_\varphi \in \mathfrak{R}$.*
   (vii) *If $\varphi \in C(\bar{B})$, ess $\sigma(T_\varphi) = \varphi(\partial B)$, the range of φ restricted to the boundary of B.*

Proof.   The first three statements follow easily from the definition of a Toeplitz operator. Proofs of (iv) and (v) can be found in [4]. Statement (vi) follows from (v). For a proof of (vii), see [1] or [5]. (These last two papers are basic, and should be consulted, as should [2] for the corresponding results on the circle.)

2. Main theorems

Let dV be the usual Lebesgue measure on B,

$$V(B) = (\pi^n)^{1/2}(n!)^{-1/2},$$

so that for f and g in B the inner-product is defined by $(f, g) = \int f\bar{g}\, dV$, the integral taken over B.

Let $\mathfrak{T}(C(\bar{B}))$ and $\mathfrak{T}(HC(B))$ be the closed subalgebras of $\mathfrak{L}(A^2(B))$

generated, respectively, by

$$\{T_\varphi\colon \varphi \in C(\bar{B})\} \quad \text{and} \quad \{T_\varphi\colon \varphi \in HC(B)\}.$$

For λ on ∂B we define $\mathfrak{F}_\lambda$ to be the closed ideal in $\mathfrak{T}(HC(B))$ generated by $\{T_\varphi\colon \varphi \in C(\bar{B}), \varphi(\lambda) = 0\}$.

LEMMA 2.1.   (i)   *The algebra $\mathfrak{T}(C(\bar{B}))$ is irreducible and contains $\mathfrak{R}$.*

(ii)   *$\mathfrak{T}(C(\bar{B}))/\mathfrak{R}$ is *-isometrically isomorphic to $C(\partial B)$ and hence has maximal ideal space ∂B.*

(iii)   *Each ideal $\mathfrak{F}_\lambda$ contains $\mathfrak{R}$.*

Proof.   See [1] for proofs of (i) and (ii). Since $\mathfrak{T}(C(\bar{B}))$ is irreducible and is contained in $\mathfrak{T}(HC(B))$, it follows that $\mathfrak{T}(HC(B))$ is irreducible. Each $\mathfrak{F}_\lambda$ is a closed ideal in $\mathfrak{T}(HC(B))$, and hence is irreducible. We must show $\mathfrak{F}_\lambda$ contains a non-zero compact operator. Let φ be a non-negative continuous function vanishing on a neighborhood of ∂B. It follows that T_φ is compact [5, p. 356]. Since

$$\int \varphi \, |z_1|^2 \, dV = (T_\varphi z_1, z_1),$$

T_φ is non-zero. We conclude that $\mathfrak{F}_\lambda$ contains all compact operators.

Denote the quotient algebra $\mathfrak{T}(HC(B))/\mathfrak{F}_\lambda$ by $\mathfrak{T}_\lambda$, and the quotient map from $\mathfrak{T}(HC(B))$ onto $\mathfrak{T}_\lambda$ by Φ_λ. The next proposition will be our basic tool, and should be compared with [2, Theorem 7.49].

PROPOSITION 2.2.   *Let Φ be the *-homomorphism from $\mathfrak{T}(HC(B))$ to $\sum_{\lambda \in \partial B} \oplus \, \mathfrak{T}_\lambda$ defined by $\Phi = \sum_{\lambda \in \partial B} \oplus \, \Phi_\lambda$.*

(i)   *The sequence*

$$(0) \to \mathfrak{R} \to \mathfrak{T}(HC(B)) \overset{\Phi}{\to} \sum \oplus \, \mathfrak{T}_\lambda$$

is exact at $\mathfrak{T}(HC(B))$.

(ii)   *If $\varphi \in HC(B)$, then T_φ is Fredholm if and only if $\Phi_\lambda(T_\varphi)$ is invertible in $\mathfrak{T}_\lambda$ for all λ in ∂B.*

(iii)   *If φ is continuous at λ, then $\Phi_\lambda(T_\varphi) = \varphi(\lambda)$.*

Proof.   Lemma 2.1(ii) implies that $\mathfrak{T}(C(\bar{B}))/\mathfrak{R}$ is contained in the center of $\mathfrak{T}(HC(B))/\mathfrak{R}$. We also know that the maximal ideal space of $\mathfrak{T}(C(\bar{B}))/\mathfrak{R}$ is ∂B. Parts (i) and (ii) now follow from a general C^*-algebra result [2, pp. 196–197]. Suppose $\varphi = \langle f, g \rangle$ is in $HC(B)$ and $\lambda \in \partial E$. If φ is continuous at λ then $f(\lambda) = g(\lambda)$, and we can unambiguously define $\varphi(\lambda)$ to be $f(\lambda)$ or $g(\lambda)$. (From now on if $\lambda \in \partial B$ and we write $\varphi(\lambda)$, it will be understood that φ is continuous at λ.) To prove (iii) it is sufficient to assume $\varphi(\lambda) = 0$. For $\varepsilon' > 0$, choose an open ball $N(\lambda)$ about λ in B such that $\|\varphi \mid N(\lambda)\|_\infty < \varepsilon'$. Let ψ be a continuous function on $\bar{B}$ such that $\psi = 0$ on a neighborhood of λ contained

in $N(\lambda)$, and $\psi = 1$ on $B \setminus N(\lambda)$. Let I and J be, respectively, the characteristic functions of the sets $N(\lambda)$ and $B \setminus N(\lambda)$. By Proposition 1.2 we have

$$T_\varphi = T_{\varphi I} + T_{\varphi \psi J} \quad \text{and} \quad \Phi_\lambda(T_\varphi) = \Phi_\lambda(T_{\varphi I}) + \Phi_\lambda(T_{\varphi \psi J}).$$

By Proposition 2.2(iii) we have $\Phi_\lambda(T_{\varphi \psi J}) = \Phi_\lambda(T_{\varphi J})\Phi_\lambda(T_\psi) = 0$, since $\psi \in \mathfrak{I}_\lambda$. Thus

$$\|\Phi_\lambda(T_\varphi)\| = \|\Phi_\lambda(T_{\varphi I})\| = \|T_{\varphi I} + \mathfrak{I}_\lambda\| \leq \|T_{\varphi I} + \mathfrak{R}\|,$$

since $\mathfrak{I}_\lambda$ contains $\mathfrak{R}$. The proof is complete since

$$\|T_{\varphi I} + \mathfrak{R}\| \leq \lim_{\varepsilon \to 1} \sup_{1 > |z| > \varepsilon} |\varphi I(z)|.$$

Let $\sigma(\Phi_\lambda(T_\varphi))$ denote the spectrum of $\Phi_\lambda(T_\varphi)$ in $\mathfrak{X}_\lambda$. It follows from Proposition 2.2 that for φ in $HC(B)$,

$$\text{ess } \sigma(T_\varphi) = \bigcup_{\lambda \in \partial B} \sigma(\Phi_\lambda(T_\varphi))$$

$$= \{\varphi(\lambda): \lambda \in \partial B \setminus \partial E\} \cup \left[\bigcup_{\lambda \in \partial E} \sigma(\Phi_\lambda(T_\varphi)) \right].$$

THEOREM 2.3. *Let $\varphi = \langle f, g \rangle$ be in $HC(B)$. The essential spectrum of T_φ consists of $\{\varphi(\lambda): \lambda \in \partial B \setminus \partial E\}$ together with the closed line segments joining $f(\lambda)$ to $g(\lambda)$, for each λ in ∂E.*

We postpone the proof of this theorem until the next section. All that needs to be proved is that for λ in ∂E, $\sigma(\Phi_\lambda(T_\varphi))$ is equal to the closed line segment joining $f(\lambda)$ to $g(\lambda)$. The next theorem makes use of the fact that each φ in $HC(B)$ can be written as an ordered pair of continuous functions. This should be compared with the proof of the corresponding result for PC on the circle [3, p. 21].

THEOREM 2.5. *The algebra $\mathfrak{X}(HC(B))/\mathfrak{R}$ is commutative.*

Proof. In view of Proposition 2.2, it is sufficient to show that the algebra $\Phi_\lambda(\mathfrak{X}(HC(B)))$ is commutative for each $\lambda \in \partial B$. If $\lambda \in \partial B \setminus \partial E$ then Φ_λ is evaluation at λ, and the result is easy. Suppose $\lambda \in \partial E$, and $\varphi = \langle f_1, f_2 \rangle$ and $\psi = \langle g_1, g_2 \rangle \in HC(B)$. Further suppose that φ is chosen to be discontinuous at λ. Then there exist constants $\alpha, \beta, \beta \neq 0$, such that $\alpha\varphi + \beta\psi$ is continuous at λ. Thus

$$\alpha\Phi_\lambda(\varphi) + \beta\Phi_\lambda(\psi) = (\alpha\varphi + \beta\psi)(\lambda),$$

and it follows that $\Phi_\lambda(\mathfrak{X}(HC(B)))$ is generated by $\Phi_\lambda(T_\varphi)$ and the identity. Therefore $\Phi_\lambda(\mathfrak{X}(HC(B)))$ is commutative. Theorem 2.5 now follows immediately.

COROLLARY 2.6. *If φ and ψ are in $HC(B)$, then $T_\varphi T_\psi - T_\psi T_\varphi$ is a compact operator.*

3. The proof of Theorem 2.3

Let λ be a point on ∂E, and suppose $\varphi = \langle f, g \rangle$ belongs to $HC(B)$. We set $\psi = \langle f(\lambda), g(\lambda) \rangle$. Thus ψ is constant on each half of the ball, and $\Phi_\lambda(T_\varphi) = \Phi_\lambda(T_\psi)$, by Proposition 2.2(iii). The proof of Theorem 2.3 will be completed by showing that the spectrum of $\Phi_\lambda(T_\psi)$ consists precisely of the line segment joining $f(\lambda)$ and $g(\lambda)$. This is equivalent to the following lemma.

LEMMA 3.1. *Let* $\alpha, \beta \in \mathbf{C}$, *and* $\psi = \langle \alpha, \beta \rangle$. *Then for* $\lambda \in \partial E$, $\Phi_\lambda(T_\psi)$ *is invertible if and only if the line segment joining* α *and* β *does not pass through the origin.*

We prove one half of the lemma immediately. Suppose the line segment joining α and β does not pass through the origin. Since $aT_\psi = T_{a\psi}$ for a constant a, we may assume α and β are both in the open right half-plane. Thus there exists an $\varepsilon > 0$ such that $|\varepsilon\alpha - 1| < 1$ and $|\varepsilon\beta - 1| < 1$, and hence $\|\varepsilon\psi - 1\|_\infty < 1$. Since

$$\|T_{\varepsilon\psi} - 1\| = \|T_{\varepsilon\psi - 1}\| \leq \|\varepsilon\psi - 1\|_\infty,$$

the operator T_ψ is invertible. Thus $\Phi_\lambda(T_\psi)$ is invertible.

Assume now that the line segment joining α and β passes through the origin. We will now show that for each λ on ∂E, $\Phi_\lambda(T_\lambda)$ is non-invertible in $\mathfrak{T}_\lambda$. (Note that just showing T_ψ is non-Fredholm would only imply that there exists some λ on ∂E such that $\Phi_\lambda(T_\psi)$ is not invertible.) Let

$$\Lambda_p = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \partial E : \lambda_1 = p\},$$

and note that $\bigcup \Lambda_p = \partial E$, the union taken over all p, $-1 \leq p \leq 1$. If $p = \pm 1$, *then of course* $\Lambda_p = \{(p, 0, \dots, 0)\}$.

LEMMA 3.1. *Let* $\lambda, \mu \in \Lambda_p$, *and let* U *be a unitary transformation of* $\mathbf{C}^n$ *which leaves the* z_1-*coordinate fixed, and maps* λ *onto* μ. *There is a* *-isometric isomorphism* W *from* $\Phi_\lambda(\mathfrak{T}(HC(B)))$ *onto* $\Phi_\mu(\mathfrak{T}(HC(B)))$ *such that for* φ *in* $HC(B)$,

$$W(\Phi_\lambda(T_\varphi)) = \Phi_\mu(T_{\varphi'}) \quad where \quad \varphi' = \varphi \circ U^{-1}.$$

Proof. The map $D : \varphi \to \varphi \circ U$ is an automorphism of $HC(B)$, mapping $\mathfrak{J}_\mu$ onto $\mathfrak{J}_\lambda$. Considered as a linear operator on $A^2(B)$, D is unitary, and $T_{\varphi'} = D^{-1} T_\varphi D$. Standard arguments now yield the lemma.

The above isomorphism maps $\Phi_\lambda(T_\psi)$ onto $\Phi_\mu(T_\psi)$. Thus if $\Phi_\lambda(T_\psi)$ is non-invertible for any λ in Λ_p, it is non-invertible for all λ in Λ_p. Nothing like the above lemma holds if $\mu \in \Lambda_p$, $\lambda \in \Lambda_{p'}$, and $p \neq p'$. For this would require a rotation of $\mathbf{C}^n$ which maps λ to μ, maps E onto itself (to preserve $HC(B)$), and is unitary (to preserve $A^2(B)$). No such map exists.

LEMMA 3.3. *Given* p, $-1 \leq p \leq 1$, *there exists a function* f *in* $HC(B)$ *such*

that:

   (i) T_f *is non-Fredholm.*

  (ii) $\Phi_\mu(T_f)$ *is invertible in* $\mathfrak{T}_\mu$, $\mu \in \partial B$, $\mu \notin \Lambda_p$.

 (iii) $(f - \psi)(\lambda) = 0$, $\lambda \in \Lambda_p$.

Rather than prove Lemma 3.3, we will state and prove an equivalent lemma. First however let us see how Lemma 3.3 establishes the second half of Lemma 3.1. It follows from Proposition 2.2(ii) that there exists λ' in Λ_p such that $\Phi_{\lambda'}(T_f)$ is not invertible. By (iii) of the same proposition we have $\Phi_{\lambda'}(T_f) = \Phi_{\lambda'}(T_\psi)$. By the previous remarks, $\Phi_\lambda(T_\psi)$ is non-invertible for all λ in Λ_p. Since $\partial E = \bigcup \Lambda_p$, the second half of Lemma 3.1 is complete.

LEMMA 3.4.   *Given* p, $-1 \leq p \leq 1$, *there exists a function* $f = \langle f_1, f_2 \rangle$ *in* $HC(B)$ *such that:*

   (i) T_f *is non-Fredholm.*

  (ii) f *is non-zero on* $\partial B \setminus \partial E$.

 (iii)   *The line segment joining* $f_1(\mu)$ *and* $f_2(\mu)$ *does not pass through* 0, *for* μ *in* $\partial E \setminus \Lambda_p$.

 (iv) $f_1(\lambda) = \alpha$, $f_2(\lambda) = \beta$, $\lambda \in \Lambda_p$.

The equivalence of Lemma 3.3 and Lemma 3.4 is completely straightforward. We point out that the first part of Lemma 3.1 is needed here. It now remains to construct a function f which satisfies Lemma 3.4. We will consider the cases $-1 < p < 1$ and $p = \pm 1$ separately.

So let us assume $-1 < p < 1$ and we may assume α and β are pure imaginary with $\operatorname{Im} \alpha > 0 > \operatorname{Im} \beta$. To simplify notation we will set $p = 0$. It should not be difficult to see what modifications have to be made in the definition of f for other p. We define a map f^* on T by

$$f^*(e^{i\theta}) = f^*(\theta) = \begin{cases} 1 + \dfrac{2}{\pi}(\alpha - 1)\theta, & 0 \leq \theta \leq \dfrac{\pi}{2}, \\[2mm] 1 + 2\alpha - \dfrac{2}{\pi}(\alpha + 1)\theta, & \dfrac{\pi}{2} \leq \theta \leq \pi, \\[2mm] 1 + \dfrac{2}{\pi}(1 - \beta)\theta, & \dfrac{-\pi}{2} \leq \theta \leq 0, \\[2mm] 1 + 2\beta + \dfrac{2}{\pi}(1 + \beta)\theta, & -\pi \leq \theta \leq \dfrac{-\pi}{2}. \end{cases}$$

We now define $f = \langle f_1, f_2 \rangle$ in $HC(B)$ by

$$f_1(x_1 + iy_1, z_2, \ldots, z_n) = f^*(\cos^{-1} x_1),$$

for $\operatorname{Im} y_1 \geq 0$, and

$$f_2(x_1 + iy_1, z_2, \ldots, z_n) = f^*(-\cos^{-1} x_1),$$

for $\operatorname{Im} y_1 \leq 0$. The range of both f^* and f is the quadrilateral determined by

the points 1, α, -1, and β. Notice that for λ in Λ_p, $f_1(\lambda) = f^*(\cos^{-1} p)$ and $f_2(\lambda) = f^*(-\cos^{-1} p)$, hence in particular for λ in Λ_0, $f_1(\lambda) = \alpha$, $f_2(\lambda) = \beta$. It is clear that f satisfies (ii), (iii), and (iv) of Lemma 3.4. We will now show that T_f is not Fredholm.

Let $\Delta = \{(z_1, \ldots, z_n): |z_1| < 1, z_2 = \cdots = z_n = 0\}$, define

$$L^\infty(\Delta) = \{\varphi \in L^\infty(B): \varphi \text{ is independent of } z_2, \ldots, z_n\},$$

and let $C(\bar{\Delta})$ and $HC(\Delta)$ be the collections of functions in $C(\bar{B})$ and $HC(B)$, respectively, which are independent of $z_2, \ldots, z_n$. Note that f is in $HC(\Delta)$. We will show that the adjoint of T_f has an infinite dimensional kernel.

Recall that an orthonormal basis for $A^2(B)$ is given by

$$e_k = (\pi^n)^{-1/2} \left[\frac{(n+|k|)!}{k!} \right]^{1/2} z^k,$$

where $k = (k_1, \ldots, k_n)$ is an n-tuple of non-negative integers,

$$|k| = k_1 + \cdots + k_n, \quad k! = k_1! \cdots k_n!, \quad z^k = z_1^{k_1} \cdots z_n^{k_n}.$$

For each $(n-1)$-tuple k of non-negative integers we define $A^2(k)$ to be the subspace of $A^2(B)$ generated by

$$\{e_\mu: \mu = (m, k), \quad m = 0, 1, 2, \ldots\}.$$

We have $A^2(B) = \sum \oplus A^2(k)$, the sum taken over all k.

LEMMA 3.5.   *If $\varphi \in L^\infty(\Delta)$ then*:
   (i)   *Each $A^2(k)$ is a reducing subspace for T_φ.*
   (ii)   *If there exists $\varepsilon > 0$ such that $\varphi \equiv 0$ for $\varepsilon < |z_1| < 1$, then each $T_\varphi \mid A^2(k)$ is a compact operator.*
   (iii)   *If $\varphi \in C(\bar{\Delta})$, then each $T_\varphi \mid A^2(k)$ is unitarily equivalent to a compact perturbation of $T_\varphi \mid A^2(0)$.*

   Proof.   The application of Fubini's Theorem to the integrals

$$(\varphi e_\nu, e_u) = \int \varphi e_\nu \bar{e}_\mu \, dV,$$

establishes (i). To prove (ii), let $\mu = (m, k)$, $\nu = (q, k)$ and write $e_\mu = c_\mu z^\mu$ and $e_\nu = c_\nu z^\nu$. If φ is as in (ii), then a direct computation using Fubini's Theorem and polar coordinates leads to the inequality

$$|(T_\varphi e_\mu, e_\nu)| \le A c_\mu c_\nu \varepsilon^{m+q+2},$$

where A is a constant dependent on n, k and $\|\varphi\|_\infty$. It follows that

$$\sum_m \sum_q |(T_\varphi e_\mu, e_\nu)|^2 < \infty.$$

Therefore $T_\varphi \mid A^2(k)$ is Hilbert–Schmidt, and hence compact. To prove (iii), let $\varphi \in C(\bar{\Delta})$ and $S_\varphi = T_\varphi \mid A^2(0)$. For each non-negative integer m, identify

m and the n-tuple $(m, 0, \ldots, 0)$, and let V be the unitary operator from $A^2(0)$ onto $A^2(k)$ defined by $V(e_m) = e_\mu$, where $\mu = (m, k)$. To show that $V^{-1}T_\varphi V$ is a compact perturbation of S_φ, it is sufficient to assume $\varphi = z_1$, since polynomials in z_1 and $\bar{z}_1$ are dense in $C(\bar{\Delta})$. We thus have

$$(V^{-1}T_\varphi V e_m, e_{m+1}) = (z_1 e_\mu, e_\nu)$$
$$= c_\mu(z_1^\nu, e_\nu)$$
$$= c_\mu c_\nu^{-1}(e_\nu, e_\nu)$$
$$= c_\mu c_\nu^{-1},$$

where $\mu = (m, k)$ and $\nu = (m+1, k)$. This last expression goes to 1 as m goes to ∞. All other matrix entries of $V^{-1}T_\varphi V$ are obviously 0. Thus $V^{-1}T_\varphi V$ is a compact perturbation of the unilateral shift. Since the above proof is valid if $k = 0$, S_φ is also a compact perturbation of the unilateral shift. The proof of (iii) is complete.

LEMMA 3.6.   *If $\varphi \in HC(\bar{\Delta})$ and $\varphi \equiv 0$ for $|z_1| = 1$, then $T_\varphi | A^2(k)$ is compact for all k.*

Proof. Let $I(r)$ be the characteristic function of the open ball of radius $1 - 1/r$ about the origin in $\mathbf{C}^n$. Since $\varphi \in HC(\bar{\Delta})$, $\varphi I(r)$ converges in the sup norm to φ. From

$$\|T_{\varphi - \varphi I(r)} | A^2(k)\| \leq \|T_{\varphi - \varphi I(r)}\| \leq \|\varphi - \varphi I(r)\|_\infty,$$

it follows that $T_{\varphi I(r)} | A^2(k)$ converges in norm to $T_\varphi | A^2(k)$. Since each of the former operators is compact, we conclude $T_\varphi | A^2(k)$ is compact.

If φ belongs to $C(\bar{\Delta})$, then $\varphi'(e^{i\theta}) = \varphi(e^{i\theta}, 0, \ldots, 0)$ belongs to $C(T)$. Minor modifications of the proof of Lemma 3.5(iii) show that $T_\varphi | A^2(0)$ is unitarily equivalent to a compact perturbation of T_φ, acting on H^2. Returning to our function f we note that since $p \neq \pm 1$, f^* is in $C(T)$, and hence

$$g(re^{i\theta}, z_2, \ldots, z_n) = rf^*(e^{i\theta})$$

is in $C(\bar{\Delta})$, although of course f is not, and $g' = f^*$. By Lemma 3.5(i) we can write T_f as a direct sum,

$$T_f = \sum \oplus T_f | A^2(k),$$

the sum taken over all $(n-1)$-tuples k. It follows from Lemma 3.6 that

$$(*) \qquad\qquad T_f = \sum \oplus \{T_g | A^2(k) + C_k\}$$

where C_k is a compact operator on $A^2(k)$. Combining our previous remarks with Lemma 3.5(iii), we see that for each k, $T_g | A^2(k) + C_k$ is unitarily equivalent to a compact perturbation of T_{f^*} acting on H^2. Now f^* was defined so as to have winding number 1 about the origin, hence $j(T_{f^*}) = -1$ [2; p. 185]. It follows that $j(T_g | A^2(k) + C_k) = -1$, and hence

$$\dim \ker (T_g | A^2(k) + C_k)^* > 0.$$

We conclude from (∗) that ker T_f^* is infinite dimensional, and hence T_f is not Fredholm. This completes the proof of Lemma 3.4. We should point out here that Lemma 3.5(iii) is not true in general for φ in $HC(\bar{\Delta})$, and in particular is not true for $\varphi = f$.

We now consider the case $p = -1$. The proof for $p = 1$ is analogous. As before we construct a function f satisfying Lemma 3.4. Define f^* on T by

$$f^*(e^{i\theta}) = f^*(\theta) = \begin{cases} 1 + \dfrac{(\alpha - 1)\theta}{\pi}, & 0 \le \theta < \pi, \\ 1 - \dfrac{(\beta - 1)\theta}{\pi}, & -\pi < \theta \le 0, \end{cases}$$

and define $f = \langle f_1, f_2 \rangle$ in $HC(B)$ by

$$f_1(x_1 + iy_1, z_2, \ldots, z_n) = f^*(\cos^{-1} x_1)$$

for $\operatorname{Im} y_1 \le 0$, and

$$f_2(x_1 + iy_1, z_2, \ldots, z_n) = f^*(-\cos^{-1} x_1)$$

for $\operatorname{Im} y_1 \le 0$. It is clear that f satisfies (ii), (iii), and (iv) of Lemma 3.4.

For arbitrary $\varphi = \langle \varphi_1, \varphi_2 \rangle$ in $HC(B)$, the previous work in this section shows that the line segment joining $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ is contained in ess $\sigma(T_\varphi)$ for λ in $\partial E \setminus (\Lambda_1 \cup \Lambda_{-1})$. It is easy to see from the definition of f that the interval $(0, 2)$ is contained in the union of these line segments. Since ess $\sigma(T_f)$ is closed, 0 is in ess $\sigma(T_f)$. Therefore T_f is not Fredholm.

REFERENCES

1. L. A. COBURN, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math. J., vol. 23 (1973), pp. 433–439.
2. R. G. DOUGLAS, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
3. ———, *Banach algebra techniques in the theory of Toeplitz operators*, CBMS 15, Amer. Math. Soc., Providence, 1973.
4. G. MCDONALD, *Fredholm properties of a class of Toeplitz operators on the ball*, Indiana Univ. Math. J., vol. 26 (1977), pp. 567–576.
5. U. VENUGOPALKRISHNA, *Fredholm operators associated with strongly pseudoconvex domains in $\mathbf{C}^n$*, J. Functional Analysis, vol. 9 (1972), pp. 349–373.

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