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ON THE LOCAL THEORY OF TOEPLITZ OPERATORS

BY

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There are several local theories for Toeplitz operators acting on the usual Hardy space H^2 of the unit circle [1], [6], [7] and [14]. The approach in these theories is to associate with a global operator a local operator for each subset of some partition of the maximal ideal space M_{L^∞} of L^∞. A typical result says that if for each subset in the partition the local operator is invertible, then the global Toeplitz operator is Fredholm. Obviously the nature of the partition is crucial and the above result has been established for various partitions of M_{L^∞}. The problem of understanding the local operators for a fixed element in the partition remains and this is the main concern here.

In one of these local theories [6] the local operator is an element in a quotient C^*-algebra which does not have an obvious representation on a Hilbert space. The work in [4] gives some explanation of these local operators for the case of an element in the partition of M_{L^∞} over the unit circle. In this paper further explanations and refinements are provided.

The arguments below are made easier because the authors have allowed themselves the luxury of the recent description by S. Y. Chang [2] and D. E. Marshall [10] of the closed algebras between H^∞ and L^∞. The introduction of these algebras into the local picture provides further interpretations of invertibility of the local operators back on the unit circle.

The authors are indebted to I. Gohberg for first calling attention to these local problems.

1. Preliminaries

Let L^∞ denote the Banach algebra of essentially bounded Lebesgue measurable functions on the unit circle $\mathbf{T}$ and let M_{L^∞} denote its maximal ideal space. No notational distinction will be made between $f \in L^\infty$ viewed as a function on $\mathbf{T}$ or its Gelfand transform as a continuous function on M_{L^∞}. The notation H^∞ will be used for the algebra of bounded analytic functions on the unit disc considered as a subalgebra of L^∞ and C will stand for the algebra of continuous functions on $\mathbf{T}$. A subset S of M_{L^∞} is called a peak set for H^∞ in case there is an $f \in H^\infty$ such that f equals one on S and $|f|$ is less than one off S. The set S will be called a weak peak set for H^∞ in case it is an intersection of peak sets. If the set S is a weak peak set for H^∞, then the restriction algebra $H^\infty|_S$ is a closed

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subalgebra of $L^\infty|_S$. The reader is referred to [8] for many of the basic properties of H^∞ and L^∞ and to [9] for additional properties of function algebras.

Let S be a weak peak set for H^∞. We denote by H_S^∞ the set

$$\{\phi \in L^\infty : \phi|_S \in H^\infty|_S\}.$$

It is easy to verify that H_S^∞ is a closed subalgebra of L^∞ containing H^∞. Recently, in answer to a question of R. G. Douglas, S. Y. Chang [2] and D. E. Marshall [10] have given a description of the closed algebras $\mathscr{A}$ which satisfy $H^\infty \subset \mathscr{A} \subset L^\infty$. In fact, let $\mathscr{A}$ be such an algebra and $\Sigma_{\mathscr{A}} = \{\bar{\phi} : \phi \text{ is inner}, \bar{\phi} \in \mathscr{A}\}$. The result of these authors is:

THE CHANG-MARSHALL THEOREM. *Let $\mathscr{A}$ be a closed algebra satisfying $H^\infty \subset \mathscr{A} \subset L^\infty$. Then $\mathscr{A}$ equals the smallest closed algebra generated by H^∞ and $\Sigma_{\mathscr{A}}$.*

It follows that whenever S is a weak peak set for H^∞, then the algebra H_S^∞ is the smallest closed algebra generated by H^∞ and the collection Σ_S of functions $\bar{\phi}$, where ϕ is inner and $\bar{\phi} \in H_S^\infty$. The inner functions ϕ invertible in H_S^∞ are those inner functions which are constant on the support of every representing measure for a homomorphism of H^∞, whenever this support set is a subset of S.

Actually the full strength of the Chang-Marshall theorem is not required for the purposes of this paper. The algebra $H_S^\infty = H^\infty + J_S$, where J_S is the ideal in L^∞ consisting of functions vanishing on S. Axler [1] established the equivalent of the Chang-Marshall theorem for algebras of the form $H^\infty + J_S$. The techniques employed by Axler are simpler than those found in [2] and [10].

The authors would like to thank R. Younis and the referee for the observation in the preceding paragraph.

In the later sections we will be discussing the localization of Toeplitz operators to closed subsets S of M_{L^∞}. In addition to being weak peak sets these closed subsets all share an additional property which we now describe. The weak peak set S will be said to have property $(*)$ in case:

$(*)$ ϕ inner, $\bar{\phi}$ in H_S^∞, implies ϕ is constant on S.

It is not difficult to verify that any closed antisymmetric set for H^∞ has property $(*)$. Recall that a subset S of M_{L^∞} is said to be an antisymmetric set for H^∞ in case every function in H^∞ which is real valued on S is constant on S. In Section 3 further examples of sets possessing property $(*)$ are described.

There are two expressions which describe the local distance from a function u in L^∞ to H^∞ on a weak peak set S. These are the quantities:

$$\operatorname{dist}_S(u, H^\infty) = \inf_{h \in H^\infty} \|u - h\|_S \quad \text{and} \quad \operatorname{dist}(u, H_S^\infty) = \inf_{h \in H_S^\infty} \|u - h\|_\infty .$$

These distances can readily be seen to be equal.

The following extension result will be used in the study of local invertibility of Toeplitz operators in the next section.

PROPOSITION 1. *Let S be a weak peak set for H^∞ which has property* $(*)$. *Let u be in L^∞. Assume $|u| = 1$ on S and* $\mathrm{dist}_S(u, H^\infty) = \mathrm{dist}(u, H_S^\infty) < 1$. *Then there is a unimodular $\tilde{u}$ in L^∞ such that $\tilde{u} = u$ on S and* $\mathrm{dist}(\tilde{u}, H^\infty) < 1$.

Proof. It can be assumed that $|u| = 1$ a.e. The hypothesis states that there is a function h in H_S^∞ satisfying $\|u - h\|_\infty < 1$. The Chang-Marshall Theorem implies that h can be assumed of the form $h = g\bar{\phi}$ where g is in H^∞ and ϕ is an inner function with $\bar{\phi}$ in H_S^∞. The fact that S has property $(*)$ means that ϕ is some constant λ of modulus one on S. Set $\tilde{u} = \bar{\lambda}\phi u$, then $\tilde{u} = u$ on S and $\|\tilde{u} - \bar{\lambda}g\|_\infty < 1$. This ends the proof.

2. Restricted invertible Toeplitz operators

If f is an element in L^∞, then T_f will denote the Toeplitz operator with symbol f acting on the Hardy space H^2 of the unit circle. For the basic properties of Toeplitz operators we refer the reader to [6] or [13].

Let S be a closed subset of M_{L^∞}. The Toeplitz operator T_f will be called an S-restricted invertible operator in case $f = g$ on S where g is in L^∞ and T_g is invertible. A similar definition is made for S-restricted left-invertible (right-invertible) operators. The following proposition is a local analogue of a result of Rabindranathan [11] (See, also Sarason [13]).

PROPOSITION 2. *Let S be a weak peak set for H^∞ which has property* $(*)$. *Let u be a function in L^∞ which is unimodular on S. The following are equivalent:*

(i) *The operator T_u is S-restricted left-invertible.*
(ii) $\mathrm{dist}_S(u, H^\infty) = \mathrm{dist}(u, H_S^\infty) < 1$.

Proof. The result of Rabindranathan [11] shows the equivalence of (i) and (ii) when $S = M_{L^\infty}$.

We first show (i) implies (ii). The assumption (i) implies that there is a v in L^∞ such that T_v is left-invertible and such that $v = u$ on S. Replacing v by $v|v|^{-1}$, if necessary, we can assume that v is unimodular. The result of Rabindranathan implies $\mathrm{dist}(v, H^\infty) < 1$. Clearly,

$$\mathrm{dist}_S(u, H^\infty) = \mathrm{dist}_S(v, H^\infty) < 1.$$

The proof that (ii) implies (i) uses Proposition 1. Indeed, assuming (ii), there is a unimodular extension $\tilde{u}$ of u from S to M_{L^∞} such that $\mathrm{dist}(\tilde{u}, H^\infty) < 1$. The easy half of Rabindranathan's result implies that $T_{\tilde{u}}$ is left-invertible and, therefore, T_u is S-restricted left-invertible. This completes the proof.

A similar result is true for the case where the operator T_u is S-restricted invertible. We first establish the following:

LEMMA 1. *Let S have property* $(*)$. *Then for u in L^∞*

$$\mathrm{dist}(u, (H_S^\infty)^{-1}) = \mathrm{dist}_S(u, (H^\infty)^{-1}).$$

Proof. It follows from the Chang-Marshall Theorem that the collection

$$Q = \{\psi h \bar{\phi}: h \in (H^\infty)^{-1}, \psi, \phi \text{ inner } \psi = \phi = 1 \text{ on } S\}$$

is dense in $(H_S^\infty)^{-1}$.

Let $\varepsilon > 0$ be arbitrary. Choose a function $\psi h \bar{\phi} \in Q$ such that

$$\|u - \psi h \bar{\phi}\|_\infty \leq \text{dist}\,(u, (H_S^\infty)^{-1}) + \varepsilon.$$

Then

$$\|u - h\|_S \leq \|u - \psi h \bar{\phi}\|_\infty \leq \text{dist}\,(u, (H_S^\infty)^{-1}) + \varepsilon.$$

This shows $\text{dist}_S\,(u, (H^\infty)^{-1}) \leq \text{dist}\,(u, (H_S^\infty)^{-1})$. Now let h be a function in $(H^\infty)^{-1}$ such that

$$\|u - h\|_S \leq \text{dist}_S\,(u, (H^\infty)^{-1}) + \varepsilon/2.$$

There is a clopen V containing S such that

$$\|u - h\|_V \leq \text{dist}_S\,(u, (H^\infty)^{-1}) + \varepsilon.$$

The function $\tilde{u} = u\chi_V + 1 - \chi_V$ agrees with u on S and $\tilde{h} = h\chi_V + 1 - \chi_V$ belongs to $(H_S^\infty)^{-1}$. Clearly,

$$\text{dist}\,(\tilde{u}, (H_S^\infty)^{-1}) \leq \|\tilde{u} - \tilde{h}\|_\infty \leq \text{dist}_S\,(u, (H^\infty)^{-1}) + \varepsilon.$$

It is not difficult to check that $\text{dist}\,(u, (H_S^\infty)^{-1}) = \text{dist}\,(\tilde{u}, (H_S^\infty)^{-1})$ and this gives the inequality $\text{dist}\,(u, (H_S^\infty)^{-1}) \leq \text{dist}_S\,(u, (H^\infty)^{-1})$ completing the proof.

PROPOSITION 3. *Let S be a weak peak set for H^∞ which has property* $(*)$. *Let u be a function in L^∞ which is unimodular on S. Then the following are equivalent.*

(i) *The operator T_u is S-restricted invertible.*
(ii) $\text{dist}\,(u, (H_S^\infty)^{-1}) = \text{dist}_S\,(u, (H^\infty)^{-1}) < 1$.

Rabindranathan [11] has shown the equivalence of (i) and (ii) for the case $S = M_{L^\infty}$. The proof of the proposition follows from this result and is similar to the proof of Proposition 2. We omit the details.

3. Partitions of M_{L^∞}

Let B denote a closed self-adjoint subalgebra of L^∞. There is a natural mapping π from M_{L^∞} to M_B the maximal ideal space of B, where $\pi(\gamma) = \gamma|_B$. The mapping π is continuous and surjective (see [6, page 48]). Hence $\{\pi^{-1}(\lambda)\}_{\lambda \in M_B}$ is a partition of M_{L^∞} into nonempty disjoint closed subsets. This partition will be referred to as the partition of M_{L^∞} over the subalgebra B. The sets in such a partition are obviously the sets of constancy for the functions in B.

In the later sections of this paper we shall be particularly interested in the partitions of M_{L^∞} over C and over $QC = (H^\infty + C) \cap (H^\infty + C)^-$. The latter algebra is referred to as the algebra of quasi-continuous functions on $\mathbf{T}$. For the case $B = C$, M_B is homeomorphic to $\mathbf{T}$, and the partition element $\pi^{-1}(\lambda)$, $\lambda \in \mathbf{T}$, is often denoted by X_λ. It is referred to as the fiber of M_{L^∞} over the point $\lambda \in \mathbf{T}$ (see [8, page 161]).

Since there are real valued $H^\infty + C$ functions which are discontinuous at $\lambda = 1$ (see [12]) it follows that the partition of M_{L^∞} over QC is a proper refinement of the partition over C. One further refinement of the partition over QC is the partition of M_{L^∞} into maximal antisymmetric sets for $H^\infty + C$. It is easy to verify that each maximal antisymmetric set for $H^\infty + C$ is contained in one of the fibers $\pi^{-1}(\xi_0)$, $\xi_0 \in M_{QC}$. A nontrivial example in an unpublished note of D. Sarason shows that the partition of M_{L^∞} into maximal antisymmetric sets for $H^\infty + C$ is a proper refinement of the partition of M_{L^∞} over QC.

We remark that for each of the above partitions, the partition elements are weak peak sets for $H^\infty + C$. Each fiber X_λ is actually a peak set for H^∞ since the function $e_\lambda(z) = \frac{1}{2}[1 + \bar{\lambda} z]$ peaks on X_λ. It is well known ([9, page 162] for example) that maximal antisymmetric sets for $H^\infty + C$ are weak peak sets for $H^\infty + C$. Finally, let $\pi^{-1}(\xi_0)$, $\xi_0 \in M_{QC}$, denote an element of the partition over M_{QC}. If V is any neighborhood of ξ_0 in M_{QC}, then there is an element f in QC satisfying $0 \leq f \leq 1$, $f(\xi_0) = 1$, and $f \equiv 0$ on $M_{QC} \backslash V$. Considered as an element in L^∞ this function will equal one on $\pi^{-1}(\xi_0)$ and be zero on sets of the form $\pi^{-1}(\xi)$, $\xi \in M_{QC} \backslash V$. This shows $\pi^{-1}(\xi_0)$ is a weak peak set for $H^\infty + C$.

It follows that if S is a maximal antisymmetric set for $H^\infty + C$, a fiber X_λ, $\lambda \in \mathbf{T}$, or of the form $\pi^{-1}(\xi_0)$, $\xi_0 \in M_{QC}$, then $H^\infty + C|_S = H^\infty|_S$ is a closed subalgebra of $L^\infty|_S$. Actually, this forces S to be a weak peak set for H^∞ (see [9, page 192]). In particular we may consider the algebras H^∞_S. The notations H^∞_λ (respectively, $H^\infty_{\xi_0}$) will be used in case S equals X_λ, $\lambda \in \mathbf{T}$ (respectively, $\pi^{-1}(\xi_0)$, $\xi_0 \in M_{QC}$).

If λ belongs to $\mathbf{T}$, then it is well known that an inner function is invertible in H^∞_λ if and only if it is constant on X_λ. Thus X_λ has property (*). Actually, the result corresponding to the Chang-Marshall Theorem for the algebras H^∞_λ was obtained by Davie Gamelin and Garnett [5]. For the partition over QC we have the following:

PROPOSITION 4. *Let ξ_0 be in M_{QC}. Then $\pi^{-1}(\xi_0)$ has property (*).*

Proof. The maximal ideal space of $H^\infty_{\xi_0}$ is homeomorphic to the collection of homomorphisms γ of $H^\infty + C$ whose representing measures μ_γ are either point masses or supported in $\pi^{-1}(\xi_0)$. Let ϕ be an inner function satisfying $\bar{\phi} \in H^\infty_{\xi_0}$. Then $\phi(\gamma) \neq 0$, for all $\gamma \in M_{H^\infty + C}$ with supp $\mu_\gamma \subset \pi^{-1}(\xi_0)$. There exists a neighborhood U of ξ_0 in M_{QC} such that $\phi(\gamma) \neq 0$ for any γ in $M_{H^\infty + C}$ with supp $\mu_\gamma \subset \pi^{-1}(\xi)$, $\xi \in U$. Assuming the existence of such a neighborhood, the proof may be completed as follows. Choose an element f in QC such that $0 \leq f \leq 1$, $f(\xi_0) = 1$ and $f(\xi) = 0$, $\xi \notin U$. Then $f\bar{\phi}|_{\pi^{-1}(\xi)} \in H^\infty + C|_{\pi^{-1}(\xi)}$ for all $\xi \in M_{QC}$. Since each maximal antisymmetric set for $H^\infty + C$ is contained in a single element of the partition over QC, function $\overline{f\phi}$ belongs to $H^\infty + C$ by Bishop's

Theorem (see [9, page 39]). It follows that $f\phi$ belongs to QC and, therefore, $\phi|_{\pi^{-1}(\xi_0)} = f\phi|_{\pi^{-1}(\xi_0)}$ is constant.

We now discuss the existence of the neighborhood U. Let

$$\Gamma: M_{H^\infty+C} \to M_{QC}$$

denote the restriction map. Then Γ is continuous and surjective ($M_{H^\infty+C} \supset M_{L^\infty}$). The set $\Gamma^{-1}(\xi_0)$, for $\xi_0 \in M_{QC}$, is exactly the collection of homomorphisms of $H^\infty + C$ with representing measures supported in $\pi^{-1}(\xi_0)$. Using this fact and the continuity of Γ one obtains the desired neighborhood. This ends the proof.

We now turn to factorizations of Blaschke products relative to these partitions. We first consider the partition over C. Let λ_0 be in $\mathbf{T}$ and let ϕ be an arbitrary Blaschke product with zero sequence $\{z_n\}$. Let U and V be open subarcs of $\mathbf{T}$ with $\lambda_0 \in U \subset \bar{U} \subset V$. Let Ω be an open rectangle of the unit disc such that $\partial\Omega \cap \mathbf{T} = \bar{U}$. Let ϕ_1 denote the Blaschke product with zero sequence $\{z_n\} \cap \Omega$ and $\phi_2 = \phi\bar{\phi}_1$. The function ϕ_2 is continuous at λ_0 and hence constant on X_{λ_0}, the fiber over λ_0. We may assume that $\phi_2 \equiv 1$ on X_{λ_0} and hence $\phi = \phi_1$ on X_{λ_0}. Likewise ϕ_1 is continuous on the set $\mathbf{T}\backslash V$ and therefore ϕ_1 is constant on the fibers X_λ, $\lambda \notin V$. We now show that a similar factorization is possible for elements in the partition over QC with extra assumptions on ϕ.

PROPOSITION 5. *Let ξ_0 be in M_{QC} and assume $\xi_0 \in V$ with V open in M_{QC}. If ϕ is a Blaschke product with interpolating zero sequence $\{z_n\}$, then ϕ can be factored as $\phi_1\phi_2$ where ϕ_1 equals ϕ on $\pi^{-1}(\xi_0)$ and ϕ_1 is constant on each set $\pi^{-1}(\xi)$, wherever $\xi \notin V$.*

Proof. Let $[H^\infty + C]^*$ denote the dual space of $H^\infty + C$. Choose an open U with $\xi_0 \in U \subset \bar{U} \subset V$. Let $\mathscr{U} = \Gamma^{-1}(U)$ and $\mathscr{V} = \Gamma^{-1}(V)$. There is an open subset $\mathscr{H} \subset [H^\infty + C]^*$ with $\mathscr{H} \cap M_{H^\infty+C} = \mathscr{U}$ and

$$\bar{\mathscr{H}} \cap (M_{H^\infty+C} \backslash \mathscr{V}) = \varnothing$$

where the bar denotes closure in $[H^\infty + C]^*$. Let $\mathscr{D}$ denote the unit disc as a subset of $[H^\infty + C]^*$ and set $\Omega = \mathscr{H} \cap \mathscr{D}$. Let ϕ_1 be the Blaschke product with zero set $\{z_n\} \cap \Omega$ and $\phi_2 = \phi\bar{\phi}_1$. We now show that ϕ_2 does not vanish on $\mathscr{U}$. Suppose $\gamma \in \mathscr{U}$ and $\phi_2(\gamma) = 0$. Since the zeros of ϕ_2 are interpolating, there exists a net $\{z_\alpha\}$ consisting of zeros of ϕ_2 with z_α with $z_\alpha \to \gamma$ in $[H^\infty + C]^*$ (see [8, page 206]). Then some z_{α_0} is in $\mathscr{H}$ which is impossible by construction. It follows that ϕ_2 is constant on $\pi^{-1}(\xi_0)$ and we may assume this constant value is one. Similarly it follows that ϕ_1 is constant on $\pi^{-1}(\xi)$ for $\xi \notin V$. This completes the proof.

4. Local Toeplitz operators

There are other techniques of localization for Toeplitz operators different than the notion of restricted invertibility. One of these techniques, first introduced in [6] (see, also [1] and [7]) will be developed further below.

Let $\mathscr{T}$ be the C^*-algebra generated by all Toeplitz operators and $\mathscr{T}^\circ$ the algebra $\mathscr{T}/\mathscr{K}$ where $\mathscr{K}$ is the ideal of compact operators on H^2. For S a closed subset of M_{L^∞} let $a(S)$ be the smallest closed two sided ideal in $\mathscr{T}^\circ$ generated by elements of the form T_f° where f is in L^∞ and f vanishes on S. The quotient algebra $\mathscr{T}^\circ/a(S)$ will be denoted by $\mathscr{T}_S$ and if T is in $\mathscr{T}$, then T^S will denote its projection in $\mathscr{T}_S$. For $f \in L^\infty$, the element T_f^S is called a local Toeplitz operator. Axler [1] has observed that the basic identity $\|f\|_S = \|T_f^S\|$ holds for f in L^∞.

Definition. Let f be in L^∞. The Toeplitz operator T_f will be said to be S-locally invertible in case T_f^S is invertible in $\mathscr{T}_S$.

A definition similar to the preceding is made for S-local left-invertibility (right-invertibility).

In this section we are concerned with the meaning of invertibility of T_f^S in terms of T_f back on H^2. We consider only the case where S is an element in the partition over C or QC.

Let B be a closed self-adjoint subalgebra of L^∞ that satifies $C \subset B \subset QC$. The notation π will continue to denote the natural surjection $\pi\colon M_{L^\infty} \to M_B$. For the case where $S = \pi^{-1}(b)$ we will use the notation $\mathscr{T}_b$ for the algebra $\mathscr{T}^\circ/a(\pi^{-1}(b))$ and T_f^b for the local operator T_f^S, $f \in L^\infty$. The following restatement of a result of Douglas [6] will play a role in the sequel.

Theorem 1. *Let B be a self-adjoint subalgebra of L^∞ satisfying $C \subset B \subset QC$. Then $\bigcap_{b \in M_B} a(\pi^{-1}(b)) = (0)$ and for $T \in \mathscr{T}$ the mapping $b \to \|T^b\|$ is an upper-semi-continuous function on M_B.*

The above theorem implies:

Corollary 1. *Let B be a self-adjoint closed subalgebra of L^∞ satisfying $C \subset B \subset QC$. Suppose $f \in L^\infty$ and for each b in M_B the local Toeplitz operator T_f^b is invertible, then the operator T_f is Fredholm on H^2. Similarly, if for each b in M_B the local Toeplitz operator T_f^b has a left-inverse (right-inverse), then T_f is left-Fredholm (right-Fredholm) on H^2.*

The following result was obtained in the case $B = C$ in [4].

Proposition 6. *Assume $B = C$ or $B = QC$ and let b_0 be a fixed point in M_B. Let $f \in L^\infty$. The following are equivalent.*

(i) *The operator T_f is restricted $\pi^{-1}(b_0)$-left-invertible (right-invertible).*
(ii) *The local Toeplitz operator $T_f^{b_0}$ is left-invertible (right-invertible).*

Proof. If either (i) or (ii) holds, then f cannot vanish on $\pi^{-1}(b_0)$. It can therefore be assumed that $|f| = 1$ a.e. The equivalence of statements (i) and (ii) for the case of left-invertibility implies the equivalence for the case of right-invertibility as can easily be seen by taking adjoints. It is clear that (i) implies (ii).

Assume that $|f| = 1$ a.e. and $T_f^{b_0}$ has a left-inverse L^{b_0} for some L in $\mathcal{T}^\circ$. From the semi-continuity of $\|(I - LT_f)^b\|$ it follows that there is a neighborhood U of b_0 in M_B such that T_f^b is left-invertible for $b \in U$.

Let $0 < \varepsilon < 1/2$. A result of Ziskind [15] implies that there is a g in H^∞ and a finite product of interpolating Blaschke products ϕ such that $\|f - g\bar{\phi}\| < \varepsilon$. Using Proposition 5, for the case $B = QC$, or the remark preceding Proposition 5, for the case $B = C$, we can factor $\phi = \phi_1 \phi_2$ where $\phi = \phi_1$ on $\pi^{-1}(b_0)$ and ϕ_1 is constant on each set of the form $\pi^{-1}(b)$ where $b \notin U$.

Set $\tilde{f} = \phi_2 f$ and note that $\|\tilde{f} - g\bar{\phi}_1\|_\infty < \varepsilon$. This forces

$$|g(t)| \geq 1 - \varepsilon \quad \text{a.e.}$$

and therefore $T_{g^{-1}}$ is a left-inverse for T_g. For $b \notin U$ we have $T^b_{g\bar{\phi}_1} = c(b)T^b_g$, where $c(b)$ denotes the constant value of $\bar{\phi}_1$ on $\pi^{-1}(b)$. Thus $\overline{c(b)}\, T^b_{g^{-1}}$ is a left-inverse for $T^b_{g\bar{\phi}_1}$ and this left-inverse has norm at most $(1 - \varepsilon)^{-1}$. The inequality $\|[T_{\tilde{f}} - T_{g\bar{\phi}_1}]^b\| < \varepsilon < 1/2$ insures that $T^b_{\tilde{f}}$ has a left-inverse for $b \notin U$. For $b \in U$, the local Toeplitz operator $T^b_{\tilde{f}}$ equals $T^b_f T^b_{\phi_2}$ and hence is left-invertible. Corollary 1 implies that $T_{\tilde{f}}$ is a left-Fredholm Toeplitz operator.

If $T_{\tilde{f}}$ is not left-invertible, then $T_{\tilde{f}}$ is actually Fredholm. This follows because of a result of Coburn (see, e.g., [6]) which shows that one of the kernels of $T_{\tilde{f}}$ or $T^*_{\tilde{f}}$ is trivial. Let n be the index of $T_{\tilde{f}}$. Set $\chi^n(\theta) = e^{in\theta}$ and $g = \chi^n \tilde{f}$. The Toeplitz operator T_g is invertible. Every continuous function is constant on $\pi^{-1}(b_0)$ and by modifying g by a constant, if necessary, one can assume $g = \tilde{f} = f$ on $\pi^{-1}(b_0)$. This completes the proof.

5. Two-sided invertibility

We are now in a position to prove the following theorem which shows that the various notions of local invertibility are equivalent. We remark that for the case of a fiber X_λ, S. Y. Chang has given an independent proof along the lines of classical function theory that statements (i) and (ii) are equivalent. Further R. G. Douglas has observed the equivalence of (iii) and (iv) follows from his work in [7]. The equivalence of (iii) and (i) is the main contribution here.

THEOREM 2. *Let B be one of the algebras C or QC and let b_0 be a fixed point in M_B. Let f belong to L^∞ and assume that f is unimodular on $\pi^{-1}(b_0)$. The following are equivalent.*

(i) *The operator T_f is $\pi^{-1}(b_0)$-restricted invertible.*
(ii) *The operator T_f is $\pi^{-1}(b_0)$-restricted left-invertible and $\pi^{-1}(b_0)$-restricted right-invertible.*
(iii) *The local Toeplitz operator $T_f^{b_0}$ is invertible.*
(iv) $\text{dist}_{\pi^{-1}(b_0)}(f, (H^\infty)^{-1}) = \text{dist}(f, (H^\infty_{b_0})^{-1}) < 1$.

Proof. Statements (i) and (iv) are equivalent by Proposition 3. The equivalence of (ii) and (iii) is provided by Proposition 6. Statement (i) trivially implies (ii). The proof will be complete when it is shown that (ii) implies (iv).

Assume T_f is both $\pi^{-1}(b_0)$-restricted left-invertible and $\pi^{-1}(b_0)$-right-invertible. It follows from Proposition 2 and the Chang-Marshall Theorem that there is an h in H^∞ and an inner function ϕ which equals one on $\pi^{-1}(b_0)$ such that $\|f - h\bar{\phi}\|_\infty < 1$. Let $h = \psi g$ be the factorization of h into a product of an outer function g and an inner function ψ. The inequality $\|1 - \overline{f\phi\psi g}\|_\infty < 1$ shows that $g \in (L^\infty)^{-1}$ and hence belongs to $(H^\infty)^{-1}$. Moreover, the local Toeplitz operator

$$T^{b_0}_{\bar{f}\phi\psi g} = T^{b_0}_{\bar{f}} T^{b_0}_{g} T^{b_0}_{\psi}$$

is invertible. Since $T^{b_0}_g$ and $T^{b_0}_{\bar{f}}$ are invertible, then $T^{b_0}_\psi$ is invertible. In particular, Proposition 2 implies dist $(\bar{\psi}, H^\infty_{b_0}) < 1$ and so $\bar{\psi} \in H^\infty_{b_0}$. The function $h\bar{\phi} = \psi g\bar{\phi}$ is in $(H^\infty_{b_0})^{-1}$. This ends the proof.

6. Conclusion

The results in [4] establish the equivalence of invertibility of the local operator T^λ_f, $\lambda \in \mathbf{T}$, and a local factorization of the symbol f at λ. This notion of local factorization was first defined in [3]. Theorem 2 yields a simpler (but less direct) argument to establish that the invertibility of the local operator T^λ_f implies the desired local factorization.

The results in Proposition 6 and Theorem 2 can be established in case $\pi^{-1}(b_0)$ is replaced by a maximal antisymmetric set for $H^\infty + C$. The arguments in this case use a suitable variation of Proposition 5 and the transfinite induction approach to maximal antisymmetric sets. The transfinite method is explained in [1].

Finally we note that the proof of the extension property (Proposition 1) depends on property (*). The authors do not know whether this proposition can be extended to the case where S is an arbitrary weak peak set for H^∞. There are examples of function algebras $A \subset C(X)$ and subsets $S \subset X$ where $A|_S$ is closed and the extension property fails. These examples do not have the strong logmodularity behavior of the algebras discussed in [1].

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