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POLYNOMIAL IDENTITIES OF NONASSOCIATIVE RINGS PART I: THE GENERAL STRUCTURE THEORY OF NONASSOCIATIVE RINGS, WITH EMPHASIS ON POLYNOMIAL IDENTITIES AND CENTRAL POLYNOMIALS

BY

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General introduction

A (polynomial) identity (resp. central polynomial) of a ring R is, intuitively, a polynomial in several noncommuting, nonassociating indeterminates, for which every substitution of elements in R yields 0 (resp. yields an element in the center of R). In this paper, we study the structure of rings with 1, not necessarily associative, which satisfy "suitable" polynomial identities; such rings are called PI-rings. The specific aim is to obtain a theory which can be applied to associative, alternative, and Jordan rings. Central polynomials will be fundamental in the study.

The motivating idea of this paper came from a letter of M. Slater in 1973, where he asked whether the superficial similarities between results of mine for associative PI-rings, and results of his for nonassociative alternative rings, might go further beneath the surface. Upon investigation, the following analogue was immediate: Prime alternative, nonassociative rings actually satisfy a central polynomial $[X_1, X_2]^2$, whereas prime, associative PI-rings also satisfy central polynomials (first discovered by Formanek).

The application of this fact led to an extension of the associative PI-theory to alternative rings, and the nice properties of nonassociative prime alternative rings could be derived "because" these rings are alternative PI-rings. Such considerations were also applicable to many kinds of Jordan algebras, leading to a theory of classes of rings satisfying central polynomials, with immediate applications for associative rings, alternative rings, and some classes of Jordan rings. The basic idea was to show that, for suitable classes of PI-rings, in a ring with no nil ideals, every nonzero ideal intersects the center nontrivially. (In particular, if the center is a field then the ring has no proper nonzero ideals.) Repeated use of this fact yields very much information, along the lines of the

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known theory of associative rings with central polynomial. (In fact the theory is based, for a large part, on generalizing known associative PI-theory.)

Over the years since the original version of this paper was submitted in 1974, the general theory it contained was made tighter and more extensive. Presently, the paper stands in three parts:

I. The general structure theory of nonassociative rings, with emphasis on polynomial identities and central polynomials.

II. Elementary sentences, universal rings, and the various radicals.

III. Applications of the structure theory to associative, alternative, and Jordan rings.

We shall refer to these parts as I, II, and III. Clearly the applications in III are the main motivation of this research. On the other hand, as long as a comprehensive theory was being developed, it seemed worthwhile to put the results of I and II in a framework broad enough to house other work on identities.

This decision is reflected immediately, in the set Ω from which we draw the coefficients of the identities of a ring R. The obvious selection for such a set is the integers $\mathbf{Z}$; by a theorem of Amitsur, identities with coefficients in $\mathbf{Z}$ yield all the information of the classical PI-theory for associative rings. On the other hand, much of the classical literature on nonassociative rings is written about algebras over fields, so we would certainly want the possibility that Ω is a commutative ring Φ, with the stipulation that R is a Φ-algebra. Moreover, since many Jordan rings are built from the symmetric elements of a ring with involution $(*)$, we want the possibility of identities "for symmetric elements," i.e., identities involving $(*)$, so Ω should contain a symbol representing the action of $(*)$ on R. (A "symmetric" indeterminate would then be $X + X^*$.) At this juncture, it seemed worthwhile to make Ω a general set of actions on R, in this manner we could also include the following well-studied situations: R is an associative ring with an automorphism whose ring of fixed points satisfies a polynomial identity, or R is an associative ring satisfying an identity with coefficients in R (called a "generalized polynomial identity" in the literature). These ramifications will be treated in some detail in the body of I, although, for the application, we will be concerned primarily with the case that R is an algebra, possibly with involution.

In the same philosophy, we have included some results which seem to contain enough interest to be stated in their full generality, whereas only a special case would be sufficient for the applications in III. For example, universal Ω-rings are used only in the study of the identities of the universal Cayley-Dickson algebra, which could be accomplished directly in less than a page. Even so, the question of which sentences pass from a ring to its universal ring is crucial if we want to exploit the universal ring to its fullest. Ample motivation is provided by Amitsur's use of a universal ring, as an example of a finite dimensional division algebra without any maximal subfields Galois over the center.

Introduction to Part I

As stated in the general introduction, we shall be interested in a ring R with 1, and a set Ω acting distributively (over addition) on R from both the left and right. Ω will be fixed, and we call R an Ω-ring. We shall consider polynomials with "coefficients" in Ω, the coefficients possibly interspersed throughout the polynomial. The precise definitions are given in Section 1A, along with some other basic notions, defined in the category of Ω-rings. For example, the center of a Ω-ring R, written $Z(R)$, is the set of elements of R satisfying every possible associativity and commutativity condition with Ω and R. Examples of Ω-rings and certain identities and central polynomials will be given in Section 1B.

If $Z = Z(R)$ and H is a commutative, associative Z-algebra, then the tensor product $R \otimes_Z H$ is a Ω-ring, under the operations induced canonically. Many kinds of identities of R pass to $R \otimes_Z H$. The most important example is the multilinear identity, because there is a procedure by which an identity can be "multilinearized" and still preserve many of its properties. A definition of R-stable identities is given; these identities are characterized as the identities which are also identities of $R \otimes_Z H$ for every commutative, associative Z-algebra H. Multilinearization and R-stable identities and central polynomials are treated in Section 1C.

The most useful instance of $R \otimes_Z H$ is when S is a multiplicative subset of Z and $H = Z_S$, the localization of Z by S. In this case we designate $R \otimes_Z (Z_S)$ as R_S. Passing from R to R_S is called central localization, the subject of Section 2. In 2A, we see some general properties of central localization; in particular every identity of R is an identity of R_S. One very pleasant property of central localization is that the localization of a prime ideal is prime, and in fact there are several nice ways of obtaining information about prime ideals of R by passing to R_S. These theorems are taken largely from [23], and are collected in 2B. Actually, we are interested mainly in prime ideals P of R such that R/P has no nonzero nil ideals. Call such a prime ideal "strongly prime"; R is said to be strongly prime if 0 is a strongly prime ideal.

In Section 3, we get to the heart of the paper by using central localization to examine certain classes of rings with central polynomial. Define a *Kaplansky class* $\mathscr{C}$ as a class of Ω-rings having the following properties:

(1) If $R \in \mathscr{C}$ then $R/\text{Nil } R$ is the subdirect product of strongly prime rings in $\mathscr{C}$.

(2) If $R \in \mathscr{C}$ then $R[\lambda] \in \mathscr{C}$ (where λ is a commuting, associating indeterminate over R).

(3) If $R \in \mathscr{C}$ then every central localization of R is in $\mathscr{C}$.

(4) If $R \in \mathscr{C}$ and $\text{Nil }(R) = 0$, then the intersection of the maximal ideals of $R[\lambda]$ is 0.

Conditions (1)–(3) are included to insure that $\mathscr{C}$ resembles a variety; these conditions hold in any variety defined by identities of degree ≤ 2 in each indeterminate, for example. Condition (4) is much more restrictive, but seems to be necessary for our purposes. The class is called "Kaplansky" because, by theorems of Amitsur and Kaplansky, {associative PI-rings} satisfies (4). (Actually, Kaplansky's theorem is generalized to any rings with central polynomial.)

In a Kaplansky class, the presence of a central polynomial has many consequences, the most important being that, in rings with no nil ideals, all nonzero ideals intersect the center nontrivially. Most of the results of [23] can be transferred verbatim; in particular there is a correspondence between the lattices of prime ideals of R and Z.

Another related approach to the correspondence between ideals of R and Z was motivated by a short proof, found independently by Amitsur [4] and Rowen, of the famous Artin-Procesi theorem (characterizing Azumaya algebras in terms of polynomial identities). The "heart" of the theorem can be given for nonassociative rings, with the idea of the proof patterned after [27]; the idea is to build a new central polynomial from a given one, and by very elementary manipulations to go back and forth between R and Z. This is done in §4.

In §5, a general decomposition theorem is given which enables one to separate a semiprime ring into its associative and "purely nonassociative" parts; this result will be particularly useful for alternative rings.

In §6, the special case of Ω-rings with involution is treated in considerable detail, and it is shown that the general theory described in §1 is entirely consistent with the theory of "*-identities" that has developed in the last ten years. Related Ω-rings are built from the symmetric elements of Ω-rings with involution; these Ω-rings are to be used in Jordan algebra constructions in Part III.

1. General preliminary results

1A. Definitions and examples. *Ring* will denote a nonassociative ring with 1. In a ring R, define

$$[r_1, r_2, r_3] = (r_1 r_2)r_3 - r_1(r_2 r_3) \quad \text{and} \quad [r_1, r_2] = r_1 r_2 - r_2 r_1.$$

Also define the *nucleus*

$$N(R) = \{r \in R \mid [r, R, R] = [R, r, R] = [R, R, r] = 0\}$$

and the *center*

$$Z_0(R) = \{r \in N(R) \mid [r, R] = 0\}.$$

When unambiguous, N and Z_0 will respectively denote $N(R)$ and $Z_0(R)$. For convenience, write $r_1 \cdots r_m$ for $(\cdots((r_1 r_2)r_3)\cdots)r_m$.

Now let Ω be a set; we say R is a *Ω-ring* if there are operations

$$\Omega \times R \to R \quad \text{and} \quad R \times \Omega \to R,$$

both of which distribute over addition in R. Note that R is always a $\mathbf{Z}$-algebra and a Z_0-algebra, so we could take Ω to be either $\mathbf{Z}$ or Z_0. When we call R a "ring," without specifying Ω, we tacitly take $\Omega = \mathbf{Z}$. Usually Ω will be some commutative, associative ring over which R is an algebra, but we give the general definition to include the following possibilities (which have been treated extensively in the literature when R is associative): (i) Ω is a subring of R; (ii) Ω contains a symbol denoting a given involution of R; (iii) Ω contains a symbol denoting a given automorphism of R. The first two cases are treated later in the paper, and will be included in the examples below.

In this theory, we fix Ω and deal with the category of Ω-rings. Accordingly, an *ideal* A of a Ω-ring R is a ring ideal of R, such that $\Omega A \subseteq A$ and $A\Omega \subseteq A$. Homomorphisms, epimorphisms, etc., will be viewed in the context of preserving Ω-structure. Given an element r in R, we will let $\langle r \rangle$ denote the ideal (of R) generated by r. Clearly, we may assume that Ω has an "identity" action e given by $er = re = r$ for all r in R; if we do not yet have such an element in Ω, we add it to Ω formally.

One can construct formally a "free" Ω-ring, which we call $\Omega\{X\}$, as follows. *Monomials* are strings consisting of elements of Ω, the formal element 1, and (nonassociating, noncommuting) indeterminates X_i, with parentheses indicating the order of multiplication; every monomial is stipulated to contain either 1 or some X_i, and we identify the monomial h with $(1h)$ and with $(h1)$. $\Omega\{X\}$ is the set of formal sums of monomials, with addition defined formally and multiplication given by $(\sum h_i)(\sum h_j) = \sum (h_i h_j)$; to make $\Omega\{X\}$ into an Ω-ring, define $\omega(\sum h_i) = \sum (\omega h_i)$ and $(\sum h_i)\omega = \sum (h_i \omega)$. Elements of $\Omega\{X\}$ are called *polynomials*. $\Omega\{X\}$ is free in the sense that, for any countable set of elements $r_1, r_2, \ldots$ in a Ω-ring R, there is a unique homomorphism $\Omega\{X\} \to R$ sending $X_1 \to r_1$, $X_2 \to r_2, \ldots$. A polynomial in the kernel of all homorphisms $\Omega\{X\} \to R$ is called an *identity* of R.

Given an element f of $\Omega\{X\}$, denote f by $f(X_1, \ldots, X_m)$ if f is in the Ω-subring of $\Omega\{X\}$ generated by $X_1, \ldots, X_m$; $f(r_1, \ldots, r_m)$ denotes the image of f in R, under the homomorphism sending $X_i \to r_i$ for $1 \le i \le m$, and $X_i \to 0$ for all other i. Let

$$R(f) = \{f(r_1, \ldots, r_m) \mid \text{all } r_i \text{ in } R\},$$

the set of "evaluations" of f in R. Clearly f is an identity of R iff $R(f) = 0$. Also, let $R^+(f)$ be the additive subgroup of R generated by $R(f)$.

If the indeterminates of a monomial h occur (from left to right) in the order $X_{\mu(1)}, X_{\mu(2)}, \ldots, X_{\mu(t)}$, call $X_{\mu(1)} X_{\mu(2)} \cdots X_{\mu(t)}$ the *fingerprint* of h. The sum of the monomials of f with a given fingerprint is called a *generalized monomial*. In other words, a generalized monomial is a sum of monomials with the indeterminates arranged in the same order, but with (possibly) different placement of parentheses and different operators from Ω.

Many important identities are generalized monomials, given in §1B; however, motivated by the associative theory, we shall focus on the opposite situation. Accordingly, say f is *R-proper* if some generalized monomial of f is *not* an identity of R; f is *R-strong* if f is proper for every nonzero homomorphic image of R. If R has an R-strong identity, we call R a *PI* (Ω-)*ring*.

Some remarks are in order. Suppose a polynomial f is written as a sum of generalized monomials $\sum f_i$. Clearly f is R-strong iff the ideal generated by all $\sum R(f_i)$ contains 1. Thus, if f is R-strong and R_1 is a Ω-ring containing R and having the same 1, then f is R_1-strong. By a similar argument using the "diagonal map" of R into a direct power R' of copies of R, one sees easily that if f is R-strong then f is R'-strong; thus any direct power of a PI-ring is a PI-ring. However, this argument fails for direct products; a polynomial which is R_i-strong for each i, need not be $\prod R_i$-strong. Also, every homomorphic image of a PI-ring is a PI-ring.

For $\mathscr{S} \subseteq \Omega\{X\}$, we define $\mathscr{V}(\mathscr{S})$, the *variety determined by* $\mathscr{S}$, as the class of Ω-rings for which each element of $\mathscr{S}$ is an identity. Note, for any identity f of R, that every endomorphic image of f is an identity of R. Hence, if we let $\tilde{\mathscr{S}}$ be the ideal of $\Omega\{X\}$ generated by all endomorphic images of $\mathscr{S}$ in $\Omega\{X\}$, we have $\mathscr{V}(\tilde{\mathscr{S}}) = \mathscr{V}(\mathscr{S})$. Conversely, $\Omega\{X\}/\tilde{\mathscr{S}} \in \mathscr{V}(\mathscr{S})$. Such a Ω-ring $\Omega\{X\}/\tilde{\mathscr{S}}$ is called a *universal PI-ring* or a "relatively free" ring and will be one subject of careful investigation in Part II of this series. Define

$$Z(R;\Omega) = \{z \in Z_0 \mid (\omega r)z = \omega(rz) \quad \text{and} \quad (zr)\omega = z(r\omega) \text{ for all } r \text{ in } R \text{ and all } \omega \text{ in } \Omega\}.$$

Since Ω is usually understood, we shall write $Z(R)$ for $Z(R;\Omega)$. When R is unambiguous, we merely write Z. A polynomial f is *R-central* if $0 \neq R(f) \subseteq Z$. (Central polynomials for matrix algebras over a field were found independently by Formanek [9] and Razmyslov [21], and revolutionized the theory of associative PI-rings. In this paper, we also build the general PI-theory around central polynomials.)

Let $\mathscr{S}(R)$ denote {identities of R}. If $R_1 \subseteq R_2$ or R_1 is a homomorphic image of R_2 then $\mathscr{S}(R_2) \subseteq \mathscr{S}(R_1)$. Two Ω-rings R_1 and R_2 are *equivalent* if $\mathscr{S}(R_1) = \mathscr{S}(R_2)$. For example, R_1 is equivalent to any direct product of copies of R_1. Note that equivalent Ω-rings also have the same set of central polynomials. Write $\mathscr{V}(R)$ for $\mathscr{V}(\mathscr{S}(R))$.

Given two polynomials f_1 and f_2, say $f_1 \leq f_2$ if $f_1 \in \Omega\{X\}^+(f_2)$, i.e., if f_1 is a sum of evaluations of f_2 in $\Omega\{X\}$ (and their negations). It is easy to see that $f_1 \leq f_2$ iff $R^+(f_1) \subseteq R^+(f_2)$ for all Ω-rings R. In particular, if $f_1 \leq f_2$, then whenever f_2 is an identity of R (resp. R-central) then f_1 is an identity of R (resp. either R central or an identity of R).

(This definition is more restrictive than saying that f_1 is in the ideal of $\Omega\{X\}$ generated by $\Omega\{X\}(f_2)$, and is used so that we may treat central polynomials as well as identities.)

The associative definitions of [23] and [24] are generalized easily. For example, the *degree* of X_i in a generalized monomial h of f is the degree of X_i in the fingerprint of h; f is *homogeneous in* X_i if X_i has the same degree in each generalized monomial of f, and f is *completely homogeneous* if f is homogeneous in each X_i. The *height* of f is the maximal height of the fingerprints of its generalized monomials (cf. [13], where *height* of a monomial is its degree minus the number of indeterminates occurring in the monomial). If f is homogeneous of degree 1 in every X_i occurring in f, then f is *multilinear*; note all multilinear polynomials have height 0.

A polynomial $f(X_1, \ldots, X_m)$ is *t-linear* if f is homogeneous of degree 1 in each X_i, $1 \leq i \leq t$. Say $f(X_1, \ldots, X_m)$ is *t-alternating* if whenever $1 \leq i \leq j \leq t$, $f(\ldots, X_i, \ldots, X_j, \ldots)$ vanishes when we substitute the same indeterminate for both X_i and X_j. If f is t-alternating then

$$f(\ldots, X_i, \ldots, X_j, \ldots) = -f(\ldots, X_j, \ldots, X_i, \ldots)$$

whenever $1 \leq i \leq j \leq t$; the converse is true if $\frac{1}{2} \in \Omega$. A polynomial f is *t-normal* if f is t-multilinear and t-alternating.

1B. Examples, with remarks. This section consists of a number of examples of the above concepts, with remarks to illustrate how they "fit" into the general theory to be developed.

(i) Any expression which is 0 in the free nonassociative ring $Z\{X\}$ is an (improper) identity of all rings. (Such identities are called *trivial.*) The following two examples are very useful:

$$(\alpha) \quad X_1[X_2, X_3, X_4] + [X_1, X_2, X_3]X_4 - [X_1X_2, X_3, X_4] + [X_1, X_2X_3, X_4] - [X_1, X_2, X_3X_4],$$

$$(\beta) \quad [X_1, X_2, X_3] - [X_1, X_3, X_2] + [X_3, X_1, X_2] + X_1[X_2, X_3] + [X_1, X_3]X_2 - [X_1X_2, X_3].$$

(ii) Using identity (β), one can see that $f(X_1, \ldots, X_m)$ is R-central iff $[X_{m+1}, f]$, $[X_{m+1}, X_{m+2}, f]$, $[X_{m+1}, f, X_{m+2}]$, $(\omega X_{m+1})f - \omega(X_{m+1}f)$, and $f(X_{m+1}\omega) - (fX_{m+1})\omega$ (for all ω in Ω), but not f, are identities of R.

(iii) If Ω is a ring Σ, and e is the multiplicative unit of Σ, we say R is a Σ-*algie* when the map $\Sigma \to \Sigma 1$ is a ring homomorphism sending $Z(\Sigma)$ into $Z(R)$, such that $\omega r = (\omega 1)r$ and $r\omega = r(\omega 1)$ for all r in R, ω in Σ. (This last condition shows that "ideal" means the same thing, viewing R as Σ-algie or as ring.)

The condition "R is a Σ-algie" is varietally defined as $\mathscr{V}(\mathscr{S}_0)$, where

$$\mathscr{S}_0 = \{(\omega_1 + \omega_2)X_1 - \omega_1 X_1 - \omega_2 X_1, X_1(\omega_1 + \omega_2) - X_1\omega_1 - X_1\omega_2, [\omega_1, \omega_2, 1], [1, \omega_1, \omega_2], [\omega_1, 1, X_1], [X_1, 1, \omega_1], [1, \omega_1, X_1], [X_1, \omega_1, 1], [z, X_1], [X_1, X_2, z], [X_1, z, X_2] \text{ for all } \omega_1, \omega_2 \text{ in } \Sigma, \text{ all } z \text{ in } Z(\Sigma)\}.$$

(The identities $e1 - 1$ and $1e - 1$ have already been assumed.)

Algies are useful because they give a generalization of algebra, which (for example) enables us to take "coefficients" of the identity from R itself. (Note that the coefficients are interspersed throughout the polynomial so that elements of Σ work more like "ring elements" than "scalars.") Such identities are called "generalized polynomial identities" and are very interesting (cf. [1], [17], [18], [24], [25] for representative results). The "algie" approach to generalized polynomial identities is given in [24].

(iv) If Ω is a commutative ring Φ, we say R is an *algebra* if R is a Φ-algie with $\Phi 1 \subseteq Z(R)$. Obviously these extra conditions are varietally defined, so the class of Φ-algebras is a variety. Every ring is an algebra. We shall be concerned with the following varieties of algebras, defined by additional identities:

(1) power-associative (satisfying $[X_1^i, X_1^j, X_1^k]$ for all i, j, k in $\mathbf{Z}^+$),
(2) right alternative (satisfying $[X_1, X_2, X_2]$),
(3) alternative (satisfying $[X_1, X_1, X_2]$ and $[X_1, X_2, X_2]$),
(4) associative (satisfying $[X_1, X_2, X_3]$),
(5) commutative algebras (satisfying $[X_1, X_2]$),
(6) Jordan algebras (satisfying $[X_1, X_2]$ and $[X_1^2, X_2, X_1]$).

(v) For any generalized monomials f_1, f_2, and f_3, obviously $[f_1, f_2, f_3]$ is a generalized monomial. Consequently, power-associative, right alternative, alternative, and associative algebras are defined by improper identities.

(vi) In general, improper identities have no bearing on the dimension of an algebra; they often give information about individual elements, instead of collective information. Some examples: If nX_1 is an identity then each element of R is n-torsion (i.e., R has characteristic n). If $\Omega = R$ and R is an R-algie with identities $[r, X_1, X_2]$, $[X_1, r, X_2]$, and $[X_1, X_2, r]$, then $r \in N(R)$; if $[r, X_1]$ is also an identity of R then $r \in Z(R)$. Finally, every associative ring which is not prime satisfies some improper generalized polynomial identity $r_1 X_1 r_2$, for suitable elements r_1, r_2 of R.

(vii) Suppose f is a polynomial with coefficients only from $\{1, -1\}$ having the property that there is a suitable fingerprint with only one monomial corresponding to it. Then f is R-strong for every R. Proof: just substitute "1" for every X_i. (Example: $[X_1, X_2]$ is R-strong for every R.)

(viii) It is simple to separate an identity f of R into the sum of an R-proper identity f_1 and an R-improper identity f_2: let f_2 be the sum of those generalized monomials of f which are identities of R. Incidentally, the "constant term," the part of f having degree 0, is $f(0, \ldots, 0) = 0$, so f_1 has "constant term" 0. Thus, we will assume in our study of proper identities that all monomials have positive degree.

(ix) If $Z(R)$ is a field then every R-central polynomial is obviously R-strong.

(x) It is easy to see that every t-normal polynomial is an identity of every Φ-algebra spanned (as module over Φ) by fewer than t elements. The most famous example of a t-normal polynomial is the *standard polynomial*

$$S_t(X_1, \ldots, X_t) = \sum (\text{sg } \pi) X_{\pi 1}, \ldots, X_{\pi t},$$

summed over all permutations π of $(1, \ldots, t)$; by (vii), S_t is R-strong for all rings R.

(xi) It is easy, using the result of Formanek [9] (or that of Razmyslov [21]) to construct a t^2-normal central polynomial for every central simple associative algebra of degree t (i.e., dimension t^2 over its center). This was observed by Amitsur [4], Goldie, and Rowen [27], and has important implications. (Details will be given in Section 4.)

(xii) Suppose, for convenience of notation, that R is a power-associative Φ-algebra. If r is an element (of R) which is algebraic of degree t over Φ, then $\Phi(r)$ is spanned by t elements, so $S_{t+1}(r^tX_1, r^{t-1}X_1, \ldots, X_1)$ is an improper generalized identity of R. If *every* element of R is algebraic of degree $\leq t$ over Φ, we say that R is *algebraically algebraic* of degree $\leq t$; in this case,

$$S_{t+1}(X_2^tX_1, X_2^{t-1}X_1, \ldots, X_1)$$

is an identity of R, which is R-strong, by (vii).

(xiii) Suppose Ω has an element $(*)$ which is an anti-automorphism of degree ≤ 2 over R. We call $(*)$ an *involution* of R. We write r^* for the action of $(*)$ on R (defined formally to be the same on both left and right). Such a situation is very important, with representative results (in the associative case) in [2], [10], [16], [18], [24], and [26]. We can treat the involution varietally, by noting that $(*)$ is an involution on R iff R satisfies the identities $(*X_1 - X_1*)$ (so that we write X^* for $(*X)$ or $(X*)$), $((X_1^*)^* - X_1)$, $(X_1X_2)^* - X_2^*X_1^*$, and $(1^* - 1)$. In this case, we call the corresponding universal ring $(\Omega\{X\}, *)$, the *"free" ring with involution*. In virtually every situation, Ω will be itself a ring with involution $(\Sigma, *)$, with $(\omega r)^* = r^*\omega^*$ and $(r\omega)^* = \omega^*r^*$ for all ω in Σ, all r in R. However, to preserve complete generality, we may not have a priori symbols ω^* when $\omega \in \sum$. This is easily remedied. Just define ω^* formally via the action $\omega^*r = (r^*\omega)^*$ and $r\omega^* = (\omega r^*)^*$. Now $(\Omega\{X\}, *)$ is in fact a ring with involution.

We will treat involutions in detail in §5, including discrepancies between the definitions presented here and the definitions in the literature.

1C. Central extensions of rings, multilinearization, and stable identities. If R is a ring with center Z and if H is a commutative, associative Z-algebra, then $R \otimes_Z H$ is a Ω-ring, with operations $\omega(\sum_i r_i \otimes h_i) = \sum_i (\omega r_i \otimes h_i)$ and $(\sum_i r_i \otimes h_i)\omega = \sum_i (r_i\omega \otimes h_i)$, seen via the definition of tensor product. Call $R \otimes_Z H$ a *tensor extension* of R. There is a canonical homomorphism $R \to R \otimes_Z H$ given by $r \to r \otimes 1$, although there may be a nonzero kernel. A very important example when this map is an injection is when $H = Z[\lambda]$, where λ is a commuting, associating indeterminate over Z. In this case we write $R[\lambda]$ for $R \otimes_Z H$, and call $R[\lambda]$ the *polynomial ring over R*.

It was recognized early that, for R associative, all multilinear identities of R are identities of every tensor extension of R. (We shall soon present this fact more generally for nonassociative rings.) Since tensor extensions are crucial to much of the structure theory, the process of obtaining multilinear identities from arbitrary identities is fundamental. Fortunately there is the following very successful, well-known procedure: Given $f(X_1, \ldots, X_m)$, define

$$\Delta_j f(X_1, \ldots, X_{m+1}) = f(X_1, \ldots, X_j + X_{m+1}, \ldots, X_m) \\ - f(X_1, \ldots, X_j, \ldots, X_m) - f(X_1, \ldots, X_{m+1}, \ldots, X_m).$$

Clearly $\Delta_j f \leq f$, $\Delta_j f$ has lower height than f, and, for any generalized monomial f_π of f, there is a generalized monomial f'_π of $\Delta_j f$ with $f_\pi \leq f'_\pi$. By induction on height, we see that one obtains, after a finite number of applications of various Δ_k, a multilinear polynomial h, and an easy induction argument (on height) yields the following properties:

PROPOSITION 1.0. *With notation as above, $h \leq f$,* ht $(h) \leq$ ht (f) *(with equality holding only if f is multilinear), and for any generalized monomial f_π of f there is a generalized monomial h_π of h, with $f_\pi \leq h_\pi$. Thus, if f is a proper (resp. strong) identity of R then h is a proper (resp. strong) identity of R.*

Proposition 1.0 shows that *every PI Ω-ring has a strong multilinear identity.* We shall often use this fact tacitly. The situation for central polynomials is not nearly as nice. Multilinearizations of central polynomials may not be central. (The problem is that the Δ_i operator might take nonidentities of R to identities of R. For example, if R is a commutative, associative $\mathbf{Z}$-algebra satisfying the identity $X_1^2 - X_1$ (i.e., R is Boolean), then X_1^2 is R-central, but $X_1X_2 + X_2X_1$ is an identity of R. If R is a Q-algebra then a trivial specialization argument shows that for any polynomial f, $R(\Delta_i f) = R(f)$, so multilinearizations of central polynomials remain nonidentities whenever R has characteristic 0.) In general, one has a partial result parallel to [23, Section 1]. A polynomial f is *blended* if for each X_i occurring (nontrivially) in f, X_i occurs in every monomial of f. For example, $X_1X_2 - X_1$ is not blended, but $X_1X_2 - (X_2X_1)X_1$ is blended.

PROPOSITION 1.1. *Suppose G is a given additive subgroup of R, and f is a polynomial with $R(f) \nsubseteq G$. Then there exists a blended polynomial $h \leq f$ with the following additional properties:* (1) ht $(h) \leq$ ht (f); (2) *if f is t-normal then h is t-normal;* (3) $R(h) \nsubseteq G$.

Proof. Use the argument of [23, Lemma 1.1]. Q.E.D.

In particular, for $G = 0$, we can use "blended" central polynomials in place of arbitrary central polynomials. We return now to identities (and central polynomials) which pass to central extensions. For this purpose, let us introduce the concept of *R-stable*, defined by induction on height:

(i) All multilinear polynomials are R-stable.

(ii) For f completely homogeneous, f is R-stable if, for every j, $\Delta_j f$ is R-stable.

(iii) For f *not* completely homogeneous, f is R-stable if, when f is written as a sum of completely homogeneous polynomials $f_1, \ldots, f_k$ with k minimal then, for all u, $1 \le u \le k$, f_u is R-stable and $R^+(f_u) \subseteq R^+(f)$.

Remark 1.2. (i) Every completely homogeneous identity of degree ≤ 2 in each indeterminate is R-stable; for example the identities defining alternative algebras (§1B, example (iv)) are stable.

(ii) All identities of R are R-stable, iff every identity of R is the sum of completely homogeneous identities of R (proof by induction on height). Note that if f is a sum of completely homogeneous nontrivial polynomials $f_1, \ldots, f_k$, with k minimal, then ht $(f_u) \le$ ht (f) for all u, $1 \le u \le k$.

(iii) If $R \in \mathscr{V}(R_1)$ then every R_1-stable identity of R_1 is an R-stable identity of R. (In particular, this is true if $R \subseteq R_1$.)

The proof of [23, Proposition 1.3 (ii)] can be adapted to yield:

PROPOSITION 1.3. *Every R-stable identity of a Ω-ring R is an identity of every tensor extension of R.*

If $R \subseteq R_1$ and $R_1 = RZ(R_1)$ then we say R_1 is a *central extension* of R. Clearly, in this case, R_1 is a homomorphic image of $R \otimes_Z Z(R_1)$, so we have:

COROLLARY 1.4. *Every central extension R_1 of a PI-ring R is PI.*

Proof. R has a strong multilinear identity, which, by proposition 1.3, is an identity of R_1, and is obviously R_1-strong. Q.E.D.

PROPOSITION 1.5. (i) *Every identity of* $R[\lambda]$ *is* $R[\lambda]$*-stable.*

(ii) *An identity f of R is R-stable iff f is an identity of* $R[\lambda]$.

Proof. (i) By Remark 1.2 (ii), it suffices to show that any identity $f(X_1, \ldots, X_m)$ of $R[\lambda]$ is the sum of completely homogeneous identities of $R[\lambda]$. Let X_i have degree d_i in f, let t be a t-tuple $(t_1, \ldots, t_m)$ with $0 \le t_i \le d_i$ for each i, and let f_t be the sum of all monomials of f in which $X_1, \ldots, X_m$ have respective degrees $t_1, \ldots, t_m$. We aim to show each f_t is an identity of $R[\lambda]$.

Suppose $x_1, \ldots, x_m$ are in $R[\lambda]$, i.e., each x_i has the form $\sum_{j=1}^{n_i} r_{ij}\lambda^j$. Then define $\alpha_1 = 0$ and, inductively, $\alpha_{i+1} = d_i(\alpha_i + n_i) + 1$. Checking coefficients of powers of λ in

$$f(\lambda^{\alpha_1}x_1, \ldots, \lambda^{\alpha_m}x_m)$$

shows each $f_t(x_1, \ldots, x_m) = 0$; we conclude that each f_t is an identity of $R[\lambda]$, as desired.

(ii) If f is an R-stable identity of R, then f is an identity of $R[\lambda]$, by Proposition 1.3. The converse follows immediately from (i) and Remark 1.2 (iii). Q.E.D.

THEOREM 1.6. *For any given positive integer t, a polynomial f is an R-stable identity of R iff f is an identity of* $R[\lambda_1, \ldots, \lambda_t]$. *(In this case, f is* $R[\lambda_1, \ldots, \lambda_t]$*-stable.)*

Proof. Write $R^{(t)} = R[\lambda_1, \ldots, \lambda_t]$, and note that $R^{(t)} = R^{(t-1)}[\lambda_t]$. Suppose f is an R-stable identity of R. By Proposition 1.3, f is an identity of $R^{(1)}$, and f is $R^{(1)}$-stable by Proposition 1.5(i). Using induction on t, we may assume f is an $R^{(t-1)}$-stable identity of $R^{(t-1)}$. But then, by Proposition 1.3, f is an identity of $R^{(t)}$. Every identity of $R^{(t)}$ is $R^{(t)}$-stable, by Proposition 1.5(i) (since $R^{(t)} = R^{(t-1)}[\lambda_t]$).

The opposite direction is immediate; any identity of $R^{(t)}$ is $R^{(t)}$-stable, and thus R-stable, by Remark 1.2. Q.E.D.

COROLLARY 1.7. *R is equivalent to* $R[\lambda_1, \ldots, \lambda_t]$, *iff every identity of R is the sum of completely homogeneous identities.*

Stable identities are very important, because, in what follows, we will need to pass identities to polynomial rings. For this reason, we give a class of rings in which *every* identity is stable (which, by Remark 1.2(i), is equivalent to saying that every identity is the sum of completely homogeneous identities).

Say an element z of Z is *regular* if $zr \neq 0$ *for all nonzero elements r of R.*

Remark 1.8. If Z contains an infinite subring Z_1 of regular elements, then every identity of R is the sum of completely homogeneous identities.

Proof. We sketch a very well-known argument. Write $f = \sum f_t(X_1, \ldots, X_m)$, where each f_t is homogeneous in X_1, of degree t. For any c in Z and $r_1, \ldots, r_m$ in R, we have

$$f(cr_1, \ldots, r_m) = \sum c^t f_t(r_1, \ldots, r_m).$$

Thus, using enough different values of c, we can apply a Vandermonde determinant argument to conclude that each $f_t(r_1, \ldots, r_m) = 0$; thus f_t is an identity of R. The proof is concluded by continuing this procedure for each X_i. Q.E.D.

In Part II, more classes of rings are given in which every identity is stable.

1D. Proving Ω-rings satisfy given canonical identities. By kth *power* of an element, we mean some product of the element by itself k times, under suitable placement of parentheses. (For example, $(aa)a$ and $a(aa)$ are third powers of a.) A *power* of an element is a kth power for some suitable natural number k. An element is *nilpotent* if some power is 0, and an ideal is *nil* if every element is nilpotent. Standard arguments show that the sum of two nil ideals is nil; by an application of Zorn's lemma, any Ω-ring R has a unique maximal nil ideal, which we call Nil (R).

A certain result of Amitsur [2] can be stated in a very general context (with the same proof).

THEOREM 1.9 ("Amitsur's method"). *Suppose R is a Ω-ring with cardinality α, and let $\omega = \max(\alpha, \omega_0)$, where ω_0 is the cardinality of the integers. If f is an identity of $R^\omega/\text{Nil}(R^\omega)$, then some power of f is an identity of R.*

Sketch of Proof. Write f as $f(X_1, \ldots, X_m)$, and let $R' = R^\omega$. Since $\omega = \omega^m$, we can index the components of R' by the m-tuples of elements of R, and the conclusion of the theorem follows forthwith, as in [2]. Namely, take the element $\hat{a}_u$ of R' whose $(r_1, \ldots, r_m)$-component is r_u, for $1 \leq u \leq m$. By assumption, $f(\hat{a}_1, \ldots, \hat{a}_m) \in \text{Nil}(R^\omega)$, so some power of $f(\hat{a}_1, \ldots, \hat{a}_m)$ is 0. In other words, $h(\hat{a}_1, \ldots, \hat{a}_m) = 0$ for some power h of f. But for each $(r_1, \ldots, r_m)$, the $(r_1, \ldots, r_m)$-component of $h(\hat{a}_1, \ldots, \hat{a}_m)$ is $h(r_1, \ldots, r_m)$, proving h is an identity of R. Q.E.D.

Theorem 1.9 is used as follows. Suppose $\mathscr{V}$ is a variety of Ω-rings. If R is a PI-ring in $\mathscr{V}$ then so is $R^\omega/\text{Nil}(R^\omega)$, and often one can show that $R^\omega/\text{Nil}(R^\omega)$ satisfies a standard identity. In this case, R satisfies a power of a standard identity. For a sampling of the many uses of this method, see [2], [3], [23], [25], [26]. We shall also be using this method in this paper.

A similar treatment of the theory can be carried out, by considering only identities in a fixed number of indeterminates. Namely, let $\Omega\{X\}^{(m)}$ be the Ω-subring of $\Omega\{X\}$ generated by $X_1, \ldots, X_m$. Given elements of $r_1, \ldots, r_m$ of a Ω-ring R, one has a unique homomorphism $\Omega\{X\}^{(m)} \to R$ which sends $X_1 \to r_1, \ldots, X_m \to r_m$. The intersection of the kernels of all homomorphisms $\Omega\{X\}^{(m)} \to R$ is the set of *m-identities* of R, denoted as $\mathscr{S}^{(m)}(R) = \Omega\{X\}^{(m)} \cap \mathscr{S}(R)$, where $\mathscr{S}(R) = \{\text{identities of } R\}$. Of course $\Omega\{X\}$ and $\mathscr{S}(R)$ are the respective direct limits of $\Omega^{(m)}\{X\}$ and $\mathscr{S}^{(m)}(R)$ as $m \to \infty$. There is an injection $\Omega\{X\} \to \Omega^{(2)}\{X\}$, given by $X_k \to X_1^k X_2$, where X_1^k denotes a kth power of X_1 (chosen arbitrarily). Hence R has a (strong) identity if and only if R has a (strong) 2-identity. Set $\mathscr{U}^{(m)}(R) = \Omega^{(m)}\{X\}/\mathscr{S}^{(m)}(R)$. There may fail to be an injection from $\mathscr{U}(R)$ to $\mathscr{U}^{(m)}(R)$. Indeed, if (R) is the universal alternative algebra, then $[\bar{X}_1, \bar{X}_2, \bar{X}_3] \neq 0$ but lies in the kernel of every homomorphism $\mathscr{U}(R) \to \mathscr{U}^{(2)}(R)$ (since all alternative algebras generated by 2 elements are associative). Still, the interplay between universal algebras is often useful.

2. Central localization of Ω-rings, and its effect on the lattice of prime ideals

2A. General properties of central localization. In this section we develop the concept of *central localization*, a special case of tensor extension, defined as follows: Let R be a Ω-ring with center Z, and let S be a multiplicatively closed subset of Z, containing 1, and let $H = Z_S$, the classical localization of the commutative, associative ring Z at S; we define $R_S = R \otimes_Z H$. R_S can be characterized very neatly as the classical localization (as Z-module) of R with respect to S, made into a Ω-ring via the operations

$$(r_1 s_1^{-1})(r_2 s_2^{-1}) = (r_1 r_2)(s_1 s_2)^{-1},$$
$$\omega(r_1 s_1^{-1}) = (\omega r_1)s_1^{-1},$$

and

$$(r_1 s_1^{-1})\omega = (r_1 \omega)s_1^{-1}$$

for all r_1, r_2 in R, all s_1, s_2 in S, and all ω in Ω. Note that any set of m elements in R_S can be put in the form $r_1 s^{-1}, \ldots, r_m s^{-1}$ for suitable r_i in R, s in S. The *canonical homomorphism* ψ_S: $R \to R_S$ is given by $\psi_S(r) = r1^{-1}$.

If B is a subset of R, we write B_S for $\{bs^{-1} \mid b \in B, s \in S\}$. If $T = \{$regular elements of $Z\}$ then we call R_T the *Ω-ring of central quotients of* R, and note that ψ_T is an injection, and $Z(R_T) = Z_T$. (This concept is most interesting in the case $T = Z - \{0\}$, because then $Z(R_T)$ is the field of fractions of Z.) Central localization works very well in PI-theory, because we have:

THEOREM 2.0. *For every multiplicative subset S of Z, and for every polynomial f, $R_S^+(f) \subseteq (R^+(f)S)_S$.*

Proof. Use a Vandermonde determinant argument from [23, Theorem 3.1], as follows. Let $G = R^+(f)$, and let $G' = (GS)_S$. It suffices to show

$$f(r_1 s^{-1}, \ldots, r_m s^{-1}) \in G'$$

for r_i in R and s in S. Let f_t be the sum of all monomials of f having total degree t, and let $y_t = f_t(r_1, \ldots, r_m)$. Letting d be the total degree of f, we have $\sum_{t=0}^{d} s^{jt} y_t = f(s^j r_1, \ldots, s^j r_m) \in G$. Using the usual Vandermonde argument on this system of $d + l$ equations, with y_t as the variables, $0 \le t \le d$, we get $h(s)y_t \in GS$, where the Vandermonde determinant $h(s)$ is a product of terms of the form $s^p - s^q$, $p < q$. Evidently $h(s)$ is of the form $s^k h'(s)$, where $h'(s)$ is a polynomial in s, with integral coefficients and constant term 1. Applying ψ_S yields $h(s)y_t 1^{-1} \in G'$; multiplying by $1s^{-k}$ gives $h'(s)y_t 1^{-1} \in G'$. Let $h'(s) = 1 - sh''(s)$. Then, for all t,

$$y_t 1^{-1} - sh''(s)y_t 1^{-1} \in G'.$$

Thus, by induction, $y_t 1^{-1} - s^k(h''(s))^k y_t 1^{-1} \in G'$, yielding

$$\begin{aligned} f(r_1 s^{-1}, \dots, r_m s^{-1}) &= \sum_{t=0}^{d} f_t(r_1 s^{-1}, \dots, r_m s^{-1}) \\ &= \sum_{t=0}^{d} y_t s^{-t} \in \sum_{t=0}^{d} (h''(s))^t y_t 1^{-1} + G' \\ &= \sum_{t=0}^{d} f_t(h''(s)r_1, \dots, h''(s)r_m) 1^{-1} + G' \\ &= f(h''(s)r_1, \dots, h''(s)r_m) 1^{-1} + G' \\ &\subseteq G1^{-1} + G' \subseteq G'. \end{aligned}$$

Q.E.D.

COROLLARY 2.1. *Every identity of R is an identity of R_S, for each multiplicative subset S of Z.*

Proof. When $G = 0$, clearly $(GS)_S$ also is 0. Q.E.D.

2B. Prime and strongly prime ideals. Call R *strongly semiprime* if $\text{Nil}\,(R) = 0$. As we shall see, strongly semiprime rings are quite amenable to examination in terms of central polynomials. Call an ideal B of R *strongly semiprime* if $\text{Nil}\,(R/B) = 0$. Clearly, in this case, $\text{Nil}\,(R) \subseteq B$.

Call an Ω-ring R *prime* if, for any two nonzero ideals, the product is nonzero; a prime Ω-ring R is *strongly prime* if $\text{Nil}\,(R) = 0$. An ideal P of R is *prime* (resp. *strongly prime*) if R/P is prime (resp. strongly prime). Recall that $\langle \quad \rangle$ denotes "the ideal (of R) generated by."

Remark 2.2. If R is prime then every nonzero element of Z is regular. (Proof: Suppose $rz = 0$ for elements r in R, $z \neq 0$ in Z. Then $\langle r \rangle \langle z \rangle = 0$, so $\langle r \rangle = 0$, implying $r = 0$. Hence z is regular.)

Strongly prime ideals arise very naturally, we see in the next result.

PROPOSITION 2.3. *If T is a multiplicatively closed subset of a Ω-ring R, then every ideal P (of R) maximal with respect to $P \cap T = \varnothing$ is strongly prime.*

Proof. Assume $0 \notin T$, since otherwise the assertion is vacuous. It is standard that P is prime. Suppose $\text{Nil}\,(R/P) = A/P \neq 0$. Then $A \supset P$, so there exists (nonzero) t in $A \cap T$. But then some power t_i of t lies in P. Thus $t_i \in T \cap P = \varnothing$, a contradiction. This proves $\text{Nil}\,(R/P) = 0$, so P is strongly prime. Q.E.D.

The ideals described in Proposition 2.3 exist (if $0 \notin T$), by Zorn's lemma.

PROPOSITION 2.4. $\text{Nil}\,(R) = \bigcap$ {*strongly prime ideals of R*}.

Proof. Let $\mathscr{P}$ = {strongly prime ideals of R}. For each P in $\mathscr{P}$, Nil $(R) \subseteq P$, so Nil $(R) \subseteq \bigcap \{P \in \mathscr{P}\}$. Conversely, if r is a nonnilpotent element of R, then T = {powers of r} is multiplicatively closed; by Proposition 2.3, there exists some $P \in \mathscr{P}$ with $r \notin P$. Hence $\bigcap \{P \in \mathscr{P}\}$ is nil, implying Nil $(R) = \bigcap \{P \in \mathscr{P}\}$. Q.E.D.

COROLLARY 2.5. *The intersection of the strongly prime ideals of a strongly semiprime ring is* 0.

We continue along the lines of [23, §3]. For an associative PI-ring, all prime ideals are strongly prime. If P is a prime ideal of $Z = Z(R)$, we note that $(Z - P)$ is a multiplicative subset of Z, and we write R_P for R_{Z-P} (and ψ_P for the canonical homomorphism $\psi_{Z-P}\colon R \to R_P$). It is standard to show that ψ_P^{-1} is a lattice isomorphism of {strongly prime ideals of R_P} onto {strongly prime ideals $\tilde{P}$ of $R \mid \tilde{P} \cap Z \subseteq P$}. This fact makes us very interested in lifting strongly prime ideals from Z to strongly prime ideals of R. This is achieved in the next result (when we put $R_1 = Z$).

PROPOSITION 2.6. *Suppose R_1 is an subring of an Ω-ring R, B_1 is a strongly semi-prime ideal of R_1, and $\tilde{B}$ is an ideal of R, maximal with respect to $\tilde{B} \cap R_1 \subseteq B_1$. Then $\tilde{B}$ is strongly semiprime.*

Proof. If Nil $(R/\tilde{B}) = A/\tilde{B} \neq 0$, then $\tilde{B} \subset A$, so $A \cap R_1 \not\subseteq B_1$. Hence $((A \cap R_1) + B_1)/B_1$ is a nonzero nil ideal of R_1/B_1, contrary to Nil $(R_1/B_1) = 0$.

COROLLARY 2.7. *In the notation of Proposition 2.6, if B_1 is a strongly prime ideal of R_1 then $\tilde{B}$ is a strongly prime ideal of R. Conversely, if $\tilde{P}$ is a prime ideal of R, then $\tilde{P} \cap Z$ is a prime ideal of Z.*

Proof. Straightforward.

We record two more results, with proofs paralleling those of [23, Lemma 3.5 and Lemma 3.6].

LEMMA 2.8. *If R is strongly prime then, for any prime ideal P of Z, R_P is strongly prime.*

LEMMA 2.9. *Let P be a prime ideal of Z and let $\tilde{P}$ be an ideal of R, maximal with respect to $\tilde{P} \cap Z \subseteq P$. We have $P \subseteq \tilde{P}$ iff $P_P \subseteq \tilde{P}_P$.*

Now, we know that any prime, associative, commutative ring has a field of fractions. Accordingly, call a prime Ω-ring *absolutely prime* if its ring of central quotients is simple; an ideal $\tilde{P}$ of R is *absolutely prime* if $R/\tilde{P}$ is absolutely prime.

Remark 2.10. A prime ring R is absolutely prime iff every nonzero ideal of R intersects Z nontrivially. (Proof: pass to the ring of central quotients, using Remark 2.2.)

THEOREM 2.11. *Let P be a prime ideal of Z and let $\tilde{P}$ be an ideal of R, maximal with respect to $\tilde{P} \cap Z \subseteq P$.*

(i) *$\tilde{P}_P$ is maximal in R_P.*

(ii) *Let $\bar{R} = R/\tilde{P}$, $S = Z - P$, $\bar{S} = (S + \tilde{P})/\tilde{P}$. Then there is a canonical embedding $\bar{R} \to R_P/\tilde{P}_P$ which induces an isomorphism $R_P/\tilde{P}_P \approx \bar{R}_{\bar{S}}$, which is the ring of central quotients of $\bar{R}$. In particular, $\tilde{P}$ is absolutely prime; $\tilde{P}$ is maximal if and only if $Z(R/\tilde{P})$ is a field, in which case $R/\tilde{P} \approx R_P/\tilde{P}_P$.*

Proof. As in [23, Theorem 3.7].

There is a simple fact about intersections of ideals in prime rings which we record now for later use.

Remark 2.12. Suppose R is prime and $\{A_\gamma \mid \gamma \in \Gamma\}$ is a set of ideals of R, with intersection 0. If $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\bigcap \{A_\gamma \mid \gamma \in \Gamma_2\} \neq 0$ then $\bigcap \{A_\gamma \mid \gamma \in \Gamma_1\} = 0$. (Proof:

$$(\bigcap \{A_\gamma \mid \gamma \in \Gamma_1\})(\bigcap \{A_\gamma \mid \gamma \in \Gamma_2\}) \subseteq \bigcap \{A_\gamma \mid \gamma \in \Gamma\} = 0.$$

Since R is prime, we conclude $\bigcap \{A_\gamma \mid \gamma \in \Gamma_1\} = 0$.)

3. Application of central polynomials and central localization to structure theory

3A. Centrally admissible rings. The object of Section 3 is to obtain a structure theory of *PI* (Ω-)rings, based on absolutely prime ideals. Our starting point is an attempt to learn more about "absolutely prime," i.e., when are strongly prime rings absolutely prime? To answer this question, one needs to study various radicals of a Ω-ring.

The flavor is set by the following result. Recalling that, by convention, R is a Ω-ring with center Z, say a subset B of R *hits* Z if $B \cap Z \neq 0$.

LEMMA 3.1. *Suppose $\{M_\gamma \mid \gamma \in \Gamma\}$ is a set of maximal ideals of R with zero intersection such that, for any $\Gamma' \subseteq \Gamma$, $R/\bigcap \{M_\gamma \mid \gamma \in \Gamma'\}$ satisfies a central polynomial. Then every nonzero ideal of R hits Z.*

Proof. Consider an ideal $A \neq 0$ of R. We wish to prove $A \cap Z \neq 0$. Let

$$\Gamma' = \{\gamma \in \Gamma \mid A \nsubseteq M_\gamma\}, \quad B = \bigcap \{M_\gamma \mid \gamma \in \Gamma'\},$$

$\bar{R} = R/B$, $R_\gamma = R/M_\gamma$, and let $\pi_\gamma = R \to R_\gamma$ be the canonical projection, for each γ in Γ. Since R_γ is simple, $\pi_\gamma(A) = R_\gamma$ for each γ in Γ'.

By hypothesis, $\bar{R}$ satisfies some central polynomial $g(X_1, \ldots, X_m)$. Then, for some γ in Γ', $R_\gamma(g) \neq 0$, so we can pick $a_1, \ldots, a_m$ in A such that $g(\pi_\gamma(a_1), \ldots, \pi_\gamma(a_m)) \neq 0$. Let $a = g(a_1, \ldots, a_m) \neq 0$ (since $a_\gamma \neq 0$). Clearly $\bar{a} \in Z(\bar{R})$. But $\pi_\gamma(a) = 0$ for all γ in $\Gamma - \Gamma'$, so $a \in Z \cap A$. Q.E.D.

DEFINITION 3.2. R is *centrally admissible* if the following two conditions hold.

(1) $\bigcap$ {maximal ideals of $R[\lambda]$} $= 0$.

(2) Given a set $\{M_\gamma \mid \gamma \in \Gamma\}$ of maximal ideals of $R[\lambda]$ with intersection 0, there is a subset $\Gamma' \subseteq \Gamma$ such that, for any $\Gamma'' \subseteq \Gamma'$, $R[\lambda]/\bigcap \{M_\gamma \mid \gamma \in \Gamma''\}$ satisfies a central polynomial, and $\bigcap \{M_\gamma \mid \gamma \in \Gamma'\} = 0$.

THEOREM 3.3. *If R is centrally admissible then every nonzero ideal of R hits Z.*

Proof. Immediate from Definition 3.2 and Lemma 3.1. Q.E.D.

The major objective of Section 3 is to build a structure theory based on "centrally admissible," using Theorem 3.3. One major goal is to determine which rings are centrally admissible.

Brown-McCoy [7] and Smiley [31] defined a radical of an arbitrary ring to be the intersection of its maximal ideals; in the context of Ω-rings, let $BM(R) = \bigcap$ {maximal ideals of R} for a Ω-ring R. If $BM(R) = 0$, we say R is *semisimple* (i.e., a subdirect product of simple Ω-rings). Looking at Z as a ring, we see that $BM(Z)$ is the Jacobson radical of Z. This concept arises in Definition 3.2, since condition (1) says "$BM(R[\lambda]) = 0$". In fact, for prime Ω-rings, we have:

THEOREM 3.4. *If R is prime with R-stable central polynomial and if $R[\lambda]$ is semisimple, then R is centrally admissible and absolutely prime.*

Proof. It suffices to check condition (2) of Definition 3.2; for then we are done by Theorem 3.3.

We assume $\{M_\gamma \mid \gamma \in \Gamma\}$ is a set of maximal ideals of $R[\lambda]$ with intersection 0. Let g be an R-stable central polynomial of R. Then g is $R[\lambda]$-central. Let

$$\Gamma' = \{\gamma \in \Gamma \mid R[\lambda](g) \nsubseteq M_\gamma\},$$

and let $\Gamma_2 = \Gamma - \Gamma'$. For each γ in Γ_2, $R[\lambda](g) \subseteq M_\gamma$, so $\bigcap \{M_\gamma \mid \gamma \in \Gamma_2\} \neq 0$. Hence, by Remark 2.12, $\bigcap \{M_\gamma \mid \gamma \in \Gamma'\} = 0$ (since $R[\lambda]$ is prime). But g is $R[\lambda]/M_\gamma$-central for every γ in Γ', so condition (2) holds. Q.E.D.

Thus, given a strongly prime Ω-ring R with R-stable central polynomial, the obstruction to proving R is absolutely prime is $BM(R[\lambda])$. Most of this section will be spent on overcoming this obstruction.

3B. Rings whose centers are local. Recall that a commutative ring is *local* if it has a unique maximal ideal, easily seen to be the set of noninvertible elements. Accordingly, we call a Ω-ring R *local* if $BM(R)$ is maximal. Our object will be to study local Ω-rings in this subsection, but we start with a general result.

PROPOSITION 3.5. *Suppose $g(X_1, \ldots, X_m)$ is a blended polynomial. For any subset N of R and for any ideal B of R, $(B + N)(g) \subseteq N(g) + B \cap R^+(g)$.*

Proof. For any $b_1, \ldots, b_m$ in B and $x_1, \ldots, x_m$ in N, we have

$$\begin{aligned} g(b_1 + x_1, \ldots, b_m + x_m) &= (g(x_1, b_2 + x_2, \ldots, b_m + x_m) \\ &\quad + g(b_1, b_2 + x_2, \ldots, b_m + x_m) \\ &\quad - \Delta_1 g(x_1, b_2 + x_2, \ldots, b_m + x_m, b_1)) \\ &\in g(x_1, b_2 + x_2, \ldots, b_m + x_m) + B \cap R^+(g) \end{aligned}$$

(since the last two terms are in $B \cap R^+(g)$). Doing this procedure for each indeterminate yields

$$g(b_1 + x_1, \ldots, b_m + x_m) \in g(x_1, \ldots, x_m) + B \cap R^+(g),$$

proving $(B + N)(g) \subseteq N(g) + B \cap R^+(g)$. Q.E.D.

The most common application will be merely $(B + N)(g) \subseteq N(g) + B \cap Z$ whenever g is R-central. We continue with another elementary computation which will set the flavor of this section. (Generalizing the notation of §1, given a subset W of R, and a polynomial $f(X_1, \ldots, X_m)$, we let

$$W(f) = \{f(w_1, \ldots, w_m) \mid \text{all } w_i \text{ in } W\},$$

and let $W^+(f)$ be the additive subgroup of R generated by $W(f)$.)

THEOREM 3.6. *Suppose Z is local with maximal ideal P, and let R have a blended central polynomial g with $R(g) \nsubseteq P$. Then R is local. Moreover, for any subset N of R, if $(BM(R) + N)(g) \nsubseteq P$, then $N(g) \nsubseteq P$.*

Proof. Pick elements $r_1, \ldots, r_m$ of P, with $g(r_1, \ldots, r_m) \notin P$, and choose an ideal M of R, maximal with respect to $r_1 \notin M$. (Such M exists, by Zorn's lemma.) We claim that M contains every proper ideal A of R.

Indeed, suppose we have an ideal $A \nsubseteq M$. By hypothesis there exist elements a in A and x in M, with $r_1 = a + x$. Thus

$$\begin{aligned} g(r_1, \ldots, r_m) &= g(a + x, r_2, \ldots, r_m) \\ &= g(a, r_2, \ldots, r_m) + g(x, r_2, \ldots, r_m) \\ &\quad + \Delta_1 g(a, r_2, \ldots, r_m, x) \in g(a, r_2, \ldots, r_m) + M \cap Z, \end{aligned}$$

implying $g(a, r_2, \ldots, r_m) \notin P$ (since $M \cap Z \subseteq P$). But

$$g(a, r_2, \ldots, r_m) \in A \cap Z,$$

implying $A \cap Z \nsubseteq P$; hence $1 \in A$. In other words, if A is proper then $A \subseteq M$, proving the claim.

Next, suppose N is a subset of R, with $(M + N)(g) \nsubseteq P$. Then by Proposition 3.5, $N(g) + M \cap Z \nsubseteq P$. But $M \cap Z \subseteq P$; hence $N(g) \nsubseteq P$. Q.E.D.

COROLLARY 3.7. *If Z is local and if R has a central polynomial g, with $R(g) \nsubseteq BM(Z)$, then R is local.*

Proof. By Proposition 1.1, we may assume g is a blended R-central polynomial, so we can apply Theorem 3.6. Q.E.D.

If W is an additive subset of R, let core $(W) = \sum$ (ideals of R contained in W), which is clearly the "largest" ideal of R contained in W. R is *primitive* if there is a maximal left ideal, with core 0.

Remark 3.8. Primitive Ω-rings are prime.

Proof. If B_1, B_2 are nonzero ideals, of a Ω-ring R having maximal left ideal L with core 0, then $B_2 \nsubseteq L$, so $R = B_2 + L$ (by maximality of L). Hence

$$B_1 = B_1 R = B_1 B_2 + B_1 L \subseteq B_1 B_2 + L.$$

Since $B_1 \nsubseteq L$, we conclude $B_1 B_2 \nsubseteq L$, so $B_1 B_2 \neq 0$. Q.E.D.

Define Jac $(R) = \bigcap$ {core $(M) \mid M$ maximal left ideal of R}. Note that Jac $(R) = 0$ if R is primitive. We shall make frequent use of the connection between $BM(R)$ and Jac (R), which is therefore the next object of attention.

Remark 3.9. If $z \in Z$ is regular, then $Z \cap zR = zZ$. In particular, if z^{-1} exists then $z^{-1} \in Z$.

Proof. Suppose $zr \in Z \cap zR$. Then for all r_1, r_2 in R, we have

$$0 = [zr, r_1, r_2] = z[r, r_1, r_2],$$

showing $[r, r_1, r_2] = 0$. Similar verifications yield $r \in Z$. Now if z^{-1} exists then $1 = zz^{-1} \in Z \cap zR$, so $1 \in zZ$, implying $z^{-1} \in Z$. Q.E.D.

In [7, Definition on p. 51 and Theorem 7], an ideal B is shown to be contained in $BM(R)$ iff $b \in \langle 1 - b \rangle$ for all elements b of B. (Actually [7] deals with associative rings, but the proofs do not need associativity.)

PROPOSITION 3.10. *For any R, $BM(R) \cap Z \subseteq$ Jac (Z). In particular, if R is centrally admissible and* Jac $(Z) = 0$, *then R is semisimple.*

Proof. We will demonstrate the first assertion, since the second assertion is a direct consequence. Suppose $z \in BM(R) \cap Z$. Then, for all elements z' of Z,

$$zz' \in \langle 1 - zz' \rangle = R\langle 1 - zz' \rangle.$$

Thus, for some r in R, $zz' = r(1 - zz')$, implying $(1 + r)(1 - zz') = 1$. By Remark 3.9, $(1 - zz')^{-1} \in Z$ for all z' in Z, so we conclude that zZ is a quasi-regular ideal of Z, proving that $BM(R) \cap Z \subseteq \text{Jac}(Z)$. Q.E.D.

THEOREM 3.11. *Suppose R is a Ω-ring with central polynomial.*

(i) *If R is primitive then R is absolutely prime.*
(ii) *If R is prime and* $\text{Jac}(Z) = \text{Jac}(R) = 0$, *then R is semisimple.*
(iii) *If R is primitive and semisimple then R is simple.*

Proof. First we make the following straightforward observations. Let S be a multiplicative subset of Z, and let L be a maximal left ideal of R, with $L \cap S = \varnothing$. Then L_S is a maximal left ideal of R_S, and core $(L_S) = (\text{core}(L))_S$. Also, by hypothesis, we have an R-central polynomial g which, by Proposition 1.1, we may assume is blended. Clearly g is R_S-central, by Corollary 2.1.

(i) Let $S = Z - \{0\}$, and let L be a maximal left ideal of R having 0 core. By Remark 3.8, R is prime, so, by Remark 2.2, all elements of S are regular. Moreover, $R(L \cap Z) \subseteq \text{core}(L) = 0$, implying $L \cap Z = 0$. Thus, as observed above, L_S is a maximal left ideal of R_S, with 0 core. But $Z_S = Z(R_S)$ is a field, which is certainly local with maximal ideal 0. Thus, we can apply Theorem 3.6 to obtain R_S is local with $BM(R_S) \subseteq \text{core}(L_S) = 0$. Hence R_S is simple, proving R is absolutely prime.

(ii) Let $\mathscr{L} = \{\text{maximal left ideals } L \text{ of } R \mid R(g) \nsubseteq L\}$. By Remark 3.8, for any L in $\mathscr{L}$, core (L) is a prime ideal of R, so $P = L \cap Z$ is a prime ideal of Z, and L_P is a maximal left ideal of R_P. Since $Z(R_P) = Z_P$ is local and $R_P(g) \nsubseteq P_P$, Theorem 3.6 yields R_P local with $BM(R_P) \subseteq \text{core}(L_P) = (\text{core}(L))_P$. Thus core (L) is *the* largest ideal (of R) whose intersection with Z is contained in P. By Proposition 3.10, $BM(R) \cap Z = 0$, so, in particular, $BM(R) \subseteq \text{core}(L)$. Since this holds for each L in $\mathscr{L}$, we conclude (via Remark 2.12) that $BM(R) \subseteq \bigcap \{\text{core}(L) \in \mathscr{L}\} = 0$.

(iii) By assumption, there is a maximal ideal $\tilde{P}$ of R with $R(g) \nsubseteq \tilde{P}$. Let $P = \tilde{P} \cap Z$, and let L be a maximal left ideal of R with core 0. As in (i), $L \cap Z = 0$, so L_P is a maximal left ideal of R_P with core 0. Also Z_P is local and $R_P(g) \nsubseteq \tilde{P}_P$, so, by Theorem 3.6, R_P is local, with $BM(R_P) \subseteq \text{core}(L_P) = 0$. Hence R_P is simple, implying $\tilde{P}_P = 0$. Thus $\tilde{P} = 0$; i.e., 0 is a maximal ideal of R. Therefore R is simple. Q.E.D.

In Theorem 3.11(ii) we could remove the hypothesis that R be prime, by refining Theorem 3.6, but the present form is sufficient for the following result.

COROLLARY 3.12. *If R is prime with an R-stable central polynomial, and if* Jac $(R[\lambda]) = 0$, *then* $BM(R[\lambda]) = 0$.

Proof. Jac $(Z(R[\lambda])) =$ Jac $(Z[\lambda]) = 0$, by a famous theorem of Amitsur (cf. [12, p. 12]). Thus by Theorem 3.11(ii), $BM(R[\lambda]) = 0$.

Thus, we would like to show that if R is strongly prime then Jac $R[\lambda] = 0$. This is known in the associative case, from a theorem of Amitsur [12, p. 12]. Unfortunately, we have been unable to prove the theorem in general, due to the fact that generation of left ideals in the nonassociative case is very complicated.

3C. Rings with regular central polynomial. Call a polynomial f *regular* if f is linear in X_1 (in each monomial). Let the *central kernel* I of R be $\sum \{R^+(g) \mid$ regular R-central polynomial $g\}$, easily seen to be an ideal of Z. An ideal of Z (similarly of R) not containing I is called *identity-faithful.* Then [23, Theorem 4.16] becomes the following powerful theorem.

THEOREM 3.13. *Suppose R is a Ω-ring. Let P be an identity-faithful prime ideal of Z, and let $\tilde{P}$ be an ideal of R, maximal with respect to $\tilde{P} \cap Z \subseteq P$. Then $\tilde{P}$ is absolutely prime, and:*

(a) $Z(R_P) = Z_P = I_P$.

(b) $P \subseteq \tilde{P}$, i.e., $P = \tilde{P} \cap Z$.

(c) *$\tilde{P}$ is the union of all ideals of R whose intersection with Z is P. In particular, $\tilde{P}$ is the only absolutely prime ideal of R such that $\tilde{P} \cap Z = P$. This yields a* 1:1 *order-preserving correspondence $P \mapsto \tilde{P}$ between {identity-faithful prime ideals of Z} and {identity-faithful absolutely prime ideals of R}.*

(d) *If P is maximal then $\tilde{P}$ is maximal; hence the correspondence given in* (c) *yields a canonical* 1:1 *order-preserving correspondence $P \mapsto \tilde{P}$ between {identity-faithful maximal ideals of Z} and {identity-faithful maximal ideals of R}.*

(e) *For any identity-faithful strongly prime ideal $\tilde{P}$ of R, and for $P = \tilde{P} \cap Z$, $R_P/\tilde{P}_P$ is isomorphic to the algebra of central quotients of $R/\tilde{P}$ (and is clearly simple).*

Proof. By Theorem 2.11, $\tilde{P}$ is absolutely prime, and the rest of the theorem follows, as in the proof of [23, Theorem 4.16]. Q.E.D.

Let us make a brief digression, to use Theorem 3.13 to compare Jac (R) and Jac (Z).

COROLLARY 3.14. *If $BM(R) = 0$ then $I \cap$* Jac $(Z) = 0$, *where I is the central kernel of R.*

Proof. Let $J =$ Jac (Z). For every maximal ideal $\tilde{P}$ of R, either $I \subseteq \tilde{P}$ or $\tilde{P} \cap Z$ is identity faithful (and is thus a maximal ideal of Z). Thus $I \cap J \subseteq \tilde{P}$ for every maximal ideal $\tilde{P}$ of R, proving $I \cap J \subseteq BM(R) = 0$. Q.E.D.

COROLLARY 3.15. *If R is prime with regular R-central polynomial, and if $BM(R) = 0$, then* $\text{Jac}(Z) = 0$.

Proof. Immediate, since $I \neq 0$ in this case, and Z is a domain. Q.E.D.

3D. Central classes of rings. At many points we have related prime ideals of Z to absolutely prime ideals of R, most notably in Theorem 3.13. Also, we have used central polynomials at various points, and, in view of Theorem 3.4 we would like to work in classes of rings for which $\text{Nil}(R) = 0$ implies $\text{Jac}\, R[\lambda] = 0$. Since we have been unable to prove this result in general, let us postulate it in a certain class of Ω-rings.

DEFINITION 3.16. A class of Ω-rings, $\mathscr{C}$, is called *central* if:

(i) Every strongly prime Ω-ring R in $\mathscr{C}$ has an R-stable central polynomial.
(ii) If $R \in \mathscr{C}$ then $R/\text{Nil}(R) \in \mathscr{C}$ and $R/\text{Nil}(R)$ is a subdirect product of strongly prime images in $\mathscr{C}$.
(iii) If $R \in \mathscr{C}$ then $R[\lambda] \in \mathscr{C}$.

THEOREM 3.17. *In a central class $\mathscr{C}$, the following conditions are equivalent:*

(1) *For all R in $\mathscr{C}$,* $\text{Nil}(R) = 0$ *implies* $\text{Jac}(R[\lambda]) = 0$.
(2) *For all R in $\mathscr{C}$,* $\text{Nil}(R) = 0$ *implies* $BM(R[\lambda]) = 0$.
(3) *Every strongly prime member of $\mathscr{C}$ is absolutely prime.*

Proof. (1) $\Rightarrow$ (2). Suppose $\text{Nil}(R) = 0$. We can write R as a subdirect product of strongly prime images R_γ in $\mathscr{C}$; by (1) and Corollary 3.12,

$$0 = \text{Jac}(R_\gamma[\lambda]) = BM(R_\gamma[\lambda]) \quad \text{for each } \gamma,$$

implying $BM(R[\lambda]) = 0$.

(2) $\Rightarrow$ (3). This is Theorem 3.4 (since any $R[\lambda]$-central polynomial is R-stable).

(3) $\Rightarrow$ (1). Let $J = \text{Jac}(R[\lambda])$. Suppose $\text{Nil}(R) = 0$. Given a strongly prime ideal P of R such that $R/P \in \mathscr{C}$, let $\bar{R} = R/P$ and $J' = \text{Jac}(\bar{R}[\lambda])$. We have

$$J' \cap Z(\bar{R}[\lambda]) \subseteq \text{Jac}(Z(\bar{R})[\lambda]) = 0$$

(since $Z(\bar{R})$ is associative). But $\bar{R}[\lambda]$ is strongly prime, and thus absolutely prime; hence $J' = 0$ (since every nonzero ideal hits the center). This implies $J \subseteq P[\lambda]$ (since $\bar{R}[\lambda] \approx R[\lambda]/P[\lambda]$), so

$$J \subseteq \bigcap \{P[\lambda] \mid R/P \text{ is strongly prime and is in } \mathscr{C}\} = \left(\bigcap P\right)[\lambda] = 0.$$

Q.E.D.

DEFINITION 3.18. A class of Ω-rings is *Kaplansky* if it satisfies (i), (ii), and (iii) of Definition 3.16, as well as:

(iv) If $R \in \mathscr{C}$ and $\text{Nil}(R) = 0$ then $\text{Jac}(R[\lambda]) = 0$.

In part II, we will get a partial result along the lines of (iv).

Let us make some observations concerning the verification of (i)–(iv) of Definition 3.18. Of course, given (i)–(iii), any of the conditions of Theorem 3.17 are equivalent to (iv).

Remark 3.19. Condition (ii) holds in every variety. Moreover, when $\mathscr{C}$ is a variety, one gets the property that if $R \in \mathscr{C}$ then every central localization of R is in $\mathscr{C}$.

Remark 3.20. If $\mathscr{C}$ is a variety defined by completely homogeneous identities and central polynomials of degree ≤ 2 in each indeterminate, then (iii) holds, in view of Remark 1.2(i). In fact, in this case, we have the stronger property that $\mathscr{C}$ is closed under tensor extensions.

Remark 3.21. If Ω is an infinite field then, by Remark 3.8, (iii) holds, and $\mathscr{C}$ is closed under tensor extensions.

Our use of the name "Kaplansky" for the class of Definition 3.18 was motivated by Kaplansky's fundamental structure theorem in associative PI-rings, that every primitive associative PI-ring is *central simple*, i.e., is simple and is a finite-dimensional algebra over its center. The fact that simple associative PI-rings have central polynomials enables one to apply Kaplansky's theorem very successfully, in a manner similar to the way we shall treat Kaplansky classes.

The obvious way that Kaplansky classes are nice is that strongly prime rings are absolutely prime. This fact makes Theorem 3.13 quite useful in Kaplansky classes, and also gives an easy way of obtaining a good deal of information. For example, say a *central annihilator* of R is a set of the form $\mathrm{Ann}_R\, B$, where $B \subseteq C$. Obviously central annihilators are ideals.

PROPOSITION 3.22. *If R is in a Kaplansky class, if* $\mathrm{Nil}\,(R) = 0$, *and if R satisfies the ascending chain condition on central annihilators, then the Ω-ring of central quotients of R is a direct sum of simple PI-rings.*

Proof. Since $\mathrm{Ann}_Z\, B = Z \cap \mathrm{Ann}_R\, B$, we see that Z satisfies the ascending chain condition on annihilators, so we can proceed as in [23, Theorem 5.4]. Q.E.D.

In Kaplansky classes, it is often very easy to obtain canonical identities, as seen by the following result.

THEOREM 3.23. *Suppose $\mathscr{C}$ is a Kaplansky class in which every direct power of an Ω-ring of $\mathscr{C}$ is in $\mathscr{C}$, and assume every semisimple member satisfies an identity f. Then, for every Ω-ring R in $\mathscr{C}$, a suitable power of f is an identity of R.*

Proof. By Theorem 1.9, it suffices to show f is an identity of $R/\mathrm{Nil}\,(R)$ whenever $R \in \mathscr{C}$. But $R/\mathrm{Nil}\,(R)$ is a subdirect product of strongly prime Ω-rings in $\mathscr{C}$, so we need only show that f is an identity of every strongly prime Ω-ring R_1 in $\mathscr{C}$. Now $BM(R_1[\lambda]) = 0$ by Definition 3.18 ((i), (iv)) and Corollary 3.12. Thus, by hypothesis, f is an identity of $R_1[\lambda]$ and is thus an identity of R_1. Q.E.D.

In a variety, if f is an identity of every simple member then f is an identity of every semisimple member. Thus, in varieties $\mathscr{C}$ which are Kaplansky classes, one needs only check that f is an identity of every simple Ω-ring in $\mathscr{C}$, to conclude that some power of f is an identity of every Ω-ring in $\mathscr{C}$.

4. A nonassociative generalization of the Artin-Procesi theorem, and its consequences

In this section we give an alternate (to Section 3) method of examining the relationship between the ideal structures of R and Z. The basic motivating idea is the (associative) Artin [5]-Procesi [20] theorem which characterizes Azumaya algebras R of rank t, by certain polynomial conditions:

(i) R satisfies all identities of some simple Ω-ring R_0 which is t-dimensional over its center;
(ii) For every simple Ω-ring R_1 of dimension $< t$ over its center, no nonzero homomorphic image of R satisfies all identities of R_1.

With the advent of central polynomials, it was recognized that (ii) could be replaced by:

(ii)′ There is a 1-normal R_0-central polynomial g, such that $1 \in R^+(g)$.

The proof of the Artin-Procesi theorem was made considerably simpler using (ii)′ instead of (ii), and so it is natural to ask what can be proven about nonassociative rings satisfying (i) and (ii)′.

Indeed, we shall obtain three properties which, in the associative case, imply R is Azumaya: (1) For every maximal ideal P of Z, there exists c in $Z - P$, such that R_c is a free Z_c-module, of degree t; (2) $\tilde{A} \to \tilde{A} \cap Z$ gives a lattice injection from {ideals of R} to {ideals of Z}; (3) $A \to AR$ gives a lattice injection from {ideals of Z} to {ideals of R}, and is the inverse correspondence to the correspondence given in (2). The method of proof will use t-normal central polynomials in a condition replacing (ii)′, and use this polynomial to compute elements yielding conditions (1), (2), (3). This method was discovered for the associative case independently by Amitsur [4] and Rowen [27].

Recall from Section 1A that "t-normal" means alternating and linear in the first t indeterminates. In particular, "1-normal" means linear in X_1. We shall use the term "regular" for "1-normal." If V_1, V_2 are subsets of R, we write $V_1 V_2$ for $\{\sum x_{i1} x_{i2} \mid x_{i1} \in V_1, x_{i2} \in V_2\}$.

PROPOSITION 4.1. *Let g be a regular R-central polynomial, let A, B be additive subgroups of Z, and write G for $R(g)$.*

(i) *$R^+(g)$ is an ideal of Z.*
(ii) *$((AR + BR) \cap Z)G = AG + BG$.*
(iii) *If $\tilde{A}$, $\tilde{B}$ are ideals of R with $\tilde{A} \cap G \subseteq A$ and $\tilde{B} \cap G \subseteq B$, then*

$$((\tilde{A} + \tilde{B}) \cap Z)G \subseteq A + B.$$

Proof. (i) If $r_1, \ldots, r_m \in R$ and $c \in Z$ then

$$g(r_1, \ldots, r_m)c = g(cr_1, \ldots, r_m) \in G,$$

and the assertion follows immediately.

(ii) Suppose $c = (\sum_i a_i r_i + \sum_j b_j r_j) \in Z$, for suitable a_i in A, b_i in B; let $c' \in G$. We can write $c' = g(x_1, \ldots, x_m)$ for suitable $x_1, \ldots, x_m$ in R, and then

$$\begin{aligned} cc' &= cg(x_1, \ldots, x_m) = g(cx_1, \ldots, x_m) \\ &= g((\textstyle\sum a_i r_i + \sum b_j r_j)x_1, x_2, \ldots, x_m) \\ &= \sum_i a_i g(r_i x_1, \ldots, x_m) + \sum_j b_j g(r_j x_1, \ldots, x_m) \\ &\in AG + BG. \end{aligned}$$

Hence $((AR + BR) \cap Z)G \subseteq AG + BG$, and the reverse inclusion is immediate.

(iii) Let $c = a + b \in Z$, for a in $\tilde{A}$, b in $\tilde{B}$. For any $c' = g(x_1, \ldots, x_m)$, we have

$$\begin{aligned} cc' &= g(cx_1, \ldots, x_m) = g(ax_1, \ldots, x_m) + g(bx_1, \ldots, x_m) \\ &\in \tilde{A} \cap G + \tilde{B} \cap G \subseteq A + B. \end{aligned}$$

Q.E.D.

This computation yields some general information. Let I be the additive group generated by all $R^+(g)$, for all regular R-central polynomials g.

THEOREM 4.2. *Suppose $1 \in I$. For every ideal A of Z, there is a unique ideal $\tilde{A}$ of R which is maximal with respect to $\tilde{A} \cap Z \subseteq A$. The correspondence $A \to \tilde{A}$ is a lattice injection of {ideals of Z} into {ideals of R}, and of {maximal ideals of Z} into {maximal ideals of R}. This map has a right inverse, from {ideals of R} to {ideals of Z}, given by $\tilde{B} \to \tilde{B} \cap Z$.*

Proof. Let $\mathscr{A}$ = {ideals of R, whose intersection with Z is contained in A}. By Zorn's lemma, $\mathscr{A}$ has a maximal member $\tilde{A}$. If $\tilde{B} \cap Z \subseteq A$, then, by Proposition 4.1(iii), $(\tilde{A} + \tilde{B}) \cap Z \subseteq A$; hence, $\tilde{A} + \tilde{B} = \tilde{A}$, so $\tilde{B} \subseteq \tilde{A}$. Hence $\tilde{A}$ is the unique maximal member of $\mathscr{A}$.

If $A_1 \subseteq A$ and $\tilde{A}_1$ is maximal with respect to $\tilde{A}_1 \cap Z \subseteq A_1$, then $\tilde{A}_1 \cap Z \subseteq A$, so $\tilde{A}_1 \subseteq \tilde{A}$. Thus, $A \to \tilde{A}$ is a lattice injection from {ideals of Z} to {ideals of R}. By Proposition 4.1(ii), $AR \cap Z \subseteq A$, so $AR \subseteq \tilde{A}$, implying $A = \tilde{A} \cap Z$. The other assertions follow immediately. Q.E.D.

To improve on Theorem 4.2, we shall need to build t-normal central polynomials for simple algebras of dimension t over their center. This is accomplished in three steps, based on a result of Erickson, Martindale, and Osborn [8]. Call a Ω-ring R_0 *central simple* (of dimension t) if R_0 is simple and is a vector space over $Z(R_0)$ of finite dimension (t).

Given a set of elements $V = \{r_1, \ldots, r_m\}$, say a set of regular polynomials $\{f_i(X_1, \ldots, X_u) \mid 1 \leq i \leq m\}$ *separates* V if there exist elements $x_{i1}, \ldots, x_{i,u-1}$ such that $f_i(x_{i1}, \ldots, x_{i,u-1}, r_j) = \delta_{ij}$, for all i, j. (Here $\delta_{ij} = 1$ if $i = i$, 0 otherwise.) In this case, we say V is *polynomial separated.*

LEMMA 4.3. *If R is simple then every Z-independent set is polynomial separated.*

Proof. Suppose $\{r_1, \ldots, r_n\}$ is Z-independent. By [8, Theorem 3.1], adding dummy indeterminates, we get $f_i(X_1, \ldots, X_u)$ and $x_{i1}, \ldots, x_{i,u-1}$, such that

$$f_i(x_{i1}, \ldots, x_{i,u-1}, r_j) = 0$$

if $j \neq i$, and

$$f_i(x_{i1}, \ldots, x_{i,u-1}, r_i) \neq 0.$$

Since R is simple,

$$1 \in \langle f_i(x_{i1}, \ldots, x_{i,u-1}, r_i)\rangle,$$

whereas each $\langle f_i(x_{i1}, \ldots, x_{i,u-1}, r_j)\rangle = 0$ for $j \neq i$. The assertion follows easily. Q.E.D.

PROPOSITION 4.4. *Suppose R is simple and $\{r_1, \ldots, r_t\}$ is Z-independent. Then there is a t-normal polynomial $F(X_1, \ldots, X_v)$, with $1 \in R(F)$.*

Proof. By Lemma 4.3, there are polynomials $\{f_i(X_1, \ldots, X_u) \mid 1 \leq i \leq t\}$ which separate $r_1, \ldots, r_t$. Let

$$\begin{aligned} F = \sum (\operatorname{sg} \pi) & f_1(X_{t+1}, X_{2t+1}, \ldots, X_{(u-1)t+1}, X_{\pi 1}) \\ & \times f_2(X_{t+2}, \ldots, X_{(u-1)t+2}, X_{\pi 2}) \cdots f_t(X_{2t}, \ldots, X_{ut}, X_{\pi t}), \end{aligned}$$

summed over all permutations π of $(1, \ldots, t)$. Clearly

$$F(r_1, \ldots, r_t, x_{11}, \ldots, x_{t,1}, \ldots, x_{1,t-1}) = \prod_{i=1}^{t} f_i(x_{i1}, \ldots, x_{i,u-1}, r_i) = 1.$$

Moreover, F is clearly t-normal. Q.E.D.

Note that the polynomial F depends on our simple Ω-ring.

THEOREM 4.5. *Suppose R is a central simple Ω-ring of dimension t. For any regular R-central polynomial g, there is a t-normal polynomial $g' \leq g$ with $R(g) = R(g')$.*

Proof. With $F(X_1, \ldots, X_v)$ as in Proposition 4.4, let

$$g' = g(F(X_1, \ldots, X_v)X_{v+1}, X_{v+2}, \ldots). \qquad \text{Q.E.D.}$$

Theorem 4.5 converts regular central polynomials to t-normal central polynomials. Note that this procedure generalizes §1B, Example (xi). Our next goal is to get a partial converse to §1B, Example (x), to show that "t-normality" is intimately connected with base elements of an algebra.

PROPOSITION 4.6. *Suppose $f(X_1, \ldots, X_u)$ is t-normal. Let K be a field, and suppose R is a K-algebra of dimension t, with $R(f) \neq 0$. A set*

$$\{r_1, \ldots, r_t\} \subseteq R$$

is a K-base if and only if $f(r_1, \ldots, r_t, x_{t+1}, \ldots, x_u) \neq 0$ for suitable $x_{t+1}, \ldots, x_u$ in R.

Proof. ($\Rightarrow$) Suppose $\{r_1, \ldots, r_t\}$ is a K-base and

$$f(r_1, \ldots, r_t, x_{t+1}, \ldots, x_u) = 0 \quad \text{for all } x_{t+1}, \ldots, x_u \text{ in } R.$$

It would follow easily that f is an identity of R, contrary to $R(f) \neq 0$.

($\Leftarrow$) If $f(r_1, \ldots, r_t, x_{t+1}, \ldots, x_u) \neq 0$ then $\{r_1, \ldots, r_t\}$ is K-independent and therefore must be a K-base. Q.E.D.

For the next result, given a t-normal polynomial f, define

$$f^{(i)}(X_1, \ldots, X_{u+1}) = f(X_1, \ldots, X_{i-1}, X_{u+1}, X_{i+1}, \ldots, X_u)X_i,$$

and define the *associated polynomial* of f to be

$$\hat{f}(X_1, \ldots, X_{u+1}) = f(X_1, \ldots, X_u)X_{u+1} - \sum_{i=1}^{t} f^{(i)}(X_1, \ldots, X_{u+1}).$$

THEOREM 4.7. *Suppose $f(X_1, \ldots, X_u)$ is t-normal. Let K be a field, and suppose R is a K-algebra of dimension t.*

(i) *If $r = \sum_{i=1}^t \alpha_i r_i$, α_i in K, r_i in R, then, for any $x_{t+1}, \ldots, x_u$ in R, for all $i \leq t$,*

$$f(r_1, \ldots, r_{i-1}, r, r_{i+1}, \ldots, r_t, x_{t+1}, \ldots, x_u) = \alpha_i f(r_1, \ldots, r_t, x_{t+1}, \ldots, x_u).$$

(ii) *The associated polynomial $\hat{f}$ of f is an identity of R.*

Proof. (i) is a straightforward computation. To prove (ii), we want to show $\hat{f}(r_1, \ldots, r_{u+1}) = 0$ for all r_i in R. By (i), this is true whenever $\{r_1, \ldots, r_t\}$ is a K-base of R. Thus, we may assume $\{r_1, \ldots, r_t\}$ is *not* a K-base of R. Hence

$f(r_1, \ldots, r_u)r_{u+1} = 0$, by Proposition 4.6, so we are done *unless* $f^{(i)}(r_1, \ldots, r_{u+1}) \neq 0$ for some i, which implies

$$\{r_1, \ldots, r_{i-1}, r_{u+1}, r_{i+1}, \ldots, r_t\}$$

is a K-base (by Proposition 4.6). It follows that

$$\{r_1, \ldots, r_{i-1}, r_{u+1} + r_i, r_{i+1}, \ldots, r_t\}$$

is a K-base. But then

$$\hat{f}(r_1, \ldots, r_{i-1}, r_{u+1}, r_{i+1}, \ldots, r_{u+1}) = 0$$

and

$$\hat{f}(r_1, \ldots, r_{i-1}, r_{u+1} + r_i, r_{i+1}, \ldots, r_{u+1}) = 0,$$

implying $f(r_1, \ldots, r_{u+1}) = 0$ (since $\hat{f}$ is linear in X_i). Q.E.D.

Theorem 4.7 leads to the next major structural result of this section. Given two Ω-rings R_1 and R_2, say $R_1 \leq R_2$ if R_1 satisfies every identity of R_2.

THEOREM 4.8. *Suppose that the t-normal polynomial $f(X_1, \ldots, X_u)$ is R-central, and that the associated polynomial $\hat{f}$ is an identity of R. Then, for any c in $R(f)$, R_c is a free Z_c-module, of dimension t.*

Proof. Choose $r_1, \ldots, r_t, x_{t+1}, \ldots, x_u$ in R, such that $f(r_1, \ldots, r_t, x_{t+1}, \ldots, x_u) = c$. We claim that $r_1 1^{-1}, \ldots, r_t 1^{-1}$ is a Z_c-base of R_c. Indeed given r in R, let

$$\alpha_i = f(r_1, \ldots, r_{i-1}, r, r_{i+1}, \ldots, r_t, x_{t+1}, \ldots, x_u).$$

Since $\hat{f}$ is an identity of R, we have $cr = \sum_{i=1}^t \alpha_i r_i$, so

$$rc^{-k} = \sum_{i=1}^{t} \alpha_i c^{-(k+1)} r_i 1^{-1},$$

proving that the $r_i 1^{-1}$ span R_c. On the other hand, if $\sum_{i=1}^t (\alpha_i c^{-k}) r_i 1^{-1} = 0$, for suitable α_i in Z, then, for each i,

$$0 = \alpha_i 1^{-1} f(r_1 1^{-1}, \ldots, r_t 1^{-1}, x_{t+1} 1^{-1}, \ldots, x_u 1^{-1}),$$

so $\alpha_i 1^{-1} = (\alpha_i c 1^{-1})(1 c^{-1}) = 0$. Hence the $r_i 1^{-1}$ are Z_c-independent. Q.E.D.

The final portion of our structure theory is based on a trivial observation.

LEMMA 4.9. *Suppose $g(X_1, \ldots, X_u)$ is t-normal and R-central; let $c \in R(g)$, and let $\hat{g}$ be the associated polynomial of g. If $\hat{g}$ is an identity of R then, for any ideal $\tilde{A}$ of R, $c\tilde{A} \subseteq (\tilde{A} \cap Z)R$.*

Proof. For any a in $\tilde{A}$, we have

$$ca = g(r_1, \ldots, r_u)a = \sum_i g^{(i)}(r_1, \ldots, r_u, a) \in (\tilde{A} \cap R^+(g))R. \quad \text{Q.E.D.}$$

Let $\mathscr{G}_t = \{t$-normal, R-central polynomials, whose associated polynomials are identities of $R\}$, and let $R^+(\mathscr{G}_t) = \sum \{R^+(g) \mid g \in \mathscr{G}_t\}$. Clearly $R^+(\mathscr{G}_t)$ is an ideal of Z, contained in I.

THEOREM 4.10. *Suppose $l \in R^+(\mathscr{G}_t)$. Then the map $A \to \tilde{A}$ (in Theorem 4.2) is given by $\tilde{A} = AR$, and is a lattice isomorphism of {ideals of Z} with {ideals of R}.*

Proof. By Lemma 4.9, $\tilde{A} \subseteq (\tilde{A} \cap Z)R$, implying $\tilde{A} = (\tilde{A} \cap Z)R$, and the assertion follows from Theorem 4.2. Q.E.D.

Theorems 4.2, 4.8, and 4.10 are the three structure theorems alluded to at the beginning of this section; we now tie these results together through a formal definition.

DEFINITION 4.11. R has *Azumaya type t* if $1 \in R^+(\mathscr{G}_t)R$. R has *Azumaya type* if R has Azumaya type t for some t.

PROPOSITION 4.12. *If $1 \in R^+(\mathscr{G}_t)R$, then $1 \in R^+(\mathscr{G}_t)$.*

Proof. Let $H = R^+(\mathscr{G}_t)$, and suppose $1 \in HR$ but $1 \notin H$. Take a maximal ideal P of Z containing H; let $S = Z - P$. $Z_S \subseteq Z(R_S)$, $R_S^+(\mathscr{G}_t) = H_S$, $P_S R_S = R_S$, and P_S is the unique maximal ideal of Z_S. Let W be a maximal ideal of R_S, and let $\overline{R_S} = R_S/W$, a simple Ω-ring. $\overline{H_S} = \overline{R_S^+}(\mathscr{G}_t)$, which is an ideal of $Z(\overline{R_S})$, a field. Thus

$$Z(\overline{R_S}) = \overline{H_S} \subseteq \overline{Z_S} \subseteq \overline{Z(R_S)},$$

implying $Z(\overline{R_S}) = \overline{Z_S} = (Z_S + W)/W \approx Z_S/(W \cap Z_S)$. Therefore, $W \cap Z_S$ is a maximal ideal of Z_S, so $W \cap Z_S = P_S$, contrary to $P_S R_S = R_S$. Hence $1 \in H$, after all. Q.E.D.

THEOREM 4.13. *Suppose R has Azumaya type t. Then, for every maximal ideal P of Z, there exists c in $Z - P$ such that R_c is a free Z_c-module, of degree t. Also, there is a lattice isomorphism of {ideals of R} with {ideals of Z}, given by $\tilde{A} \to A \cap Z$, with inverse $A \to AR$. In particular, for any maximal ideal P of Z, R/PR is simple, of dimension t over $Z/P \approx Z(R/PR)$.*

Proof. By Proposition 4.12, $1 \in R^+(\mathscr{G}_t)$. Thus we can apply Theorems 4.2, 4.8, and 4.10. Q.E.D.

Theorem 4.13 says that rings of Azumaya type behave like associative Azumaya algebras, a surprising result in the nonassociative case, which contains the associative Artin-Procesi theorem (as we shall see in Part III). For the time being, we look for general ways of finding rings with Azumaya type. Recall that two Ω-rings R_1 and R_2 are *equivalent* if $R_1 \leq R_2$ and $R_2 \leq R_1$.

Remark 4.14. If R is central simple of dimension t, then, by Theorem 4.5 and Theorem 4.7, $I = R^+(\mathscr{G}_t)$. Thus, more generally, for any Ω-ring R which is equivalent to a central simple Ω-ring, $I \neq 0$ iff $\mathscr{G}_t \neq \varnothing$. (In particular, every central simple Ω-ring with regular central polynomial has Azumaya type.)

THEOREM 4.15. *Suppose R is equivalent to a central simple Ω-ring with regular central polynomial. If S is a multiplicative subset of Z and $S \cap IR \neq 0$, then R_S has Azumaya type. In particular, $R_{Z-\{0\}}$ has Azumaya type.*

Proof. The first assertion is immediate in view of Theorem 2.0; the second assertion follows from Remark 4.14.

COROLLARY 4.16. *Suppose R is torsion-free as a Z-module. If R is equivalent to a central simple Ω-ring with regular central polynomial, then $R_{Z-\{0\}}$ is simple. (In particular, R is absolutely prime.)*

Proof. Let $R' = R_{Z-\{0\}}$. Since R' has Azumaya type, there is a lattice isomorphism between {ideals of R'} and {ideals of $Z(R')$} given by $\tilde{A} \to \tilde{A} \cap Z(R')$. But $Z(R')$ is the field of quotients of Z. Hence R' has no nonzero ideals, and is thus simple. Q.E.D.

We close this section with some related digressions, which extend some of the results of this section. First, call an ideal A of Z *contracted* if $AR \cap Z \subseteq A$. Clearly $AR \cap Z \supseteq A$, so A is contracted if and only if $AR \cap Z = A$. Call an ideal A of Z *cancellable* if for ideals A_1, A_2 of Z, $A_1 A = A_2 A \Rightarrow A_1 = A_2$. For example, Z itself is cancellable; if Z is a domain, then any invertible ideal of Z is cancellable.

PROPOSITION 4.17. *If R has a regular central polynomial g, such that $R^+(g)$ is cancellable, then every ideal of Z is contracted; in fact, there is a lattice surjection of {ideals of R} $\to$ {ideals of Z}, given by $\tilde{A} \to \tilde{A} \cap Z$.*

Proof. Immediate from Proposition 4.1. Q.E.D.

Our final task is to improve Remark 4.14. To do this, we must obtain a stronger result than 4.5.

THEOREM 4.18. *If R is central simple, with multilinear central polynomial g, then there is a multilinear t-normal polynomial $g' < g$, such that g' is R-central.*

Proof. By Proposition 4.4, we have a t-normal polynomial f, with $1 \in R(f)$. Write f as a sum of completely homogeneous polynomials f_i, and let

$$g_i = g(f_i(X_1, \ldots, X_m)X_{m+1}, X_{m+2}, \ldots).$$

Clearly $g = \sum g_i$, so some g_i is not an identity of R, and is thus R-central. For each indeterminate X_k occurring of degree $d_k > 0$ in f_i, replace X_k by d_k distinct indeterminates. This yields a multilinear, t-normal polynomial $f'_i \leq f_i$, so

$$g' = \sum_i g(f'_i(X_1, \ldots, X_{m'})X_{m'+1}, X_{m'+2}, \ldots)$$

is obviously R-central. Q.E.D.

Now we can carry out the above structure theorems while considering multilinear polynomials (which are always R-stable). Namely, we say R is *mult-equivalent* to R' if R and R' satisfy the same multilinear identities. Then R and R' satisfy the same multilinear central polynomials.

THEOREM 4.19. *Suppose R is mult-equivalent to a central simple Ω-ring with multilinear central polynomial. Then $R_{Z-\{0\}}$ has Azumaya type. If R is torsion-free as a Z-module, then R is absolutely prime.*

COROLLARY 4.20. *If R is a central extension of a central simple Ω-ring and if R is prime with multilinear central polynomial, then R is absolutely prime.*

Proof. Central extensions are mult-equivalent. Q.E.D.

5. Decomposition into associative and nonassociative rings

Since the overall object of this paper is to introduce methods of associative PI-theory in a general setting, one may be interested in decomposing a ring into an associative part and a nonassociative part. This is the motivation behind Section 5, which is based on the following notions used by Slater [29] (for alternative algebras).

Recall from §1A that $N(R)$ is the nucleus of R. Define

$$U(R) = \bigcup \{\text{ideals (of } R) \subseteq N(R)\} \quad \text{and} \quad D(R) = \bigcap \{\text{ideals} \supseteq [R, R, R]\}.$$

A standard argument in [13, p. 18], based on identity (α) of §1B, Example (i), shows $U(R)D(R) = D(R)U(R) = 0$. If $U(R) = 0$, call R *purely nonassociative*; if $D(R) = R$, call R *absolutely nonassociative*.

Remark 5.1. The following conditions are equivalent: (i) R is absolutely nonassociative; (ii) $1 \in D(R)$; (iii) every homomorphic image of R is purely nonassociative.

Given two additive subgroups A, B of a Ω-ring R, one sees easily that there is a unique ideal which we call $(B:A)$, maximal with respect to the property $(B:A)A \subseteq B$. If $B = 0$ then we give to $(0:A)$ the name $\mathrm{Ann}_R\, A$. Note that if $A \subseteq Z$ then $\mathrm{Ann}_R\, A = \{r \in R \mid rA = 0\}$, which is the special definition given earlier. If there is no ambiguity, we write $\mathrm{Ann}\, A$ in place of $\mathrm{Ann}_R\, A$. Since $U(R)D(R) = 0$, we have $U(R) \subseteq \mathrm{Ann}\, D(R)$; since $D(R)U(R) = 0$, we have $D(R) \subseteq \mathrm{Ann}\, U(R)$.

LEMMA 5.2. *If R is semiprime with ideals A, B, then $B \subseteq \mathrm{Ann}\, A$ iff $B \cap A = 0$ iff $AB = 0$.*

Proof. By symmetry, we need only show $BA = 0$ iff $B \cap A = 0$. Well, if $B \cap A = 0$ then $BA \subseteq B \cap A = 0$. Conversely, if $BA = 0$ then $(B \cap A)^2 \subseteq BA = 0$, so $B \cap A = 0$ (since R is semiprime). Q.E.D.

In particular, for any ideal B of a semiprime ring R, $B \subseteq \text{Ann}\ (\text{Ann}\ B)$.

PROPOSITION 5.3. *Let B be an ideal of a semiprime Ω-ring R, and let $B' = \text{Ann}\ B$. Then $B' = \text{Ann}(\text{Ann}\ B')$, and B' is a semiprime ideal; also $(B' : B) = B'$. If R is strongly semiprime then R/B' is strongly semiprime.*

Proof. Clearly $B' \subseteq \text{Ann}(\text{Ann}\ B')$. But $B \subseteq \text{Ann}\ B'$, implying $B' = \text{Ann}\ B \supseteq \text{Ann}(\text{Ann}\ B')$, so $B' = \text{Ann}(\text{Ann}\ B')$. By Lemma 5.2, $B' \cap B = 0$.

To prove B' is a semiprime ideal, suppose $A^2 \subseteq B'$. Then

$$(A \cap B)^2 \subseteq A^2 \cap B \subseteq B' \cap B = 0,$$

so $A \cap B = 0$, implying $A \subseteq B'$. Hence B' is a semiprime ideal.

Next, let $A = (B' : B)$. Then $(A \cap B)^2 \subseteq AB \subseteq B'$, so $A \cap B \subseteq B'$, implying

$$A \cap B \subseteq B' \cap B = 0,$$

so $A \subseteq B'$. This shows $(B' : B) = B'$.

Finally, suppose $\text{Nil}\ (R) = 0$, and $A/B' = \text{Nil}\ (R/B')$. Then $A \cap B$ is a nil ideal of R, so $A \cap B = 0$; hence $A \subseteq B'$, implying $\text{Nil}\ (R/B') = 0$. Q.E.D.

Proposition 5.3 has a very nice application.

PROPOSITION 5.4. *Suppose R is semiprime. Then $U(R) = \text{Ann}\ D(R)$, and R is a subdirect product of an associative semiprime Ω-ring R_1 and a purely nonassociative semiprime Ω-ring R_2. Furthermore, if $\text{Nil}\ (R) = 0$ then R_1 and R_2 are strongly semiprime.*

Proof. We know $U(R) \subseteq \text{Ann}\ D(R)$ already. Conversely,

$$[\text{Ann}\ D(R), R, R] \subseteq D(R) \cap \text{Ann}\ D(R) = 0,$$

so $\text{Ann}\ D(R) = U(R)$. Let $R_1 = R/\text{Ann}\ U(R)$ and $R_2 = R/U(R)$. Since $D(R) \subseteq \text{Ann}\ U(R)$, R_1 is associative, and we can use Proposition 5.3 for all the assertions except R_1 being purely nonassociative. To see this, let $A/U(R) = U(R/U(R))$. Then

$$[A, R, R] \subseteq U(R) \cap D(R) = 0,$$

so $A \subseteq U(R)$, implying $U(R/U(R)) = 0$, as desired. Q.E.D.

Proposition 5.4 is useful when we want to reduce questions about identities to the respective associative and nonassociative counterparts. Let us record some facts about purely nonassociative rings which shows why they are interesting.

Remark 5.5. Every purely nonassociative, strongly semiprime Ω-ring R is the subdirect product of nonassociative strongly prime Ω-rings. (Proof: Let $\{P \mid P \in \mathscr{P}\}$ be the set of strongly prime ideals P of R such that $D(R) \nsubseteq P$.

Clearly $\bigcap \{P \mid P \in \mathscr{P}\} \subseteq \text{Ann } D(R) = 0$, and thus R is the subdirect product of the nonassociative prime Ω-rings $\{R/P \mid P \in \mathscr{P}\}$.)

Remark 5.6. Every nonassociative prime Ω-ring R is purely nonassociative. (Proof: If R is nonassociative then $D(R) \neq 0$, implying $U(R) = \text{Ann } D(R) = 0$.)

6. Ω-rings with involution

As observed in example (xiii) of Section 1B, the theory of Ω-rings with involution $(*)$ is merely a special case of the theory of Ω'-rings, when we formally adjoin $(*)$ to Ω to create a new set Ω'. There are a number of issues which must be considered, however, when we wish to work in the category of rings with involution, i.e., viewing $(*)$ as part of the structure. Since involution will play an important role in the applications in Part III, we shall use this section to discuss the relation between these two approaches, as well as other structures related to the involution. We shall assume for the time being that $(*) \notin \Omega$, and $\Omega' = \Omega \cup \{*\}$, as given in Section 1B.

The first problem is how we want to define PI-ring with involution. On the one hand, we could use the general definition in §1A. On the other hand, we may prefer the definition of [26], which treats X_i^* as a distinct indeterminate from X_i. In this setting, we would define the $(*)$*-fingerprint* of a monomial (with respect to Ω), to be the string of indeterminates X_i and X_i^*, written without parentheses and without coefficients. For example, viewing $(*)$ in Ω', the fingerprint of $X_1 X_2^*$ would be $X_1 X_2$, but the $(*)$-fingerprint is $X_1 X_2^*$ itself. Then a *generalized* $(*)$*-monomial* of f would be the sum of all $(*)$-monomials with the same $(*)$-fingerprint, and f would be $(R, *)$*-proper* if some generalized $(*)$-monomial of f is not an identity of $(R, *)$. (We write $(R, *)$ instead of R, to stress the importance of $(*)$.) For example, if F is a field with the identity involution, then $X_1 - X_1^*$ is an improper (in fact trivial) identity of F, but is $(*)$-proper. This discrepancy is resolved in the next result.

PROPOSITION 6.1. *If f is a polynomial (in $(\Omega\{X\}, *)$) which is $(*)$-proper then there exists a polynomial $f' \leq f$ having the following properties:* (i) *For each generalized $(*)$-monomial f_π of f, there is a generalized monomial f'_π of f' with $f_\pi \leq f'_\pi$;* (ii) $\deg f' = 2 \deg f$.

Proof. Given $f(X_1, X_1^*, \ldots, X_m, X_m^*)$, define

$$f(X_1, X_1^*, \ldots, X_{2m}, X_{2m}^*) = f(X_1 X_2, (X_1 X_2)^*, \ldots, X_{2m-1} X_{2m}, (X_{2m-1} X_{2m})^*)$$
$$= f(X_1 X_2, X_2^* X_1^*, \ldots, X_{2m-1} X_{2m}, X_{2m}^* X_{2m-1}^*).$$

The properties are easily verified. Q.E.D.

COROLLARY 6.2. *Use notation as in* Proposition 6.1. *If f is $(R, *)$-proper then f' is R-proper (with respect to Ω').*

An *ideal* of $(R, *)$ is an ideal of R which is invariant under $(*)$. (In other words, ideals of $(R, *)$ are the Ω'-ideals of R.) If $(A, *)$ is an ideal of $(R, *)$ then an involution $(*)$ is induced on R/A in the obvious manner (namely, $(r + A)^* = r^* + A$). Conversely, if $(\bar{R}, *)$ is a homomorphic image of $(R, *)$, then the kernel of the map from R to $\bar{R}$ is an ideal of $(R, *)$. Call a polynomial f of $(\Omega\{X\}, *)$ $(R, *)$-*strong* if f is $(\bar{R}, *)$-proper for every nonzero image $(\bar{R}, *)$ of $(R, *)$.

COROLLARY 6.3. *Notation as in* Proposition 6.1, *if* $f(X_1, \ldots, X_m)$ *is* $(R, *)$-*strong then* $f'(X_1, \ldots, X_{2m}) + (f'(X_{2m+1}, \ldots, X_{4m}))^*$ *is R-strong (with respect to* Ω').

Proof. Straightforward computation.

Note that if $(*)$ is an involution of R, then $(*)$ is an automorphism of Z, of degree 1 or 2. We say $(*)$ is of the *first kind* on $(R, *)$ if $(*)$ is the identity on Z; otherwise $(*)$ is of the *second* kind on $(R, *)$.

In the category of rings with involution, the *center of* $(R, *)$ (which we call $Z(R, *)$) is $\{z \in Z \mid z^* = z\}$; $Z = Z(R, *)$ iff $(*)$ is of the first kind on R. If Z has no nilpotent elements (in particular if R is semiprime) and if a polynomial g in $(\Omega\{X\}, *)$ takes its values in Z, then either $g + g^*$ is $(R, *)$-central or g takes only antisymmetric values, in which case gg^* is $(R, *)$-central. For the remainder of this section, write Z' for $Z(R, *)$.

If $R \approx A \oplus A^0$, where A^0 is the *opposite* Ω-ring of A (i.e., A^0 has the same additive structure as A, but all multiplications are given in the opposite order), then we define the *exchange involution* $(*)$ to be given by $(a_1, a_2)^* = (a_2, a_1)$ for (a_1, a_2) in $A \oplus A^0$. Note that A^0 can be viewed as a ring with 1, and $(1, 0)^* = (0, 1)$, so the exchange involution is always of the second kind.

Remark 6.4. It is well-known that if $(R, *)$ is a simple Ω-ring with involution (i.e., $(R, *)$ has no proper nonzero ideals), then either R is a simple Ω-ring (without involution) or $(*)$ can be written as the exchange involution, where A can be taken to be a simple Ω-ring.

We make a brief diversion to relate the identities of $(R, *)$ to the identities of R (without involution). If $f(X_1, X_1^*, \ldots, X_m, X_m^*) \in (\Omega\{X\}, *)$, we can "lift" f to a polynomial $\bar{f}(X_1, X_2, \ldots, X_{2m-1}, X_{2m})$ in $\Omega\{X\}$, by replacing X_i by X_{2i-1}, and X_i^* by X_{2i}, for all i. Say f is $(R, *)$-*special* if $\bar{f}$ is an identity of R. Clearly if f is $(R, *)$-special then f is an identity of $(R, *)$.

THEOREM 6.5. *If Z has an element z such that $z - z^*$ is regular, then every multilinear identity of* $(R, *)$ *is special.*

Proof. As in [24, Theorem 7]. Q.E.D.

We say $(R, *)$ is prime (resp. semiprime) if R is prime (resp. semiprime) with respect to Ω'.

COROLLARY 6.6. *If $(R, *)$ is prime and $(*)$ is of the second kind, then all multilinear identities of $(R, *)$ are special.*

Thus, often the involution is "interesting," in the sense of producing new identities, only when it is of the first kind. Next, let us make several observations which tie the earlier structure theorems for varieties of Ω-rings to the corresponding structure theorems for Ω-rings with involution.

Remark 6.7. Nil (R) is invariant under every involution $(*)$. Thus, R is strongly semiprime iff $(R, *)$ is strongly semiprime.

Remark 6.8. $BM(R)$ is invariant under every involution $(*)$. (This is immediate from Remark 6.4.) Thus R is semisimple iff $(R, *)$ is semisimple.

Remark 6.9. Using Remarks 6.7 and 6.8, we see that in a class of rings with involution,

$$\text{Nil } (R) = 0 \Rightarrow R[\lambda] \text{ is semisimple}$$

iff

$$\text{Nil } (R, *) = 0 \Rightarrow (R[\lambda], *) \text{ is semisimple.}$$

Using Remark 6.9, we have one crucial correspondence between Kaplansky classes with $(*)$ and Kaplansky classes without $(*)$. Another link in the theories is given by:

PROPOSITION 6.10. *If $(R, *)$ is a Ω-ring with involution and H is a commutative, associative Z-algebra, then $(R \otimes_{Z'} H, *)$ is a Ω-ring with the induced involution $(*)$, given by $(\sum_i r_i \otimes h_i)^* = \sum_i r_i^* \otimes h_i$, for $r_i \in R$ and $h_i \in H$.*

Proof. $R \otimes_{Z'} H$ is obviously an Ω-ring (under the natural operations); we see that $(*)$ extends to an involution on $R \otimes_{Z'} H$, by an argument based on the universal properties of the tensor product construction, and the involution is clearly given by our formula.

Remark 6.11. $N(R)$, $U(R)$, and $D(R)$ are all invariant under $(*)$. Also, if B is an ideal of $(R, *)$ then Ann B is an ideal of $(R, *)$. In particular, the decomposition of Proposition 5.4 is valid for Ω-rings with involution.

We finish this section with a construction of considerable interest in the applications. Suppose $(R, *)$ is a Ω-ring with involution, and let $\mathcal{S}(R, *)$ be the "symmetric" elements of R, i.e., $\{r \in R \mid r^* = r\}$. Note that $1 \in \mathcal{S}(R, *)$, (since $1 = (1^*1)^* = 1^*1 = 1^*$). *Suppose* $\frac{1}{2} \in R$. Then we make $\mathcal{S}(R, *)$ into a Ω-ring with the same addition, and new multiplications

$$\omega \cdot r = (\omega r + (\omega r)^*)/2, \qquad r \cdot \omega = (r\omega + (r\omega)^*)/2,$$

and $r_1 r_2 = (r_1 r_2 + r_2 r_1)/2$, for each ω in Ω, and each r_1, r_2 in R.

THEOREM 6.11. *Suppose* $\frac{1}{2} \in R$, *and* $(R, *)$ *is a* Ω-*ring with involution.*

(i) *The correspondence* $r \mapsto (r + r^*)/2$ *is an onto mapping from* R *to* $\mathscr{S}(R, *)$ *(as sets), whose restriction to* $\mathscr{S}(R, *)$ *is the identity map.*

(ii) *If* H *is a commutative, associative* Z'-*algebra then*

$$\mathscr{S}(R \otimes_{Z'} H, *) \approx \mathscr{S}(R, *) \otimes_{Z'} H \quad (\text{as } \Omega\text{-rings}).$$

(iii) *If* $(\bar{R}, *)$ *is a homomorphic image of* $(R, *)$, *then* $\mathscr{S}(\bar{R}, *) = \overline{\mathscr{S}(R, *)}$.

Proof. (i) If $r^* = r$ then $r = (r + r^*)/2$, so all assertions are immediate.

(ii) Suppose $x \in \mathscr{S}(R \otimes H, *)$. Then $x = \sum r_i \otimes h_i$ for suitable r_i in R and h_i in H. Hence

$$x = (x + x^*)/2 = \sum ((r_i + r_i^*)/2) \otimes h_i \in \mathscr{S}(R, *) \otimes H.$$

This proves $\mathscr{S}(R \otimes H, *) \subseteq \mathscr{S}(R, *) \otimes H$; the other direction is obvious.

(iii) Clearly $\overline{\mathscr{S}(R, *)} \subseteq \mathscr{S}(\bar{R}, *)$. Conversely, if $x \in \mathscr{S}(\bar{R}, *)$ then

$$\bar{x} = (\bar{x} + \bar{x}^*)/2 = \overline{(x + x^*)/2} \in \overline{\mathscr{S}(R, *)}, \quad \text{so} \quad \overline{\mathscr{S}(R, *)} = \mathscr{S}(\bar{R}, *).$$

Q.E.D.

One can thus analyze $\mathscr{S}(R, *)$ in terms of $(R, *)$; this analysis can be carried out in general terms, but we leave it until Part III, where there is sufficient motivation. One example is when $(R, *)$ is associative; then $\mathscr{S}(R, *)$ is Jordan.

A similar analysis can be carried out with antisymmetric elements, using a "Lie" product in place of a "Jordan" product. The analogue to Theorem 6.11 would be useful in this context, but since our theory does not work well for Lie algebras (as discussed in Part III), we do not go into this situation.

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