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THE RATIONAL HOMOTOPY GROUPS OF COMPLETE INTERSECTIONS

BY

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Introduction

A complete intersection of complex dimension n is a nonsingular subvariety of CP^{n+r} which is the transverse intersection of exactly r nonsingular hypersurfaces. In this paper we compute the rational homotopy groups of all complete intersections of complex dimension greater than one. Formality and the structure of the rational cohomology ring make this computation possible. In fact, our computation is valid for any formal space whose rational cohomology ring looks like that of a complete intersection.

Any nonsingular projective algebraic variety is a compact Kähler manifold. If it is also a complete intersection of complex dimension greater than one, then it is simply connected. By Deligne, Griffiths, Morgan, and Sullivan [2], all the rational homotopy invariants of a simply connected compact Kähler manifold are a formal consequence of the rational cohomology ring. Such a space is called formal. (Actually, [2] shows only that the real homotopy invariants are a formal consequence of the real cohomology ring, but real formality implies rational formality [3], [6], [12].) Equivalently, the rational homotopy invariants of a formal space are a formal consequence of the rational homology coalgebra. Theorem 2 below is a precise formulation of this principle for the rational homotopy groups.

The rational cohomology ring of a complete intersection is not too complicated. Except for powers of the Kähler form, the rational cohomology ring is connected up to the middle dimension [Hirzebruch, 4, Theorem 22.1.2]. Poincaré duality implies that the cup product makes the middle dimensional cohomology group into a nondegenerate bilinear form.

Let V_n be a complete intersection of complex dimension n. The rational homotopy groups $\pi(V_n) \otimes Q$ are complicated enough so that some algebraic structure is needed to describe them. This is given by the Samelson product [13]. More precisely, $\pi_k(V_n) \otimes Q$ is isomorphic to $\pi_{k-1}(\Omega V_n) \otimes Q$ and the Samelson product gives $\pi(\Omega V_n) \otimes Q$ the structure of a graded Lie algebra.

If n is greater than one, V_n has the same rational homotopy type as $X \cup_\alpha e^{2n}$ where X is a bouquet of a single copy of CP^{n-1} and copies of S^n and where $\alpha: S^{2n-1} \to X$ is the attaching map for the top cell e^{2n}. Let h_0 be the number of copies of S^n which occur in X. If h_0 is nonzero, then Theorem 1 below may be expressed as follows: The rational homotopy Lie algebra of ΩV_n is the rational homotopy Lie algebra of ΩX modulo the ideal generated by the homotopy

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class of the composition $(\Omega\alpha)i\colon S^{2n-2} \to \Omega S^{2n-1} \to \Omega X$, where i is the standard inclusion $i\colon Y \to \Omega SY$.

Without giving the proof, we also write down the rational homotopy groups of all n connected compact m dimensional manifolds M^m where $m \leq 3n + 1$, $n \geq 1$. On the one hand, the proof is omitted because this computation is similar to the one we give for complete intersections and simpler. On the other hand, these manifolds are spaces of category less than 3 and Lemaire's theory [5] can be used to give the answer.

The computation of the rational homotopy groups of a complete intersection V_n has a striking corollary which will be the subject of a subsequent paper. Let $E(V_n)$ be the group of homotopy self equivalences of V_n. If n is greater than one, then the natural representation of $E(V_n)$ into the automorphism group of the rational cohomology ring $H^*(V_n; Q)$ has a finite kernel and its image is an arithmetic subgroup.

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1. The main result

Let V_n denote a complete intersection of complex dimension n. $H^*(V_n; Q)$ has a graded basis of the following form: $1, u, u^2, \ldots, u^n$ where u is the Kähler form (degree $u = 2$) and $y_1, \ldots, y_{h_0}$ where degree $y_i = n$. By Poincaré duality, the cup product

$$H^n(V_n; Q) \otimes H^n(V_n; Q) \to H^{2n}(V_n; Q)$$

defines a nondegenerate bilinear form, which is symmetric is n is even and skew symmetric if n is odd. It is easy to see that we can choose the above basis so that $uy_i = 0$ for $i = 1, \ldots, h_0$.

Let $1, u_1, u_2, \ldots, u_n, x_1, \ldots, x_{h_0}$ be a graded basis for the rational homology $H(V_n; Q)$ which is dual to the above basis. Notice that $x_1, \ldots, x_{h_0}$ are primitive, and if $n \geq 3$ they form a basis for the primitive submodule $PH_n(V_n; Q)$. If $n \geq 3$ then $h_0 = \text{rank } PH_n(V_n; Q)$ and if $n = 2$ then $h_0 = \text{rank } PH_n(V_n; Q) - 1$.

The comultiplication on the fundamental class of V_n has the form

$$\Delta(u_n) = u_n \otimes 1 + 1 \otimes u_n + \bar{\Delta}(u_n)$$

where

$$\bar{\Delta}(u_n) = \sum_{i+j=n} u_i \otimes u_j + \sum_{1 \leq i,j \leq h_0} \varepsilon_{ij}(x_i \otimes x_j + (-1)^n x_j \otimes x_i).$$

Let $\langle g_1, \ldots, g_k \rangle$ denote the abelian graded Lie algebra and $F[g_1, \ldots, g_k]$ the free graded Lie algebra generated by graded generators $g_1, \ldots, g_k$. If L and L' are graded Lie algebras, then $L \vee L'$ denotes their coproduct (= free product).

A word of caution is in order. A free graded Lie algebra is just a bit different from a free ungraded Lie algebra. A basis for $F[g_1, \ldots, g_k]$ is the union of a Hall family $\{g_\alpha\}$ [Serre, 10] and $\{[g_\alpha, g_\alpha]\colon g_\alpha \text{ is in the Hall family and degree } g_\alpha \text{ is odd}\}$.

This is in [6]. For example, $\pi(\Omega S^n) \otimes Q$ is a free graded Lie algebra on one generator e_n of degree n. If n is even, e_n is a basis; if n is odd, e_n, $[e_n, e_n]$ is a basis.

Our main result is the next theorem and the two paragraphs which follow it.

THEOREM 1. *If V_n is a complete intersection of complex dimension $n \geq 3$ and if h_0 is nonzero, then $\pi(\Omega V_n) \otimes Q$ is isomorphic to the quotient of the graded Lie algebra*

$$F[s^{-1}x_1, \ldots, s^{-1}x_{h_0}] \vee \langle s^{-1}u, s^{-1}z \rangle$$

by the ideal I generated by

$$\sum (-1)^n \varepsilon_{ij}[s^{-1}x_i, s^{-1}x_j] + s^{-1}z$$

where degree $s^{-1}x_i = n - 1$, degree $s^{-1}u = 1$, *and* degree $s^{-1}z = 2n - 2$.

If $n = 2$ and h_0 is nonzero, Theorem 1 should be rewritten as follows. Replace $\langle s^{-1}u, s^{-1}z \rangle$ by $F[s^{-1}u]$ and let I be generated by $\sum \varepsilon_{ij}[s^{-1}x_i, s^{-1}x_j] + [s^{-1}u, s^{-1}u]$.

If $n \geq 2$ and $h_0 = 0$, then $\pi(\Omega V_n) \otimes Q$ is isomorphic to $\langle s^{-1}u, s^{-1}w \rangle$ where degree $s^{-1}u = 1$ and degree $s^{-1}w = 2n$.

2. Rational homotopy methods

Given a connected cocommutative coalgebra C over a field k of characteristic zero, let $\bar{C}$ be the submodule of C concentrated in positive dimensions. Define $\mathscr{L}(C)$ to be the free graded Lie algebra generated by the graded vector space $s^{-1}\bar{C}$ where $(s^{-1}\bar{C}) = \bar{C}_{n+1}$. Make $\mathscr{L}(C)$ into a differential graded Lie algebra by defining a differential d on generators by the formula

$$d(s^{-1}c) = -\sum (-1)^{\deg c_i'}[s^{-1}c_i', s^{-1}c_i'']$$

if

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \sum \{c_i' \otimes c_i'' + (-1)^{\deg c_i' \deg c_i''} c_i'' \otimes c_i'\}.$$

THEOREM 2 [6]. *If X is a simply connected formal space, then $\pi(\Omega X) \otimes k$ is isomorphic to the homology of the differential graded Lie algebra $\mathscr{L}(H(X; k))$.*

Let $\mathscr{L} = \mathscr{L}(H(X; k))$. The map $\mathscr{L} \to \mathscr{L}/[\mathscr{L}, \mathscr{L}] = s^{-1}\bar{H}(X; k)$ is a chain map. The composition

$$\pi_n(X) \otimes k = \pi_{n-1}(\Omega X) \otimes k = H_{n-1}\mathscr{L} \to H_{n-1}s^{-1}H(X; k) = H_n(X; k)$$

is the Hurewicz homomorphism. An easy corollary of this description is:

COROLLARY 1 [7]. *If X is a simply connected formal space, then the Hurewicz homomorphism maps $\pi(X) \otimes k$ onto the primitive submodule $PH(X; k)$.*

The next theorem is the algebraic version of the spectral sequence for the rational homotopy groups of a cofibration. Let $C' \to C \to C''$ be a sequence of maps of connected cocommutative coalgebras such that $\bar{C}' \to \bar{C} \to \bar{C}''$ is a short exact sequence of vector spaces. Then:

THEOREM 3 [8], [11]. *There exists a first quadrant homology spectral sequence of graded Lie algebras with abutment $H\mathscr{L}(C)$ and with*

$$E^2 = E^2_{*,0} \vee E^2_{0,*} = H\mathscr{L}(C'') \vee H\mathscr{L}(C').$$

The maps $C' \to C$ and $C \to C''$ induce the edge homomorphisms

$$H_q\mathscr{L}(C') \to E^2_{0,q} \to E^\infty_{0,q} \to H_q\mathscr{L}(C) \quad \text{and} \quad H_p\mathscr{L}(C) \to E^\infty_{p,0} \to E^2_{p,0}$$
$$\to H_p\mathscr{L}(C'').$$

If C and D are connected cocommutative coalgebras, let $C \vee D$ denote the coproduct coalgebra, $(\overline{C \vee D})_n = \bar{C}_n \vee \bar{D}_n$. Theorem 3 implies:

COROLLARY 2 [8]. $H\mathscr{L}(C \vee D) = H\mathscr{L}(C) \vee H\mathscr{L}(D)$.

Corollary 2 is an algebraic version of:

LEMMA 1. *If X and Y are simply connected spaces, then*

$$\pi(\Omega(X \vee Y)) \otimes k = \pi(\Omega X) \otimes k \vee \pi(\Omega Y) \otimes k$$

when k is a field of characteristic zero.

To a simply connected space X, Sullivan [2] has assigned a differential graded algebra M_X. M_X is called the minimal model of X and is characterized up to isomorphism by three properties:

(1) M_X is free commutative as a graded algebra,
(2) $d\bar{M}_X$ is contained in $(\bar{M}_X)^2$, and
(3) there is a map $M_X \to A(X)$ (= PL deRham forms on X) such that $H(M_X) \to H(A(X))$ is an isomorphism.

Let $Q(M_X) = \bar{M}_X/(\bar{M}_X)^2$ be the module of indecomposables. The dual of $Q(M_X)$ satisfies $Q^{k+1}(M_X) = \pi_k(\Omega X) \otimes Q$. The quadratic term of the differential on $Q(M_X)$ defines the dual of the Samelson product [2].

3. Special cases

Let $V_n(a_1, \ldots, a_r)$ denote a complete intersection defined in CP^{n+r} by r homogeneous equations of degrees $a_1, \ldots, a_r$, respectively.

In this section we compute the rational homotopy groups of complete intersections V_n with n greater than one and $h_0 = 0$ or 1. Let V_n be such a complete intersection. Since $H^{p,q}(V_n) = H^{q,p}(V_n)$, it follows that $H^{p,n-p}(V_n) = 0$ for $p \neq n/2$. Rapaport [9] has given a list of all such complete intersections, together with the corresponding values of h_0. The only ones with $h_0 < 2$ are: (1) $V_n(1)$ with $h_0 = 0$, (2) $V_n(2)$ with n odd and $h_0 = 0$, and (3) $V_n(2)$ with n even and $h_0 = 1$.

We compute these three cases and the results we get agree with the two paragraphs which follow the statement of Theorem 1.

First, $V_n(1) = \{[z_0, \ldots, z_{n+1}] \in CP^{n+1}: z_{n+1} = 0\} = CP^n$. The minimal model for CP^n is

$$M_{CP^n} = Q[u] \otimes \Lambda[w]$$

with $du = 0$, $dw = u^{n+1}$, degree $u = 2$, and degree $w = 2n + 1$. It follows that $\pi(\Omega CP^n) \otimes Q$ is isomorphic to $\langle s^{-1}u, s^{-1}w \rangle$ if n is greater than one and to $F[s^{-1}u]$ if n is one (degree $s^{-1}u = 1$, degree $s^{-1}w = 2n$).

$$V_n(2) = \{[z_0, \ldots, z_{n+1}] \in CP^{n+1}: z_0^2 + \cdots + z_{n+1}^2 = 0\}.$$

It is not hard to see that $V_n(2)$ is homeomorphic to the Grassman manifold of oriented 2-planes in R^{n+2}. That is, $V_n(2) = SO(n + 2)/SO(2) \times SO(n)$.

The rational cohomology ring of $SO(2m + 1)$ is $\Lambda[g_3, g_7, \ldots, g_{4m-1}]$; that of $SO(2m)$ is $\Lambda[g_3, g_7, \ldots, g_{4m-5}, b_{2m-1}]$ with degree $g_i = i$ and degree $b_i = i$ [1]. Since these are free commutative graded algebras, they are the minimal models (with $d = 0$) for $SO(2m + 1)$ and $SO(2m)$, respectively. It follows that $\pi(\Omega SO(2m + 1)) \otimes Q$ is isomorphic to

$$\langle s^{-1}g_3, \ldots, s^{-1}g_{4m-1} \rangle$$

and that $\pi(\Omega SO(2m)) \otimes Q$ is isomorphic to

$$\langle s^{-1}g_3, \ldots, s^{-1}g_{4m-5}, s^{-1}b_{2m-1} \rangle$$

with degree $s^{-1}g_i = i - 1$ and degree $s^{-1}b_i = i - 1$.

From the long exact homotopy sequence of the fibration

$$SO(2) \times SO(n) \to SO(n + 2) \to V_n(2),$$

it follows that $\pi(\Omega V_n(2)) \otimes Q$ is isomorphic to $\langle s^{-1}u, s^{-1}w \rangle$ if n is odd and greater than one (degree $s^{-1}u = 1$, degree $s^{-1}w = 2n$).

Similarly, if n is even and greater than one, it follows that $\pi(\Omega V_n(2)) \otimes Q$ has a graded basis of four elements: $s^{-1}u$, $s^{-1}x$, $s^{-1}b$, and $s^{-1}z$ with degrees 1, $n - 1$, n, and $2n - 2$, respectively. This information makes it easy to build a minimal model for $V_n(2)$. It is $Q[u, x] \otimes \Lambda[b, z]$ with $du = dx = 0$, $db = xu$, $dz = x^2 - u^n$, degree $u = 2$, degree $x = n$, degree $b = n + 1$, and degree $z = 2n - 1$. Hence, if n is even and greater than 2,

$$s^{-1}b = [s^{-1}u, s^{-1}x] \quad \text{and} \quad s^{-1}z = [s^{-1}x, s^{-1}x].$$

All other Lie brackets are zero. If n is 2, there is one additional nonzero Lie bracket, $s^{-1}z = -[s^{-1}u, s^{-1}u]$.

4. The main computation

In this section, we complete the proof of Theorem 1 and of the two paragraphs which follow it. By Section 3, we may assume that V_n is a complete intersection with n and h_0 greater than one.

We begin with a specific identification of

$$\pi(\Omega CP^{n-1}) \otimes Q = H\mathscr{L}H(CP^{n-1}; Q).$$

$H(CP^{n-1}; Q)$ has a graded basis $1, u_1, \ldots, u_{n-1}$ with degree $u_i = 2i$ and comultiplication

$$\Delta(u_k) = u_k \otimes 1 + 1 \otimes u_k + \sum_{\substack{i+j=k \\ i,j>0}} u_i \otimes u_j.$$

LEMMA 2. *$H\mathscr{L}H(CP^{n-1}; Q)$ has a basis $s^{-1}u$, $s^{-1}z$ represented by the respective cycles $s^{-1}u_1$ and*

$$ds^{-1}u_n = \tfrac{1}{2} \sum_{\substack{i+j=n \\ i,j>0}} [s^{-1}u_i, s^{-1}u_j].$$

Proof. It is clear that $s^{-1}u_1$ represents a generator $s^{-1}u$ of $H_1\mathscr{L}H(CP^{n-1}; Q)$. That $ds^{-1}u_n$ represents a generator $s^{-1}z$ of $H_{2n-2}\mathscr{L}H(CP^{n-1}; Q)$ is a consequence of the fact that $\pi_{2n-2}(\Omega CP^{n-1}) \otimes Q \to \pi_{2n-2}(\Omega CP^n) \otimes Q$ is the zero homomorphism. ∎

Define a subcoalgebra D of $H(V_n; Q)$ by $D_k = H_k(V_n; Q)$ if $k < 2n$ and $D_k = 0$ if $k \geq 2n$. Then $D = H(X; Q)$ where X is the bouquet of CP^{n-1} and h_0 copies of S^n. The coalgebra $H(X; Q)$ is the coproduct of $H(CP^{n-1}; Q)$ and h_0 copies of $H(S^n; Q)$. Since S^n and CP^{n-1} are formal spaces, Corollary 2 and Theorem 2 imply that

$$H\mathscr{L}(D) = F[s^{-1}y_1, \ldots, s^{-1}y_{h_0}] \vee \langle s^{-1}u, s^{-1}z \rangle \quad \text{if } n > 2$$

and

$$H\mathscr{L}(D) = F[s^{-1}y_1, \ldots, s^{-1}y_{h_0}] \vee F[s^{-1}u] \quad \text{if } n = 2.$$

The element $ds^{-1}u_n$ is a cycle which represents an element α in the kernel of $H\mathscr{L}(D) \to H\mathscr{L}H(V_n; Q)$. If I is the ideal generated by α, we get a map

$$f\colon H\mathscr{L}(D)/I \to H\mathscr{L}H(V_n; Q).$$

Theorem 1 and the paragraph which follows it assert that f is an isomorphism. It suffices to show that the complexification $f \otimes C$ is an isomorphism.

Since $H^n(V_n; C) \otimes H^n(V_n; C) \to H^{2n}(V_n; C)$ is a nondegenerate bilinear form, we can replace the graded basis in Section 2 by a graded basis for $H^*(V_n; C)$ with the additional property that there is an orthogonal splitting of $H^n(V_n; C)$ into

$$\langle y_1, y_2 \rangle \perp \langle y_3, \ldots, y_{h_0} \rangle$$

if n is odd and into

$$\langle y_1, y_2 \rangle \perp \langle u^{n/2}, y_3, \ldots, y_{h_0} \rangle$$

if n is even. Furthermore, we require that $y_1 y_2 = u_n$ and $y_1^2 = y_2^2 = 0$.

It is clear that y_1, y_2, u_n are a vector space basis for an ideal J in $H^*(V_n; C)$. The sequence of algebras $C \oplus J \to H^*(V_n; C) \to H^*(V_n; C)/J$ is dual to a sequence of coalgebras $C' \to H(V_n; C) \to C''$ to which Theorem 3 applies. The resulting spectral sequence abuts to $H\mathscr{L}H(V_n; C)$ and has $E^2 = H\mathscr{L}(C'') \vee H\mathscr{L}(C')$. We claim that $E^2 = E^\infty$.

Note that $C'' = H(S^n \times S^n; C)$ and $C' = H(Z; C)$ where Z is a bouquet of CP^{n-1} and $h_0 - 2$ copies of S^n. Bouquets and products of formal spaces are formal [4]; therefore, Theorem 2 implies that

$$E^2 = \pi(\Omega(S^n \times S^n)) \otimes C \vee \pi(\Omega Z) \otimes C.$$

By the edge homomorphism it is automatic that $\pi(\Omega Z) \otimes C$ $(= E^2_{0,*})$ consists of infinite cycles. $\pi(\Omega(S^n \times S^n)) \otimes C$ is generated as a Lie algebra by $s^{-1}y_1$, $s^{-1}y_2$. Corollary 1 implies that these are infinite cycles. Hence, $E^2 = E^\infty$.

Since h_0 is greater than one if V_n is not $V_n(1)$ or $V_n(2)$, we have shown:

THEOREM 4. *If V_n is a complete intersection with n greater than one and V_n is not $V_n(1)$ or $V_n(2)$, then V_n has the same rational homotopy groups as $Z \vee (S^n \times S^n)$.*

More precisely, with respect to the filtration of the spectral sequence, the associated bigraded Lie algebra

$$E^0(\pi(\Omega V_n) \otimes C) = \pi(\Omega(S^n \times S^n)) \otimes C \vee \pi(\Omega Z) \otimes C.$$

It follows that $\pi(\Omega V_n) \otimes C$ is generated as a Lie algebra by $s^{-1}y_1, \ldots, s^{-1}y_{h_0}$, $s^{-1}u$, $s^{-1}z$ if $n \geq 3$ and by $s^{-1}y_1, \ldots, s^{-1}y_{h_0}$, $s^{-1}u$ if $n = 2$. Hence, the map $f \otimes C$ is surjective.

There is an obvious filtration on $H(D)/I \otimes C$ so that $f \otimes C$ is filtration preserving and $E^\infty(f \otimes C)$ is an isomorphism. Let $s^{-1}y_1$, $s^{-1}y_2$ have filtration $n - 1$ and all other generators have filtration 0.

Therefore, $f \otimes C$ and f are isomorphisms. ∎

5. Highly connected manifolds

In this section, M^m denotes an n connected compact real manifold of dimension m with $m \leq 3n + 1$, $n \geq 1$. In [6] it is shown that such a manifold is a formal space.

Let $x_1, \ldots, x_{k_0}$ be a basis for $PH(M^m; Q)$ and let u_m be the fundamental class in $H_m(M^m; Q)$. Then the comultiplication is given by

$$\Delta(u_m) = u_m \otimes 1 + 1 \otimes u_m + \sum \varepsilon_{ij}(x_i \otimes x_j + (-1)^{\deg x_i \deg x_j} x_j \otimes x_i).$$

Using the methods of Section 4, we could prove Theorem 5 below.

THEOREM 5.' *If k_0 is greater than one, then $\pi(\Omega M^m) \otimes Q$ is isomorphic to the quotient of $F[s^{-1}x_1, \ldots, s^{-1}x_{k_0}]$ by the ideal generated by*

$$\sum (-1)^{\deg x_i} \varepsilon_{ij}[s^{-1}x_i, s^{-1}x_j].$$

The only M^m with k_0 less than two are of two types. First, there are those with $1, u_m$ as a basis for the rational homology. These have the same rational homotopy groups as S^m. Second, there are those with $1, u, u^2$ as a basis for the rational cohomology. In this case, $\pi(M^m) \otimes Q$ has a generator in dimension $m/2$ and a generator in dimension $m + 1$.

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