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INJECTIVE DIFFERENTIAL SYSTEMS

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1. Introduction

In this paper we consider linear differential systems of the form

$$w'(z) = A(z)w(z). \tag{1}$$

Here $w(z)$ is the column vector $(w_1(z), \ldots, w_n(z))$ and $A(z)$ is the $n \times n$ matrix $(a_{ik}(z))_1^n$ whose elements are holomorphic functions in a given simply connected domain D of the z-plane C^1 ($\infty \notin D$). Such systems have been studied in recent years and many geometric properties of their solutions $w(z)$, mapping D into C^n, have been considered. A solution $w(z)$ was called oscillatory if there exist points $z_1, \ldots, z_n$ in D such that $w_k(z_k) = 0$, $k = 1, \ldots, n$; in geometric language this means that the manifold $w(z)$ ($z \in D$) meets the n principal hyperplanes $w_k = 0$, $k = 1, \ldots, n$, of C^n. Another geometric notion, introduced by Nehari [6], is suborthogonality. The system (1) is said to be suborthogonal in D, if, for any nontrivial solution $w(z)$ and any two points z_1 and z_2 in D, angle $\measuredangle \{w(z_1), w(z_2)\}$ between the vectors $w(z_1)$ and $w(z_2)$ is smaller than $\pi/2$. Bounds for the solutions of (1) were considered in [10]. The null system $w'(z) = 0$ has the general solution $w(z) = c$; this system is thus nonoscillatory (i.e., no nontrivial solution is oscillatory) and suborthogonal in the whole plane and "small" systems will have these geometric properties. For example, let $\|A\|$ be the spectral norm; if $\int_{\partial D} \|A(z)\| \, |dz| \leq \pi$ then the system (1) is nonoscillatory and suborthogonal in D ([9, Theorem 3] and [6, Theorem 2.1]).

Here we consider another geometric property of the solutions $w(z)$ of (1). We require that each nontrivial solution $w(z)$ is without double points: if $z_1 \neq z_2$ ($z_1, z_2 \in D$), then $w(z_1) \neq w(z_2)$. Geometrically this means that the manifold $w(z)$ ($z \in D$), lying in C^n, is without selfpenetration. This property, which clearly does not hold for the null system, reduces in the one-dimensional case, $n = 1$, to the univalence of the (scalar) solution $w(z)$. We call vector solutions $w(z)$ ($z \in D$) of the system (1), for which $z_1 \neq z_2$ ($z_1, z_2 \in D$) implies $w(z_1) \neq w(z_2)$ *injective*; and we say that the *system* (1) *in D is injective* if every nontrivial solution $w(z)$ is injective. The term "univalent" is used for mappings from domains in C^n to C^n (see for example [8] for recent results in this area) and we thus do not use it for mappings from domains in C^1 to C^n.

In Section 2 some elementary properties of injective systems are given and these systems are characterized in two special cases, when the matrix $A(z)$ is

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triangular and when this matrix is independent of z. In Section 3 we use a geometric lemma of Horn [3] to obtain angular bounds for the solutions of a differential system. As in the one-dimensional case [11], if the domain D is convex and if the range of the derivative $w'(z)$ lies in a halfspace (of R^{2n}) not containing the origin, then $w(z)$ is injective in D. The result on the bounds (applied to $w'(z)$) yields thus a sufficient condition for injectivity of the given system (Section 4). We close with a generalization to k-valency and state also the corresponding result for real systems.

2. Some results on injective systems

Injective differential systems were defined in the introduction. Together with the vector differential equation (1) we consider also the corresponding matrix differential equation

$$W'(z) = A(z)W(z), \tag{2}$$

where $W(z) = (w_{ik}(z))_1^n$. Let $W(z)$ be any fundamental solution of (2) (i.e., $W(z)$ is a solution of (2) whose determinant $|W(z)| = |w_{ik}(z)|_1^n \neq 0$ for all z of D). *The system* (1) *is injective in D if and only if the determinant*

$$|W(z_2) - W(z_1)| \neq 0 \tag{3}$$

for all pairs of distinct points in D. Indeed, any nontrivial solution $w(z)$ of (1) is given by

$$w(z) = W(z)c, \tag{4}$$

where the constant column vector $c = (c_1, \ldots, c_n) \neq 0$. For any two points z_1 and z_2 there exists a nontrivial solution $w(z)$ satisfying $w(z_1) = w(z_2)$ if and only if the equation $(W(z_2) - W(z_1))c = 0$ has a solution $c \neq 0$, i.e., if and only if

$$|w(z_2) - w(z_1)| = 0.$$

Let T be any constant nonsingular matrix, $|T| \neq 0$, and set

$$\hat{A}(z) = TA(z)T^{-1}. \tag{5}$$

The given system (1) *and the system*

$$\hat{w}'(z) = \hat{A}(z)\hat{w}(z) \tag{$\hat{1}$}$$

are together injective or noninjective in D. This follows from the relation $\hat{w}(z) = Tw(z)$ between the solutions of the two vector equations (or—equivalently—from the relation $\hat{W}(z) = TW(z)$ between the solutions of the corresponding matrix equations).

We also note that the injectivity of the system (1) is invariant under conformal mapping of the domain D. Let $z = \phi(\tilde{z})$ map the domain $\tilde{D}$ of the $\tilde{z}$-plane onto the given domain D of the z-plane. (1) transforms into

$$\tilde{w}'(z) = \tilde{A}(\tilde{z})\tilde{w}(\tilde{z}), \tag{$\tilde{1}$}$$

where $\tilde{w}(\tilde{z}) = w(\phi(\tilde{z}))$ and $\tilde{A}(\tilde{z}) = \phi'(\tilde{z})A(\phi(\tilde{z}))$. *The system* (1) *in D and the system* ($\tilde{1}$) *in $\tilde{D}$ are thus together injective or noninjective.*

For $n = 1$ the system (1) becomes the (scalar) differential equation $w'(z) = a(z)w(z)$, where $a(z)$ is a holomorphic function (from D to C^1). Such an equation is injective if every solution

$$w(z) = c \exp\left(\int_{z_0}^{z} a(\zeta)\, d\zeta\right), \quad c \neq 0, \tag{6}$$

is univalent in D. This holds if and only if

$$\int_{z_1}^{z_2} a(\zeta)\, d\zeta \neq 0, \ \pm 2\pi i, \ \pm 4\pi i, \ldots, \tag{7}$$

for every pair of distinct points in D. This condition can be generalized to systems of arbitrary dimension n if the matrix $A(z)$ is triangular (i.e., is such that all the elements below—or above—the main diagonal vanish identically).

THEOREM 1. *Let the triangular matrix $A(z) = (a_{ik}(z))_1^n$ be holomorphic in a simply connected domain D, $\infty \notin D$. The differential system*

$$w'(z) = A(z)w(z) \tag{1}$$

is injective in D if and only if the inequalities

$$\int_{z_1}^{z_2} a_{kk}(\zeta)\, d\zeta \neq 0, \ \pm 2\pi i, \ \pm 4\pi i, \ldots \tag{7$'$}$$

hold for all diagonal elements $a_{kk}(z)$, $k = 1, \ldots, n$, of $A(z)$ and for all pairs of distinct points z_1 and z_2 in D.

Proof. Let z_0 be an arbitrary point in D and let $W(z)$ be the fundamental solution of the matrix equation (2) satisfying $W(z_0) = I$ ($I = (\delta_{ik})_1^n$). The Peano-Baker series

$$W(z) = I + \int_{z_0}^{z} A(\zeta)\, d\zeta + \int_{z_0}^{z} A(\zeta) \int_{z_0}^{\zeta} A(\zeta_1)\, d\zeta_1\, d\zeta + \cdots$$

shows that $W(z)$ is, together with $A(z)$, triangular. For each diagonal element $w_{kk}(z)$ of $W(z)$ we obtain

$$\begin{aligned} w_{kk}(z) &= 1 + \int_{z_0}^{z} a_{kk}(\zeta)\, d\zeta + \int_{z_0}^{z} a_{kk}(\zeta) \int_{z_0}^{\zeta} a_{kk}(\zeta_1)\, d\zeta_1\, d\zeta + \cdots \\ &= 1 + \int_{z_0}^{z} a_{kk}(\zeta)\, d\zeta + \frac{1}{2!}\left(\int_{z_0}^{z} a_{kk}(\zeta)\, d\zeta\right)^2 + \cdots \qquad (6') \\ &= \exp\left(\int_{z_0}^{z} a_{kk}(\zeta)\, d\zeta\right). \end{aligned}$$

As $|W(z_2) - W(z_1)| = \prod_1^n (w_{kk}(z_2) - w_{kk}(z_1))$ it follows that the system (1) is injective in D if and only if all functions $w_{kk}(z)$ are univalent there, and, by (6′), this is equivalent to the inequalities (7′).

For a bounded convex domain D of diameter d these inequalities hold if there exist arguments α_k, $0 \leq \alpha_k < 2\pi$, $k = 1, \ldots, n$, such that

$$\operatorname{Re}\,(a_{kk}(z)e^{i\alpha_k}) > 0, \quad z \in D,\ k = 1, \ldots, n, \tag{8}$$

and if also

$$|a_{kk}(z)| < 2\pi/d, \quad z \in D,\ k = 1, \ldots, n. \tag{8′}$$

Indeed, we may now integrate along the segment from z_1 to z_2 and (8) yields $J_k = \int_{z_1}^{z_2} a_{kk}(\zeta)\,d\zeta \neq 0$, while (8′) implies that $|J_k| < 2\pi$, $k = 1, \ldots, n$.

Theorem 1 and the invariance of injectivity of the system under a similarity transformation (5) yield the following result.

THEOREM 2. *Let the constant matrix $A = (a_{ik})_1^n$ have the eigenvalues $\lambda_1, \ldots, \lambda_n$. The system*

$$w'(z) = Aw(z) \tag{1c}$$

is injective in the simply connected domain D, $\infty \notin D$, if and only if

$$|A| = \prod_1^n \lambda_k \neq 0 \tag{9}$$

and

$$2m\pi i/\lambda_k \notin D - D, \quad k = 1, \ldots, n,\ m = 1, 2, \ldots. \tag{9′}$$

Proof. By a similarity transformation A can be brought into a triangular form, say into the Jordan normal form $J = TAT^{-1}$. The system $v'(z) = Jv(z)$ is together with the given system (1c) injective or noninjective in D. The diagonal elements of J are the eigenvalues $\lambda_1, \ldots, \lambda_n$ and condition (7′) of Theorem 1 thus becomes $\lambda_k(z_2 - z_1) \neq 0$, $\pm 2m\pi i$, $k = 1, \ldots, n$, $m = 1, 2, \ldots$, which is equivalent to conditions (9) and (9′).

We note that if D is starlike with respect to one of its points then $D - D$ is starlike with respect to the origin. In this case (9′) reduces to the simpler condition

$$2\pi i/\lambda_k \notin D - D, \quad k = 1, \ldots, n. \tag{9″}$$

For example, the system with constant coefficients (1c) will be injective in the unit disk, $|z| < 1$, if and only if $0 < |\lambda_k| \leq \pi$, $k = 1, \ldots, n$.

3. Horn's lemma and bounds for solutions of differential systems

Let R^n be the real n-dimensional euclidean space with points $x = (x_1, \ldots, x_n)$ and let $S(= S^{n-1})$ be its unit sphere,

$$S = \{x: \|x\| = 1\} \quad \left(\|x\|^2 = \sum_1^n x_k^2\right).$$

A spherical cap of S with central angle α $(0 < \alpha \leq \pi)$ is a set of the form $S(u, \alpha) = \{x\colon x \in S, \sphericalangle \{x, u\} \leq \alpha/2\}$, where u is a fixed point of S and the angle $\sphericalangle \{x, u\}$ is, for $\|x\| = \|u\| = 1$, defined by arc cos $(x, u)((x, u) = \sum_1^n x_k u_k)$. We state Horn's result as:

LEMMA 1. *Let Γ be a closed rectifiable curve on S of length $l(\Gamma)$. If $l(\Gamma) < 2\alpha$ then Γ is contained in an open spherical cap of central angle α; if $l(\Gamma) = 2\alpha$ then Γ is contained in a closed spherical cap of central angle α, $0 < \alpha \leq \pi$.*

Horn stated this lemma for $n = 3$ and $\alpha = \pi$ [3, Lemma 2]; his proof holds in the general case and, for completeness, we bring the proof. Let P be any point on Γ and let Q be the point on Γ such that the curve segments $\Gamma_1 = PQ$ and $\Gamma_2 = QP$ have equal length $l(\Gamma_1) = l(\Gamma_2) = l(\Gamma)/2$. Let B be the midpoint of the straight chord PQ (lying inside the unit ball). By a rotation of S we may assume that $B = (0, \ldots, 0, b)$, $0 \leq b \leq 1$, and

$$P = ((1 - b^2)^{1/2}, 0, \ldots, b), \qquad Q = (-(1 - b^2)^{1/2}, 0, \ldots, b).$$

Let N be the north pole $(0, \ldots, 0, 1)$; then $\sphericalangle \{P, N\} = \sphericalangle \{N, Q\} = \beta/2$ where $b = \cos(\beta/2)$. As the points P, N, and Q lie in the (x_1, x_n)-plane it follows that $l(\Gamma) \geq 2\beta$, hence $\beta \leq \alpha$. Set $a = \cos(\alpha/2)$; then $0 \leq a \leq b \leq 1$, $a < 1$. If Γ does not intersect the hyperplane $x_n = a$, the conclusions follow. If Γ_1 intersects this hyperplane, then we construct the unique arc Γ_2' which is symmetric to Γ_1 with respect to the x_n-axis, $l(\Gamma_2') = l(\Gamma_1)$. The closed curve $\Gamma' = \Gamma_1 + \Gamma_2'$ has the length $l(\Gamma') = l(\Gamma)$ and contains two points

$$(y_1, \ldots, y_{n-1}, a) \quad \text{and} \quad (-y_1, \ldots, -y_{n-1}, a)$$

whose spherical distance is α. So if Γ_1 intersects the hyperplane $x_n = a$ then $l(\Gamma') = l(\Gamma) \geq 2\alpha$, and if Γ_1 crosses this hyperplane to points below it then $l(\Gamma) > 2\alpha$. Thus if $l(\Gamma) < 2\alpha$ then Γ_1 cannot intersect the hyperplane, and if $l(\Gamma) = 2\alpha$, Γ_1 cannot cross this hyperplane. Since the same argument applies to Γ_2 the lemma is proved.

We shall use this lemma for the unit sphere $S = \{\omega\colon \|\omega\| = 1\}$ of the complex n-dimensional euclidean space C^n (so S corresponds to S^{2n-1} of R^{2n}),

$$\omega = (\omega_1, \ldots, \omega_n), \quad \omega_k = \xi_k + i\eta_k, \quad \|\omega\|^2 = \sum_1^n |\omega_k|^2 = \sum_1^n (\xi_k^2 + \eta_k^2).$$

The length $l(\Gamma)$ of a curve Γ in C^n, given by $\omega(t)$, $a \leq t \leq b$, equals $\int_a^b \|\omega'(t)\|\, dt$ and a spherical cap $S(u, \alpha)$ is now given by

$$S(u, \alpha) = \{\omega\colon \omega \in S, \text{arc cos Re}\,(\omega, u) \leq \alpha/2\}.$$

Here u is a fixed point on S and $(\omega, \tilde{\omega})$ is the inner product in C^n, hence $\text{Re}\,(\omega, \tilde{\omega}) = \sum_1^n (\xi_k \tilde{\xi}_k + \eta_k \tilde{\eta}_k)$.

The complex differential system will now be written in the form

$$v'(z) = B(z)v(z). \tag{10}$$

As before, $\|B\|$ denotes the spectral norm of the complex matrix $B = (b_{ik})_1^n$, i.e., $\|B\|$ is induced by the euclidean vector norm $\|\omega\|$, $\|B\| = \sup_{\omega \neq 0} (\|B\omega\|/\|\omega\|)$. We state the following theorem in a somewhat restricted form which, however, is sufficient for the later applications to injectivity.

THEOREM 3. *Let the matrix $B(z)$ be holomorphic in a closed region $\bar{D}$ whose boundary is an analytic Jordan curve γ. If*

$$\int_\gamma \|B(z)\|\,|dz| = 2k, \quad 0 \leq k \leq \pi, \tag{11}$$

then for every nontrivial solution $v(z)$ of (10) there exists a unit vector $u = u(v)$ in C^n, such that

$$\operatorname{arc\,cos} \frac{\operatorname{Re}\,(v(z), u)}{\|v(z)\|} \leq \frac{k}{2} \tag{12}$$

holds for all z in $\bar{D}$.

We note a special case, needed in the sequel. *If $k < \pi$ then*

$$\operatorname{Re}\,(v(z), u) > 0, \quad z \in \bar{D}. \tag{12'}$$

Proof. Let γ be given by $z(t)$, $0 \leq t \leq T$. For a given solution $v(z)$ of (10) we set

$$v(z(t)) = \omega(t) = (\omega_1(t), \ldots, \omega_n(t)), \quad 0 \leq t \leq T. \tag{13}$$

Then

$$\omega'(t) = \frac{dz}{dt}\, v'(z), \quad 0 \leq t \leq T. \tag{14}$$

For $0 \leq t \leq T$, $\omega(t)$ describes a closed curve Γ in C^n (not going through the origin). The projection $\tilde{\Gamma}$ of Γ on the unit sphere is given by

$$\tilde{\omega}(t) = \omega(t)/\|\omega(t)\|, \quad 0 \leq t \leq T. \tag{15}$$

For the length $l(\tilde{\Gamma})$ of $\tilde{\Gamma}$ we have

$$l(\tilde{\Gamma}) = \int_0^T \|\tilde{\omega}'(t)\|\,dt \leq \int_0^T \frac{\|\omega'(t)\|}{\|\omega(t)\|}\,dt = \int_\gamma \frac{\|v'(z)\|}{\|v(z)\|}\,|dz|.$$

Here the inequality sign follows from a well known property of the projection on the unit sphere and the last equality sign follows from (13) and (14). By (10) and (11)

$$\int_\gamma \frac{\|v'(z)\|}{\|v(z)\|}\,|dz| \leq \int_\gamma \|B(z)\|\,|dz| = 2k$$

and we thus obtain $l(\tilde{\Gamma}) \leq 2k$. By Lemma 1, applied to the unit sphere S of C^n, there exists a vector u, $\|u\| = 1$, such that $\operatorname{arc\,cos} \operatorname{Re}\,(\tilde{\omega}(t), u) \leq k/2$, $0 \leq t \leq T$.

By (13) and (15) this implies

$$\arccos \frac{\operatorname{Re}\,(v(z), u)}{\|v(z)\|} \leq \frac{k}{2}, \tag{16}$$

or,

$$\operatorname{Re}\,(v(z), u) - (\cos\,(k/2))\|v(z)\| \geq 0, \tag{16'}$$

for all $z \in \gamma$.

The function $\operatorname{Re}\,(v(z), u) = \operatorname{Re}\,(\bar{u}_1 v_1(z) + \cdots + \bar{u}_n v_n(z))$ is harmonic in $\bar{D}$, and the norm $\|v(z)\|$ is subharmonic there. As $0 \leq k/2 \leq \pi/2$ it follows that $\operatorname{Re}\,(v(z), u) - (\cos\,(k/2))\|v(z)\|$ is superharmonic in $\bar{D}$ and thus attains its minimum at the boundary γ. Hence, (16′) and (16) hold for each $z \in \bar{D}$ and we have completed the proof of the theorem.

By a standard limit procedure it can be shown that this theorem remains valid for an arbitrary simply connected domain D ($\infty \notin D$). The assumption (11) has then to be interpreted in the usual H_1 sense and the conclusion (12) holds now for all z in D. This improves a previous result which stated that assumption (11) (for $0 < k \leq \pi$) implies that for every nontrivial solution $v(z)$ of (10) and for every pair of points z_1 and z_2 in D the inequality

$$\arccos \frac{\operatorname{Re}\,(v(z_1), v(z_2))}{\|v(z_1)\|\,\|v(z_2)\|} < k$$

holds [10, Theorem 3]. A simple example established the sharpness of this former result and yields thus also the sharpness of the present, stronger, result.

4. A sufficient condition for injectivity

Let the matrix $A(z)$ be holomorphic in a domain D and let (1) and (2) be the corresponding differential systems. For given z_0 in D, the condition $|A(z_0)| \neq 0$ is equivalent to the fact that, for every nontrivial solution $w(z)$ of (1), $w'(z_0) \neq 0$, and that, for every fundamental solution $W(z)$ of (2), $|W'(z_0)| \neq 0$. These conditions imply local injectivity at z_0; it is thus not surprising that in a sufficient condition for injectivity it is assumed that the holomorphic matrix $A(z)$ is nonsingular (i.e., $|A(z)| \neq 0$) in D.

THEOREM 4. *Let D be a convex domain, $\infty \notin D$, and assume that the matrix $A(z) = (a_{ik}(z))_1^n$ is holomorphic and nonsingular in D. If*

$$\int_{\partial D} \|A(z) + A'(z)A(z)^{-1}\|\,|dz| \leq 2\pi, \tag{17}$$

then the differential system

$$w'(z) = A(z)w(z) \tag{1}$$

is injective in D.

Proof. We denote

$$A(z) + A'(z)A(z)^{-1} = B(z). \tag{18}$$

First let D be the unit disk. The assumption (17) thus means that $B(z)$ is of class H_1 and that $\int_0^{2\pi} \|B(e^{i\theta})\| \, d\theta \leq 2\pi$. As the norm $\|B(z)\|$ is a subharmonic function in D it follows that

$$\int_{|z|=r} \|B(z)\| \, |dz| < 2\pi, \tag{19}$$

for every r, $0 < r < 1$.

Let $w(z)$ be an arbitrary nontrivial solution of (1); by $|A(z)| \neq 0$ it follows that $w'(z) \neq 0$ in D. We denote

$$w'(z) = v(z). \tag{20}$$

By (1), (18), and (20), this vector $v(z)$ is a (nontrivial) solution of the differential system

$$v'(z) = B(z)v(z). \tag{10}$$

Let z_1 and z_2 be two given points in $|z| < 1$ and choose r, $0 < r < 1$, such that $|z_i| < r$, $i = 1, 2$. We use (19) and apply Theorem 3 to the system (10) in the domain $\bar{D}_r = \{z\colon |z| \leq r\}$. For the given solution $v(z)$ $(= w'(z))$ of (10) there exists thus a unit vector u such that (12′) holds (in $\bar{D}_r$); i.e., we have

$$\operatorname{Re}\,(w'(z), u) > 0, \quad |z| \leq r. \tag{21}$$

Let $\alpha = \arg\,(z_2 - z_1)$; the segment $[z_1, z_2]$ is given by

$$z = z_1 + te^{i\alpha}, \quad 0 \leq t \leq |z_2 - z_1|.$$

We integrate along this segment (and denote the inner product by a):

$$a = \left(\int_{z_1}^{z_2} w'(z)\, dz, u\right) = e^{i\alpha}\left(\int_0^{|z_2 - z_1|} w'(z_1 + te^{i\alpha})\, dt, u\right).$$

By (21), $\operatorname{Re}\,(ae^{-i\alpha}) > 0$, and it thus follows that $a \neq 0$. Hence

$$w(z_2) - w(z_1) = \int_{z_1}^{z_2} w'(z)\, dz \neq 0,$$

and we proved the theorem for the unit disk.

For an arbitrary convex domain D $(\infty \notin D)$ the assumption (17) has to be interpreted as the limit, for $r \to 1$, of integrals taken along the level lines γ_r, $0 < r < 1$, of the function $\zeta = \psi(z)$ which maps D onto $|\zeta| < 1$. Given z_1 and z_2 we can find r so near to 1, that the segment $[z_1, z_2]$ lies in the interior D_r of γ_r. (17) implies that the corresponding integral along γ_r is smaller than 2π and the proof continues as in the case of the unit disk. This completes the proof of Theorem 4.

In the last part of this proof, we could not obtain the validity of the theorem for an arbitrary convex domain $\tilde{D}$ from its validity in a special case $D(|z| < 1)$ by conformal mapping $z = \phi(\tilde{z})$ from $\tilde{D}$ to D. The corresponding systems (1) and ($\tilde{1}$) are simultaneously injective or noninjective, but the line integral

$$\int_{\partial D} \|A(z) + A'(z)A(z)^{-1}\| \, |dz|$$

on the left-hand side of (17) is not invariant under conformal mapping (except for $\phi(\tilde{z}) = a\tilde{z} + b$).

We do not claim that Theorem 4 is sharp; the constant 2π on the right-hand side of (17) may not be the best one. However, in the given statement of the theorem, i.e., *for all convex domains, the constant* 2π *in* (17) *cannot be replaced by any constant larger than* 4π. *For any given convex domain* D, 2π *in* (17) *cannot be replaced by any constant larger than* $2\pi l/d$, *where* l *is the length of* ∂D *and* d *is the diameter of* D. Note that for convex domains $2 < l/d \leq \pi$. Here equality is attained in the second inequality sign for sets of constant width, while $l/d \to 2$ as D converges to a segment [2]. The above italicized statements follow by choosing $A(z) = \lambda I$, $\lambda \neq 0$. The system (1c) becomes $w'(z) = \lambda I w(z)$ with the solution $w(z) = e^{\lambda z}c$, and the system is noninjective if $(2\pi i/\lambda) \in D - D$. Hence, for any $\varepsilon > 0$, there exists a value λ, $|\lambda| = (2\pi/d) + \varepsilon$, so that the corresponding system is noninjective in D, and the value of the integral in (17) is $|\lambda| l = ((2\pi/d) + \varepsilon)l$.

We now express this result in a form which does not explicitly mention differential systems.

THEOREM 4′. *Let* D *be a convex domain,* $\infty \notin D$, *and assume that the matrix* $W(z) = (w_{ik}(z))_1^n$ *is holomorphic in* D *and that its derivative* $W'(z)$ *is nonsingular in* D. *If*

$$\int_{\partial D} \|W''(z)W'(z)^{-1}\| \, |dz| \leq 2\pi, \tag{22}$$

then every vector function $w(z)$ *defined by*

$$w(z) = W(z)c, \quad c \neq 0, \tag{4'}$$

is injective in D.

This form is more general than the previous one. Indeed, if $W(z)$ is a fundamental solution of the differential system (2), then $|A(z)| \neq 0$ implies $|W'(z)| \neq 0$ and $A(z) + A'(z)A(z)^{-1} = W''(z)W'(z)^{-1}$. So Theorem 4′ implies Theorem 4. However, in Theorem 4′ it is not assumed that $|W(z)| \neq 0$ in D, so the n parameter family $w(z)$ given by (4′) is not necessarily the general solution of a differential system with holomorphic coefficients. For example, the matrix $W(z) = zC$, $C = (c_{ik})_1^n$, $|C| \neq 0$, yields, by (4′), vector functions which are injective in the whole plane; the coefficient matrix of the corresponding differential system is given by $A(z) = (1/z)I$. For the proof of Theorem 4′, denote $W''(z)W'(z)^{-1} = B(z)$ and $w'(z) = v(z)$. It follows that $v(z)$ is a solution of the system (10) and, as $v(z) = W'(z)c$, $|W'(z)| \neq 0$, $c \neq 0$, it is a nontrivial solution of this system. (22) implies (21) and the proof continues as before.

We state the one-dimensional case of Theorem 4′ as a corollary.

COROLLARY 1. *Let D be a convex domain, $\infty \notin D$, and assume that the function $w(z)$ (from D to C^1) is holomorphic in D and that $w'(z) \neq 0$ in D. If*

$$\int_{\partial D} |w''(z)/w'(z)|\,|dz| \leq 2\pi, \tag{23}$$

then $w(z)$ is univalent in D.

By our proof, this is a consequence of the Wolff-Noshiro univalence condition (Re $(w'(z)e^{i\alpha}) > 0$ for z in a convex domain [11]); the corresponding case of Horn's lemma is a trivial statement about closed curves lying on the unit circle $|z| = 1$. Note that if (23) is replaced by the proper inequality

$$\int_{\partial D} |w''(z)/w'(z)|\,|dz| < 2\pi, \tag{23′}$$

then the inequality $w'(z) \neq 0$ in D follows, and has thus not to be stated as an assumption. We remark that for the unit disk Corollary 1 follows from a stronger result of Becker: if $w(z)$ is holomorphic in $|z| < 1$, $w'(0) \neq 0$, then

$$\left| z\,\frac{w''(z)}{w'(z)} \right| \leq \frac{1}{1 - |z|^2}, \quad |z| < 1,$$

implies univalence of $w(z)$ in $|z| < 1$ [1, Korrolar 4.1]. For a generalization of Becker's result to n-dimensional univalence see Pfaltzgraff [8].

5. k-valent systems and real systems

A holomorphic vector function $w(z) = (w_1(z), \dots, w_n(z))$ is said to be *at most k-valent in D* if, for every given vector a, the equation $w(z) = a$ has at most k solutions $w(z_1) = \cdots = w(z_k) = a$, $z_1, \dots, z_k \in D$. (Here $k \geq 1$ and the k points need not be distinct.) The following generalization of Theorem 4′ holds.

THEOREM 5. *Let D be a convex domain, $\infty \notin D$, and assume that the matrix $W(z) = (w_{ik}(z))_1^n$ is holomorphic in D and that, for a given integer k, $k \geq 1$, the kth derivative $W^{(k)}(z)$ is nonsingular in D. If*

$$\int_{\partial D} \|W^{(k+1)}(z)W^{(k)}(z)^{-1}\|\,|dz| \leq 2\pi, \tag{24}$$

then every vector function $w(z)$ defined by

$$w(z) = W(z)c, \quad c \neq 0, \tag{4′}$$

is at most k-valent in D.

Proof. Let $w(z)$ be a given vector function, defined by (4′), and denote

$$v(z) = w^{(k)}(z) = W^{(k)}(z)c. \tag{25}$$

As $|W^{(k)}(z)| \neq 0$, $c \neq 0$, it follows that $v(z) \neq 0$, $z \in D$. $v(z)$ is a nontrivial solution of the system

$$v'(z) = B(z)v(z), \tag{10}$$

where now $B(z) = W^{(k+1)}(z)W^{(k)}(z)^{-1}$. It follows that for the given function $w(z)$ and for any given domain D_r-bounded by the level line γ_r, $0 < r < 1$, of the Riemann mapping function $\zeta = \psi(z)$ from D onto $|\zeta| < 1$—there exists a unit vector $u(= u(w, r))$ such that

$$\operatorname{Re}\,(w^{(k)}(z), u) > 0, \quad z \in \bar{D}_r. \tag{26}$$

We define divided differences for vector functions in the usual way.

$$w[z, z_1] = \frac{1}{z - z_1}\,(w(z) - w(z_1)),$$

$$w[z, z_1, \dots, z_l] = \frac{1}{z - z_l}\,(w[z, z_1, \dots, z_{l-1}] - w[z_1, z_2, \dots, z_l]), \quad l = 2, \dots, k.$$

Hermite's formula [7, p. 9] states that

$$w[z, z_1, \dots, z_k] = \int \cdots \int w^{(k)}(t_0 z + t_1 z_1 + \cdots + t_k z_k)\, dt_1 \cdots dt_k. \tag{27}$$

Here the integration is over the k-dimensional simplex $t_i \geq 0$, $i = 0, \dots, k$, $\sum_0^k t_i = 1$, and the point $t_0 z + t_1 z_1 + \cdots + t_k z_k$ varies in the convex hull H of the $k + 1$ points $z, z_1, \dots, z_k$.

Now let these $k + 1$ points be given in D and let r, $0 < r < 1$, be so near to 1 that $\bar{D}_r$ contains the convex hull H of these points. As the volume element of the integral in (27) is positive, it follows from (26) and (27) that

$$\operatorname{Re}\,(w[z, z_1, \dots, z_k], u) > 0$$

hence $w[z, z_1, \dots, z_k] \neq 0$. This excludes the equality $w(z) = w(z_1) = \cdots = w(z_k)$ and thus proves the theorem.

The generalization of the Wolff-Noshiro condition to k-valency was, for $n = 1$, proved by Lavie. If $w(z)$ is holomorphic in a convex domain D and $\operatorname{Re}\,(w^{(k)}(z)e^{i\alpha}) > 0$, $z \in D$, for some α, $0 \leq \alpha < 2\pi$, then $w(z)$ is at most k-valent in D [4, Lemma 2].

For real valued matrices $Y(x) = (y_{ik}(x))_1^n$, defined on a compact segment of the real line, the following analogue of Theorem 5 holds.

THEOREM 6. *Let the integer $k \geq 1$, be given and let the real valued matrix $Y(x) = (y_{ik}(x))_1^n$ be of class $C^{(k+1)}[a, b]$, $-\infty < a < b < \infty$, and assume that the* *kth derivative $Y^{(k)}(x)$ is nonsingular in $[a, b]$. If*

$$\int_a^b \|Y^{(k+1)}(x)Y^{(k)}(x)^{-1}\| \, dx < \pi, \tag{24'}$$

then every vector function $y(x)$ defined by

$$y(x) = Y(x)c, \quad c \text{ real and } \neq 0, \tag{4''}$$

is at most k-valent in $[a, b]$.

The proof is similar to the proof in the complex case. Instead of Horn's lemma we use the obvious assertion that an arc Γ on the unit sphere S^{n-1} of length $l(\Gamma) < \pi$ lies in an open hemisphere. (24′) implies the existence of a unit vector u such that $(y^{(k)}(x), u) > 0$, $a \leq x \leq b$, and the result follows by Hermite's formula.

For real systems there exists a simple connection between the kth derived system $s'(x) = (Y^{(k+1)}(x)Y^{(k)}(x)^{-1})s(x)$ and the n parameter family $y(x)$ given by (4″). If the kth derived system is nonoscillatory in $[a, b]$ then every vector function $y(x)$ is at most k-valent in $[a, b]$. By a result of Nehari, the assumption

$$\int_a^b \|A(x)\| \, dx < \pi/2, \tag{28}$$

implies that the real system $y'(x) = A(x)y(x)$ is nonoscillatory in $[a, b]$ [5, Theorem 3.3]. This result is sharp and its sharpness was shown by a system with a constant matrix $A(x) = A$. As for a constant matrix the given system coincides with all its derived systems, it follows from (24′) and the sharpness of (28) that there exist systems which are injective in a given interval and all their derived systems have oscillatory solutions in that interval.

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