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ON EQUISINGULAR DEFORMATIONS OF PLANE CURVE SINGULARITIES

BY

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Introduction

In [6], a theory of equisingular deformations of irreducible algebroid plane curves (over an algebraically closed field, k, of characteristic 0) based on parametrizations and characteristic numbers was introduced. It was proved there also that this is equivalent to other similar theories previously known, and several applications of those methods were given. In the present paper the case of reducible curves is studied. As is common in this type of problem, the transition from the irreducible to the reducible case is not always straightforward, and often completely different proofs must be given. The contents of this paper are the following.

In Section 1, we define the intersection number of two equisingular deformations of plane branches; with this concept we may define equisingular deformations of a reducible curve. We study basic properties of this concept, and we compare it with Zariski's and Wahl's definition of equisingularity (cf. [10] and [12]).

In Section 2 we show that an equisingular deformation is determined by a "sufficiently high truncation" (depending on the equivalence class of the curve only); see (2.1) for details. In [6], a similar theorem (for deformations of an irreducible curve) was proved. The proof given here is completely different, since apparently the proof of [6] cannot be adapted to the general case.

In Section 3, we present the main result of this paper: given a plane algebroid curve C, there is an equisingular algebraic family of curves $\mathfrak{F} = (\pi, X, V, \varepsilon)$ (see (3.1) for the definitions), with V smooth, such that for any curve D, equivalent to C, there is a closed point $y \in V$ such that D is isomorphic to Spec $(\hat{\mathcal{O}}_{X,\varepsilon(y)})$. Moreover, the induced family π_y: Spec $(\hat{\mathcal{O}}_{X,\varepsilon(y)}) \to$ Spec $(\hat{\mathcal{O}}_{V,y})$ is "versal," in the sense that any equisingular deformation of Spec $(\hat{\mathcal{O}}_{X_y,\varepsilon(y)})$ is isomorphic to some pull-back of π_y. In the construction of this family we use the results of Sections 1 and 2.

In [16], Zariski presents some interesting results about the problem of moduli for plane algebroid branches (using the techniques that inspired [6] and the present paper). We believe that the main result of Section 3 is a first step to study that problem in the reducible case.

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Related to Section 3, there is an interesting question that we are not able to answer in general: is the parameter space V irreducible (or is there a similar construction with an irreducible parameter space)?

At the end of Section 3, we indicate how the theory can be developed in the complex analytic case.

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0. Notations and terminology

In this paper we shall follow the notations and terminology of [6]. We briefly review some of them. For more details, see [6, Section 0]. The letter k will denote an algebraically closed field of characteristic zero. The category of complete local k-algebras with residue field k (resp. finite dimensional k-algebras) is denoted by $\mathscr{A}_c$ (respectively $\mathscr{A}$). If $A \in \mathscr{A}_c$, $r(A)$ denotes the maximal ideal of $\mathscr{A}$. The order of a power series ϕ is denoted by $O(\phi)$. If $\phi \in A[\![x_1, \ldots, x_n]\!]$, res (ϕ) denotes the power series in $k[\![x_1, \ldots, x_n]\!]$ obtained from ϕ by reducing the coefficients mod $r(A)$.

An algebroid plane curve (over k) is a scheme Spec $k[\![x, y]\!]/(f_0)$, where $f_0 \in k[\![x, y]\!]$ has no multiple factors. Sometimes the ring $k[\![x, y]\!]/(f_0)$ or even f_0 itself is called an algebroid plane curve (actually, to simplify the notations, we follow this convention most of the time). When f_0 is irreducible, we call it a branch.

When we talk about equivalent curves, it will be in Zariski's sense (cf. [12] or [13]). This notion of equivalence is an equivalence relation in the set of all algebroid plane curves. An equivalence class of this relation will be called an equisingular type (or just a type); if a curve f_0 belongs to the equisingular type α we say that f_0 has type α. Thus, to say that curves f_0, g_0 are equivalent is the same as saying that they have the same equisingular type.

1. Equisingular deformations of plane algebroid curves

(1.1) In this section, we study a theory of equisingular deformations of plane algebroid curves, based on parametrizations and intersection numbers.

We recall that in [6] a theory of equisingular deformations of a branch, based on parametrizations, was introduced. Throughout this section, we shall use the results of that paper.

(1.2) Let $f_0^{(i)} \in k[\![x, y]\!]$, $i = 1, \ldots, r$, be distinct branches, $A \in \mathscr{A}_c$ and $f^{(i)} \in A[\![x, y]\!]$, $i = 1, \ldots, r$, an equisingular deformation of $f_0^{(i)}$ over A (in the sense of [6]). Assume that $f^{(i)}$ has a parametrization $(t^{m_i}, \phi_i(t))$, $\phi_i \in A[\![t]\!]$. In this case, we define the intersection number $(f^{(i)} \cdot f^{(j)})$ of the deformations $f^{(i)}$ and $f^{(j)}$ $(i \neq j)$ by

$$(1.2.1) \qquad (f^{(i)} \cdot f^{(j)}) = \min\,(O(f^{(i)}(t^{m_j}, \phi_j)), O(f^{(j)}(t^{m_i}, \phi_i))).$$

(1.3) We list some basic results which are easily verified:

(a) If $f^{(i)}$, $i = 1, \ldots, r$ has a parametrization (α_i, β_i), $\alpha_i = a_{m_i} t^{m_i} + \cdots$, a_{m_i} a unit in A, then it also has a parametrization $(t^{m_i}, \phi_i(t))$ (cf. [6, Proposition 1.5]).

(b) It is easy to see that if we replace $(t^{m_i}, \phi_i(t))$ by another parametrization $(t^{m_i'}, \phi_i'(t))$ of $f^{(i)}$, for $i = 1, \ldots, r$, then $m_i' = m_i$ and the number $(f^{(i)} \cdot f^{(j)})$ does not change (cf. [6, Proposition 1.11]).

(c) The numbers $(f^{(i)} \cdot f^{(j)})$ are invariant under "changes of coordinates" $x = x'$, $y = \lambda x' + \mu y'$, λ, μ in A, μ a unit.

(d) If $f^{(i)}$ has a parametrization $(\psi_i(\tau), \tau^{n_i})$, $i = 1, \ldots, r$, in a similar way we may define intersection numbers. If $f^{(i)}$, $i = 1, \ldots, r$ admits both parametrizations as in (1.2) and as above, then the intersection numbers obtained by using either of them are the same (since $\tau = bt + \cdots$, b a unit of A).

(1.4) *Remark.* With the notation of (1.2), it could happen that

$$O(f^{(i)}(t^{m_j}, \phi_j) \neq O(f^{(j)}(t^{m_i}, \phi_i).$$

For instance, let $A = k[\varepsilon]$ where $\varepsilon^2 = 0$. Let $f^{(1)} = y^2 - x^3$ be parametrized by (t^2, t^3) and let $f^{(2)} = y - \varepsilon x$ be parametrized by $(t, \varepsilon t)$. They are equisingular deformations of $f_0^{(1)} = y^2 - x^3$ and $f_0^{(2)} = y$, respectively. We have

$$O(f^{(1)}(t, \varepsilon t)) = 3, \qquad O(f^{(2)}(t^2, t^3)) = 2.$$

(1.5) In the rest of this section, when we deal with a series $f_0 \in k[\![x, y]\!]$ we shall assume (unless it be otherwise specified) that $O(f_0(x, y)) = O(f_0(0, y))$. Geometrically, this means that the y-axis is not tangent to the curve $f_0 = 0$. This is not a real restriction, since by a linear change of the variables we can reach this situation.

(1.6) DEFINITION. Let $f_0(x, y)$ be a reduced plane curve over k (cf. (1.5)), $f(x, y) \in A[\![x, y]\!]$, $A \in \mathscr{A}_c$ a deformation of f_0. Let $f_0 = \prod_{i=1}^r f_0^{(i)}$ be the product of f_0 into its prime factors in $k[\![x, y]\!]$. We say that f is an equisingular deformation of f_0 if we can write $f = \prod_{i=1}^r f^{(i)}$, in such a way that

(a) $f^{(i)}$ is an equisingular deformation of the branch $f_0^{(i)}$, $i = 1, \ldots, r$ (in the sense of [6, Definition 2.3]) and

(b) $(f_0^{(i)} \cdot f_0^{(j)}) = (f^{(i)} \cdot f^{(j)})$ for all $i \neq j$.

Note that by (1.5) and (1.3), (b) makes sense, and that for $r = 1$, this definition reduces to Definition 2.3 of [6].

(1.7) *Remark.* (a) Assume that (using the notation of (1.6)) f is an equisingular deformation of f_0, let $(t^{m_i}, \phi_i(t))$ be a parametrization of $f^{(i)}$, $i = 1, \ldots, r$. Then

$$O(f^{(i)}(t^{m_j}, \phi_j)) = O(f^{(j)}(t^{m_i}, \phi_i)).$$

In fact, say $(f^{(i)} \cdot f^{(j)}) = O(f^{(i)}(t^{m_j}, \phi_j))$. Let $f^{(i)}(t^{m_j}, \phi_j) = at^N + \cdots$. By (b) of Definition (1.6), a must be a unit. Now, as is well known,

$$O(f_0^{(i)}(t^{m_j}, \text{res } (\phi_j)) = O(f_0^{(j)}(t^{m_i}, \text{res } (\phi_i))) = N$$

(cf. [11]). Let $f^{(j)}(t^{m_i}, \phi_i) = bt^M + \cdots, b \neq 0$. Its image in $k[\![t]\!]$ is $f_0^{(j)}(t^{m_i}, \text{res } (\phi_i))$, hence either $M = N$ and b is a unit, or $M < N$. But Definition (1.2.1) rules out the second possibility. Note that in (1.4), $f^{(1)} \cdot f^{(2)}$ is not an equisingular deformation of $y^3 - x^3y$.

(b) In checking that a deformation is equisingular, we may change the variables by $x = x'$, $y = \lambda x' + \mu y'$, $\lambda, \mu \in A$, μ a unit of A (cf. (1.3) c).

(1.8) If $\rho: A \to A'$ is a homomorphism in $\mathscr{A}_c$ and $f \in A[[x, y]]$ is an equisingular deformation of f_0 over A, then there is naturally induced equisingular deformation $\rho^*(f)$ of f_0 over A' (obtained by replacing each coefficient of A by its image in A').

The following lemma will be essential in inductive arguments. Essentially, it says that there is a bijection between the tangential components of f_0 and those of its equisingular deformation f.

(1.9) LEMMA. *Let f, g, fg be equisingular deformations over $A \in \mathscr{A}_c$ of curves $f_0, g_0, f_0 g_0$, where f_0, g_0 are irreducible and have the same tangent line $y = 0$. Then*

$$f = (y - \alpha x)^n + \cdots, \qquad g = (y - \alpha x)^m + \cdots$$

(i.e., the initial forms of f and g are powers of the same binomial $y - \alpha x$, $\alpha \in r(A)$).

Proof. We may assume that A is artinian. In fact, if the lemma is proved in this case, given $A \in \mathscr{A}_c$ and assuming

$$f = (y - \alpha x)^n + \cdots, g = (y - \alpha' x)^m + \cdots, \quad \alpha \neq \alpha',$$

then for some i large enough, the images $\bar{\alpha}, \bar{\alpha}'$ of α, α' in $\bar{A} = A/\mathscr{M}^i$ ($\mathscr{M} = r(A)$) will be different. Then, by considering the deformations $\bar{f}, \bar{g}$, induced by f and g over $\bar{A}$, we shall get a contradiction. So, we assume A artinian, and we prove the lemma by induction on $q = \dim_k \mathscr{M}$, $\mathscr{M} = r(A)$.

The lemma is trivial for $q = 0$. Assume it true for q. Given $(A, \mathscr{M})$, with $\dim A = q + 1$, consider a small extension $A \to A'$ of kernel $I = (\varepsilon)$; let $u_1, u_2, \ldots, u_q, \varepsilon$ be a basis of the k-vector space $\mathscr{M}$. We may assume, after a change of coordinates and by using induction, that

(1.9.1) $$f = y^n + \cdots, \qquad g = (y - \lambda\varepsilon x)^m + \cdots,$$

where $\lambda \in k$. We must show that $\lambda = 0$. Assume by contradiction that $\lambda \neq 0$. Consider equisingular parametrizations

(1.9.2) $$(x = t^n, y = \phi(t)) \qquad (x = \tau^m, y = \psi(\tau))$$

of f and g, respectively. In view of (1.9.1), we have

$$\phi(t) = \phi'(t) + \phi_1(t)\varepsilon, \tag{1.9.3}$$

$$\psi(\tau) = \psi'(\tau) + \psi_1(\tau)\varepsilon, \tag{1.9.4}$$

where the coefficients of ϕ' (respectively ψ') are k-linear combinations of $u_1, \ldots, u_q$, and

$$\phi_1(t) = (\delta t^d + \cdots) \in k[\![t]\!], \quad d > n, \tag{1.9.5}$$

$$\psi_1(\tau) = (\lambda\tau^m + \cdots) \in k[\![\tau]\!]. \tag{1.9.6}$$

With the relations $g(x, y) = \prod_{i=1}^{m} y - \psi(\omega^j x^{1/m})$, with ω a primitive mth root of 1, and $\varepsilon\mathcal{M} = 0$, a simple computation gives

$$g(x, \phi(x^{1/n})) = g(x, \phi'(x^{1/n})) + \varepsilon\phi_1(x^{1/n})\frac{\partial g_0}{\partial y}(x, \phi_0(x^{1/n})) + \varepsilon g_1(x, \phi_0(x^{1/n})), \tag{1.9.7}$$

with $g = g'(x, y) + \varepsilon g_1(x, y)$, where $g_1 \in k[\![x, y]\!]$ and the coefficients of g' are linear combinations of $u_1, \ldots, u_q$. Now recall that

$$(f_0 \cdot g_0) = (f \cdot g) = O(g(t^n, \phi(t)) = O(f(\tau^m, \psi(\tau));$$

note that $O(g(t^n, \phi'(t)) = O(\bar{g}(t^n, \bar{\phi}(t))$, where $\bar{g}$, $\bar{\phi}$ are induced by g, ϕ, respectively, by reducing the coefficients mod I. Thus, by induction, $(f_0 \cdot g_0) = O(g(t^n, \phi'(t))$. We shall check that either

$$O(g_1(t^n, \phi_0(t))) < \min\left\{(f_0 \cdot g_0), O\left(\phi_1(t)\frac{\partial g_0}{\partial y}(t, \phi_0)\right)\right\} \tag{1.9.8}$$

(here $\phi_0 = \text{res}\,(\phi)$) or, after "interchanging the roles of f and g" (the meaning of this is made precise below) and using notations similar to those used in (1.9.7),

$$O(f_1(\tau^m, \psi_0(\tau))) < \min\left\{(f_0, g_0), O\left(\psi_1(\tau)\frac{\partial f_0}{\partial y}(\tau, \psi_0(\tau))\right)\right\}. \tag{1.9.9}$$

Thus, in either case, $(f \cdot g) < (f_0 \cdot g_0)$, contradicting the definition of equisingularity.

By "interchanging the roles of f and g" we mean the following.

(a) Set $x' = x$, $y' = y - \lambda\varepsilon x$, so that now

$$f(x', y' + \lambda\varepsilon x) = f'(x', y') = (y' + \lambda\varepsilon x)^n + \cdots, \qquad g' = y'^m + \cdots.$$

(b) Consider the parametrizations induced by ϕ and ψ, and proceed as before (i.e., as between (1.9.1) and (1.9.8)), with f replaced by g', g by f'.

To verify (1.9.8) (or (1.9.9)), we shall study the series $g_1(x, y)$ more carefully. We have

$$g_1(x, y) = \sum_{i=1}^{m} \psi_1(\omega^i x^{1/m}) \prod_{j \neq i} (y - \psi_0(\omega^j x^{1/m})), \tag{1.9.10}$$

$$\frac{\partial g_0}{\partial y} = \sum_{i=1}^{m} \left(\prod_{j \neq i} y - \psi_0(\omega^j x^{1/m}) \right). \tag{1.9.11}$$

Let

$$\phi_0(t) = \sum_{h=p}^{\infty} a_h t^h,\ a_p \neq 0; \qquad \psi_0(\tau) = \sum_{s=l}^{\infty} b_s \tau^s,\ b_l \neq 0. \tag{1.9.12}$$

There are two possibilities: $mp \neq ln$ and $mp = ln$. Assume $mp < nl$. Then

$$\phi_0(t) - \psi_0(\omega^j t) = a_p t^p + \cdots, \quad j = 1, \ldots, m.$$

Since $\psi_1(\omega^i x^{1/m}) = \lambda x + \cdots$ we get $O(g_1(t^n, \phi_0(t)) = n + (m-1)p$. On the other hand, it is easily checked that

$$O\left(\phi_1(t) \frac{\partial g_0}{\partial y} (t^n, \phi_0(t)) \right) = (m-1)p + d$$

(cf. (1.9.5)) and $(f_0 g_0) = mp$. But $p > n$, $d > n$, then (1.9.8) holds in this case.

If $mp > nl$, we interchange the roles of f and g (cf. the explanation given after (1.9.8)) and (1.9.9) is verified with a similar argument.

Now we assume $mp = nl$. We write $P_j = \phi_0(x^{1/m}) - \psi_0(\omega^j x^{1/m})$, $j = 1, \ldots, m$. We may assume:

(1.9.13) If some P_j is not of the form $(a_h - b_s \omega^{js}) x^{h/n} + \cdots$, $a_h \neq 0$, then there is a $P_{j'} = a_{h'} x^{h'/n} + \cdots$, $a_{h'} \neq 0$ (cf. (1.9.12)).

In fact, if this does not happen, there is a j such that

$$P_j = -b_s \omega^{js} x^{s/m} + \cdots, \quad b_j \neq 0.$$

Interchanging the roles of f and g (and using the parametrization $\psi_0(\omega^j t)$ rather than ψ_0) we have, in the new situation, the analogous condition satisfied. So, assume that (1.9.13) holds and write

$$P_j(t) = \phi_0(t) - \psi_0(\omega^j t^{n/m}).$$

There are several cases to be considered. Let

$$O(P_{j_0}) = \min \{O(P_j)/j = 1, \ldots, m\}.$$

Case 1. $P_{j_0} = b_h \omega^{j_0 h} x^{h/m} + \cdots$, $b_h \neq 0$. Then,

$$P_j = b_h \omega^{jh} x^{h/m} + \cdots \quad \text{for all } j.$$

This contradicts assumption (1.9.13), so that this case is ruled out.

Case 2. $P_{j_0} = a_h x^{h/m} + \cdots$. Then, $P_j = a_h x^{h/n} + \cdots$, and all the terms of the sum (1.9.10) are of the form $\lambda a_h^{m-1} x^{(m-1)h/n+1} + \cdots$. An easy computation (using $O(\phi_1) > n$) shows that (1.9.8) holds.

Case 3. $P_{j_0} = (a_h - b_s \omega^{j_0 s}) x^{h/n} + \cdots$, $a_h b_s \neq 0$, moreover, $a_h \neq b_s \omega^{js}$ for $j = 1, \ldots, m$. An elementary argument with symmetric functions and roots of unity shows

$$\sum_{i=1}^{m} \prod_{i \neq j} (a_h/b_s - \omega^{sj}) = m(a_h/b_s)^{m-1}. \tag{1.9.14}$$

With (1.9.14) it is easily seen that $O(P_j(t)) = h$ for all j, that $g_1(t^n, \phi_0(t)) = m\lambda(a_h/b_s)^{m-1} t^{(m-1)h+n} + \cdots$, and that, using these, (1.9.8) holds.

Case 4. P_{j_0} as in Case 3, but now there is some $j \in \{1, \ldots, m\}$ such that $a_h = b_s \omega^{js}$. Let $d_0 = (m, h)$, and $\omega_0 = \omega^{d_0}$. Then there are d_0 values of j, say $j_1, \ldots, j_{d_0}$, satisfying $a_h = b_s \omega^{js}$. For the remaining indices j we have $O(P_j(t)) = h$. We may assume $O(P_{j_1}) \leq O(P_{j_i})$, $i = 1, \ldots, d_0$. Now, by (1.9.13) (cf. the argument of Case 1) there are only two possibilities.

Case 4.0. $P_{j_1} = a_l x^{l/n} + \cdots$, $a_l \neq 0$. Then, $P_j = a_l x^{l/n} + \cdots$, $j = j_1, \ldots, j_{d_0}$, and a simple computation gives

$$O(g_1(t^n, \phi_0(t))) = h(m - d_0) + l(d_0 - 1) + n,$$
$$O(g_0(t^n, \phi_0(t))) = h(m - d_0) + ld_0,$$
$$O\left(\phi_1(t) \cdot \frac{\partial g_0}{\partial y}(t, \phi_0)\right) = h(m - d_0) + l(d_0 - 1) + O(\phi_1)$$

by (1.9.11). Thus, (1.9.8) follows.

Case 4.1. $P_{j_1} = (a_l - b_{n_1} \omega^{h_1 j_1}) x^{l/n} + \cdots$, $a_l b_{h_1} \neq 0$. There are two subcases: either $a_l - b_{h_1} \omega^{h_1 j_1} \neq 0, j = j_1, \ldots, j_{d_0}$, or not. In the first case, we obtain (1.9.8) as in Case 3 (note that (ω^{j_1}) is a primitive d_0-root of 1), concluding the proof. In the second case, let $d_1 = (d_0, h_1)$; note that $d_1 < d_0$. We proceed as in Case 4. Let $j = q_1, \ldots, q_{d_1}$ be the indices $j \in \{j_1, \ldots, j_{d_0}\}$ satisfying $a_l - b_{h_1} \omega^{h_1 j} = 0$; for the remaining indices $O(P_j(t)) = h_1$, etc. Since $d_1 < d_0$, it is clear that, repeating the process a finite number of times, we shall get, eventually, relation (1.9.8) in any case. Lemma (1.9) is proved.

Note that if we assume that A is an integral domain, there is a much simpler proof (by regarding f as a curve defined over an algebraic closure of A).

(1.10) Next we want to see that equisingular deformations can be "lifted." First we need some previous results.

We extend Definition (1.2) in the following obvious way: if (using the notations found there) $f^{(i)}$ and $f^{(j)}$ are "disjoint," i.e., they belong to different power series rings, we set $(f^{(i)} \cdot f^{(j)}) = 0$. Similarly, Definition (1.6) can be extended to curves with several "connected components."

Recall that it is possible to define the quadratic transform of an algebroid curve. We shall follow the conventions and notations of [6, Remark (3.5)].

(1.11) LEMMA. *Let f_0, g_0 be plane branches, of multiplicities m and n, respectively. Assume the y-axis is not tangent to either branch. Let f, g, and fg be equisingular deformations (over $A \in \mathscr{A}_c$) of f_0, g_0, and $f_0 g_0$, respectively. Then*

$$(1.11.1) \qquad (f \cdot g) = mn + (f' \cdot g')$$

and $f'g'$ is an equisingular deformation of fg.

Proof. Let $(t^m, \phi(t))$ and $(t^n, \beta(t))$ be parametrizations of f and g, respectively, where $\phi(t) = a_m t^m + \cdots$, $\beta(t) = b_n t^n + \cdots$. Then the proper transforms of f and g are $f'(x, y')$ and $g'(x, y'')$, respectively, where f' (respectively g') is obtained from $x^{-m}f(x, xy)$ (respectively $x^{-n}g(x, xy)$) by writing $y' = y - a_m$ (respectively $y'' = y - b_m$).

If f_0 and g_0 have different tangents, then their proper transforms have different origins, a fortiori f' and g' have different origins and (1.11.1) is an obvious consequence of the classical result that says that, in this case, $(f_0 \cdot g_0) = mn$ (cf. [11]).

Assume f_0 and g_0 have the same tangent. Lemma (1.9) says that $b_n = a_m = a$. Then, f and g have the same origin $(0, a)$, and (by [6, 3.6]) f', g' have strict parametrizations $(t^m, t^{-m}\phi(t) - a)$, $(t^n, t^{-n}\alpha(t) - a)$, respectively. We obtain (writing $\alpha'(t) = t^{-m}\alpha(t) - a$ and using $x^n f'(x, y - a) = f(x, xy)$)

$$f'(t^n, \alpha'(t)) = t^{-nm} f(t^n, \alpha(t)).$$

Similarly, $g'(t^m, t^{-m}\phi(t) - a) = t^{-nm} g(t^m, \phi(t))$. This shows formula (1.11.1). The rest of (1.11) is a consequence of the definitions and (1.11.1).

Now we shall prove that it is possible to "lift" deformations.

(1.12) PROPOSITION. *Let $\eta\colon A_1 \to A$ be a surjective homomorphism of rings in $\mathscr{A}_c$, $f_0 = \prod_{i=1}^r f_0^{(i)}$ a plane curve (with $f_0^{(i)}$ irreducible, $i = 1, \ldots, r$), $f = \prod_{i=1}^r f^{(i)}$ an equisingular deformation of f_0 to A, where f_i is parametrized by $(t^{n_i}, \sum_{j=n_i}^{\infty} a_j^{(i)} t^j)$. Then we have the following.*

(a) *There is an equisingular deformation f_1 of f_0 over A_1, $f_1 = \prod_{i=1}^r f_1^{(i)}$, inducing f over A, and such that $f_1^{(i)}$, $i = 1, \ldots, r$ is parametrized by $(t^{n_i}, \sum_{j=n_i}^{\infty} b_j^{(i)} t^j)$, where $\eta(b_j^{(i)}) = a_j^{(i)}$, for all j.*

(b) *Moreover, there are integers $s_1, \ldots, s_r$, depending on the equisingular type of f_0 only, such that if $g \in A[\![x, y]\!]$ is a deformation of another curve g_0 equivalent* *to* f_0, *and* $f \equiv g \bmod (x, y)^m$, *then* g *can be lifted to deformation* g_1 *of* g_0 *over* A_1, *such that* (i) $g_1 = \prod_{i=1}^r g_1^{(i)}$, *where* $g_1^{(i)}$ *is an equisingular deformation of the ith component of* g_0 *and* (ii) $g_1^{(i)}$ *admits a parametrization* $(t^n, \psi^{(i)})$, *where* $\psi^{(i)} \equiv \sum_{j=p_i}^{\infty} b_j^{(i)} t^j \bmod (t^{m-s_i})$.

Proof. We shall prove (a) by induction on $\sigma(f_0)$, the minimum numbers of quadratic transforms needed to desingularize f_0. By (1.7) (b), we may assume:

(1.12.1) Neither axis is tangent to f_0 (then it follows that a_{n_i} is a unit, $i = 1, \ldots, r$).

Write $f^{(i)} = u^{(i)} h^{(i)}$, with $h^{(i)} = y^{n_i} + \sum_{j=1}^{n_i} A_j^{(i)}(x) y^{n_i - j}$ and $u^{(i)}$ a unit of $A[\![x, y]\!]$. Then (cf. [6, Remark 1.10])

$$h^{(i)} = \prod_{j=1}^{n_i} (y - \phi_j(\omega_i x^{j/n_i}))$$

where ω_i is a primitive n_i-root of unity. Consider the quadratic transforms $f_0^{(i)\prime}$ of $f_0^{(i)}$ and $f^{(i)\prime}$ of $f^{(i)}$, $i = 1, \ldots, r$. By formula (1.11.1) and the results of [6], it follows that the (not necessarily connected) curve $f' = \prod_{i=1}^r f^{(i)\prime}$ is an equisingular deformation of $f'_0 = \prod_{i=1}^r f_0^{(i)\prime}$. Moreover, $f^{(i)\prime}$ has a parametrization

$$t^{n_i}, \phi^{(i)} = \sum_{j=n_i}^{\infty} a_j t^{j-n_i} - a_{n_i}.$$

By Lemma 3.6 of [6], the initial coefficient of $\phi^{(i)}$ is a unit (so if $O(\phi_i) < n_i$ we can pass to a parametrization of the type required in the definitions with no problems). By induction, we can lift f' to f'_1 over A_1, $f'_1 = \prod_{i=1}^r f_1'^{(i)}$, and we may assume that $f_1'^{(i)}$ admits a parametrization $(t^{n_i}, \phi_1^{(i)} = \sum b_j t^j)$, with $\phi_1^{(i)}$ inducing $\phi^{(i)}$, $i = 1, \ldots, r$. Let $b_{n_i} \in A_1$ be such that $\eta(b_{n_i}) = a_{n_i}$ (and $b_{n_i} = 0$ if $a_{n_i} = 0$), $\alpha_1^{(i)}(t) = t^{n_i} \phi_1^{(i)} + b_{p_i}$. Let

$$h_1^{(i)} = \prod_{l=1}^{n_i} (y - \alpha_1^{(i)}(\omega_i^l x^{1/n_i})) \in A_1[\![x, y]\!]$$

and let $u_1^{(i)}$ be a lifting of $u^{(i)}$, $i = 1, \ldots, r$. Then, $f_1 = \prod_{i=1}^r (u_1^{(i)} h_1^{(i)})$ lifts f to A_1. We claim that f_1 is an equisingular deformation. In fact, by Lemma 2.5 of [6], $f_1^{(i)}$ is an equisingular deformation of $f_0^{(i)}$, $i = 1, \ldots, r$; on the other hand, by applying Lemma (1.11) we check (b) of Definition (1.6). (Note that $f_1'^{(i)}$ is the quadratic transform of $f_1^{(i)}$). The assertion on the parametrizations is clear from the constructions.

Now we prove (b). Again, we assume (1.12.1) holds. Write f and g as sums of homogeneous polynomials, $f = f_n + \cdots$, $g = g_d + \cdots$. If $f \equiv g \bmod (x, y)^m$, $m > n$, then $d = n$, and $f_n = g_d$. It follows that $g = \prod_{i=1}^r g^{(i)}, g_i = (y - a_{n_i} x)^{n_i} + \cdots$, where g is a deformation of the ith branch $g_0^{(i)}$ of g_0. Hence, the connected components of the quadratic transform g' of g have the same centers as those of f. There is an induced congruence $f' \equiv g' \bmod (x, y)^{m-n}$. By induction on $\sigma(f_0)$

there are numbers $s'_1, \dots, s'_r$ such that the statement analogous to (1.12) (b) holds. It is easy to check, using arguments as in (a), that by taking $s_i = s'_i + n - n_i$, we get numbers with the property stated in (1.12) (b) (cf. [6, Lemma 3.9] for more details).

(1.13) Next, we want to see that Definition (1.6) agrees with the definitions of equisingularity given by Wahl and Zariski (cf. [10], [12], or [6, Remark 2.7]). It is known that Wahl's and Zariski's definitions are equivalent for deformations over a regular local ring (the only case when Zariski's notion is defined). As in the proof of Theorem 2.8 of [6], by using Proposition (1.11), it is enough to prove that Zariski's definition and Definition (1.6) coincide for deformations over a regular local ring $A \in \mathscr{A}_c$, i.e., $A = k[\![t_1, \dots, t_d]\!]$.

Definition (1.6) implies Zariski's definition. In fact, given the deformation $f(x, y, t)$ of a plane curve $f_0(x, y)$ over $A = k[\![t]\!]$, $t = (t_1, \dots, t_d)$, it is Zariski-equisingular if the "general" curve $f \in K[\![x, y]\!]$ (with K an algebraic closure of $k((t))$) and $f_0 \in K[\![x, y]\!]$ are equivalent (cf. [12]). But Lemma 7.1 of [13] says that f and f_0 are equivalent if and only if there is a pairing of their branches such that corresponding branches are equivalent and intersection multiplicities are preserved. It is obvious that (1.6) implies this version of Zariski equisingularity. To see the other implication, we use this result of Zariski: with notations as above, if f is a Zariski-equisingular deformation of f_0, and $f_0 = \prod_{i=1}^r f_0^{(i)}$, $f_0^{(i)} \in k[\![x, y]\!]$ and irreducible for all i, then $f = \prod_{i=1}^r f^{(i)}$, $f^{(i)} \in k[\![t, x, y]\!]$, $f^{(i)}(0, x, y) = f_0^i$, $f^{(i)}$ is equivalent to $f_0^{(i)}$ (as curves over K) and $(f^{(i)} \cdot f^{(i)}) = (f_0^{(i)} \cdot f_0^{(i)})$ (as curves over K). This is proved in [12, Section 6]. In view of Theorem 2.8 of [6], this easily implies Definition (1.6).

2. A theorem on truncations

In this section we want to prove the following result:

(2.1) THEOREM. *Fix an equisingular type α. Then, there are nonnegative integers t, r (depending on α only) such that if f_0 (respectively g_0) is an algebroid plane curve of type α, f, g are deformations of f_0 and g_0 over $A \in \mathscr{A}_c$ respectively, with f equisingular, and the congruence $f \equiv g \bmod (x, y)^v$ holds, with $v \geq t$, then there is an automorphism ϕ of $A[\![x, y]\!]$ such that $\phi(f) = g$, and such that the automorphism induced by ϕ in $A[\![x, y]\!]/(x, y)^{v-r}$ is the identity. If, moreover, $f_0 = g_0$, we may choose ϕ in such a way that the induced automorphism of $k[\![x, y]\!]$ is the identity.*

(2.2) This is a theorem of the type discussed in [4] and [5]. Actually, when $A \in \mathscr{A}_c$ is regular, Theorem (2.1) is essentially well known. In (2.3) to (2.6) we review some known results in this direction, which we shall use.

(2.3) Let $D = k[\![x_0, \dots, x_n]\!]$. By a hypersurface in $(n + 1)$-space we mean a power series $f_0 \in D$ without multiple factors. We say that f_0 has an isolated singularity at the origin if the radical of $(f, \partial f_0/\partial z_0, \ldots, \partial f_0/\partial z_n)$ is $M = (x_0, \ldots, x_n)$. If f_0 has an isolated singularity, it is known that the ideal $J_0 = (\partial f_0/\partial z_0, \ldots, \partial f_0/\partial z_n)$ has M as its radical, and hence D/J_0 is a finite dimensional k-vector space (cf. [9, 2.2]). The integer $\mu = \dim_k (D/J_0)$ is called the Milnor number of f_0.

A deformation of f_0 over $A \in \mathscr{A}_c$ (cf. (1.1)) is a series

$$f \in A[\![x_0, \ldots, x_n]\!]$$

which reduces to f_0 over k. We shall consider deformations such that $f(0, \ldots, 0) = 0$ (i.e., which "admit the trivial section") only. If A is an integral domain, and F is an algebraic closure of its field of fractions, the hypersurface defined by

$$f \in F[\![x_0, \ldots, x_n]\!]$$

is called the general member of the deformation. Given a hypersurface f_0 with an isolated singularity, we say that a deformation f of f_0 over an integral domain A has constant Milnor number (or that it is a μ-constant deformation) if f_0 and the general member have the same Milnor number.

(2.3) The following results are known. In this paragraph, A denotes a regular ring in $\mathscr{A}_c$, $B = A[\![x_0, \ldots, x_n]\!]$, f_0 a hypersurface with an isolated singularity and Milnor number μ, f a deformation of f_0 over A (where $f(0, \ldots, 0) = 0$)

$$J = (\partial f/\partial x_0, \ldots, \partial f/\partial x_n)B, \qquad I = (x_0, \ldots, x_n)B.$$

We have:

(a) $(\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$ is a regular sequence in B, and B/J is a finite free A-module.

(b) There is an integer t such that $I^t \subset J$ if and only if f is a μ-constant deformation.

For the proofs, see for instance [9, Section (2.3)] (there, Teissier works with convergent complex series, but the arguments are algebraic and they apply, with minor changes, to the formal case).

(2.4) *Remark.* The statement (2.3) (b) admits the following refinement: the number t which occurs there can be taken to be $t = \mu =$ constant Milnor number. In fact, first of all if h is a hypersurface defined over a field K, with an isolated singularity, it is clear that since the K-algebra $K[\![x_0, \ldots, x_n]\!]/J$ has dimension μ (as a K-vector space), then

(2.4.1) $$(x_0, \ldots, x_n)^\mu \subset J.$$

To prove Remark (2.4), we must show (notations as in (2.3)) that if $R = B/J$, then

(2.4.2) $$I^\mu R = 0.$$

But R is a finite free module over the integral domain A, hence $I^{\mu}R \subset R$ is torsion-free, so (2.4.2) is equivalent to having $I^{\mu}R$ be torsion. To see that $I^{\mu}R$ is torsion, consider the exact sequence of (finite) A-modules

$$0 \to I^{\mu}R \to R \xrightarrow{\alpha} R/I^{\mu}R \to 0.$$

After tensoring (over A) with F (an algebraic closure of A), we get an exact sequence

$$0 \to I^{\mu}R \otimes F \to R \otimes F \xrightarrow{\alpha'} (R/I^{\mu}R) \otimes F \to 0.$$

But α' can be identified to the natural map

$$F[\![x]\!]/J \to (F[\![x]\!]/J)/I^{\mu}(F[\![x]\!]/J)$$

which, by (2.4.1), is an isomorphism. So, $I^{\mu}R \otimes_A F = 0$ and $I^{\mu}R$ is torsion.

Next we shall present an important lemma, due to Samuel (cf. [8, Lemma 2]). In [8], Samuel assumes that A is a field, but his arguments in Lemma 2 apply to arbitrary commutative rings. We include a sketch of the proof, because from it we shall draw some consequences.

(2.5) LEMMA. *Let A be a ring, $B = A[\![x_0, \dots, x_n]\!]$, f, g elements of B, $I = (x_0, \dots, x_n)$, $J = (\partial f/\partial x_0, \dots, \partial f/\partial x_n)$, and assume that $f - g \in IJ^2$. Then, there is an automorphism ϕ of B, such that $\phi(f) = g$.*

Sketch of the proof. The automorphism ϕ will be given by $\phi(x_i) = x_i + h_i$, where (let $f_i = \partial f/\partial x_i$)

$$h_i = \sum_{j=0}^{n} u_{ij} f_j, \quad i = 0, \dots, n. \tag{2.5.1}$$

The series $u_{ij} \in B$ are obtained as follows. Write, in some way,

$$g = f + \sum_{i,j=1}^{n} a_{ij} f_i f_j, \quad a_{ij} \in B. \tag{2.5.2}$$

Consider formal variables U_{ij}, $i, j = 0, \dots, n$. Write, in some specific way, $f' = f(x_0 + \sum_j U_{0j} f_j, \dots, x_n + \sum U_{nj} f_j)$ as

$$f' = f + \sum U_{ij} f_i f_j + \sum f_i f_j(-G_{ij}(U)) \tag{2.5.3}$$

where $G_{ij} \in B[\![U_{11}, \dots, U_{nn}]\!]$, $i, j = 0, \dots, n$, and $O(G_{ij}) \geq 2$. Then, the elements $u_{ij} \in B$ that we are looking for are solutions of the equations

$$U_{ij} = a_{ij} + G_{ij}(U), \quad i, j = 0, \dots, n. \tag{2.5.4}$$

(This system can be solved inductively by writing $u_{ij}^{(0)} = a_{ij}$, $a_{ij}^{(q+1)} = a_{ij} + G_{ij}(u^{(q)})$.) Then $u_{ij} = \lim_{q \to \infty} u_{ij}^{(q)}$. In fact, it is possible to show that such a limit exists, and that the homomorphism ϕ defined using this u's satisfies the requirements. For details see [8].

(2.6) COROLLARY. *The notations are as in* (2.3), *but here* A *denotes an arbitrary ring in* $\mathscr{A}_c$. *Assume that, for some integer* s, $I^s \subset J$. *Then, there is a pair of numbers* (t, r) *such that given any* $g \in B$, *satisfying* $f \equiv g \bmod I^v$, $v \geq t$, *there is an automorphism* ϕ *of* B *such that* (i) $\phi(f) = g$ *and* (ii) *the automorphism of* B/I^{v-r} *induced by* ϕ *is the identity. Moreover, if* $f_0 = g_0$, *we may choose* ϕ *so that* (iii) ϕ *induces the identity automorphism of* $k[\![x_1, \ldots, x_n]\!]$.

This is an easy consequence of Lemma (2.9) (and its proof). Note that we may take $t = 2s + 1$, $r = 2s$.

(2.7) According to (2.3) (b) and (2.4), if f is a deformation of the hypersurface f_0 with constant Milnor number μ, then f satisfies the hypothesis of (2.6), consequently the conclusion of (2.6) holds (with $t = 2\mu + 1$, $r = 2\mu$). For another proof, see [3].

(2.8) It is well known that plane algebroid curves which are equivalent (in Zariski's sense) have the same Milnor number (e.g., see [13, p. 531] and [7, Lemma 4]). Hence, an equisingular deformation of a curve f_0, over a regular ring $A \in \mathscr{A}_c$, has constant Milnor number.

(2.9) *Proof of Theorem* (2.1). Let μ be Milnor number corresponding to the equisingular type α, $A \in \mathscr{A}_c$, f_0 a curve, and f a deformation of f_0 over A. Let A' be a regular local ring in $\mathscr{A}_c$ such that $A = A'/L$, L an ideal of A', $B = A[\![x, y]\!]$, $B' = A'[\![x, y]\!]$. By Proposition (1.12), we may find a deformation f' of f_0 over A' (such that $f'(0, 0) = 0$), inducing f. By (2.4) and (2.8),

$$(x, y)^\mu B' \subset (f'_x, f'_y)B'.$$

Clearly, this inclusion induces an inclusion

$$(x, y)^\mu B \subset (f_x, f_y)B.$$

But then, according to (2.7), given any g such that

$$f \equiv g \bmod (x, y)^v, \quad v \geq 2\mu + 1,$$

there is an automorphism ϕ as claimed in (2.1) (with $r = 2\mu$). As remarked in (2.8), μ depends on α only, and the theorem is proved.

(2.10) *Remark.* From the proof of Theorem (2.1) it follows that, in (2.1), we may take $(t, r) = (2\mu + 1, 2\mu)$, where μ is the Milnor number of any curve of type α.

(2.11) *Remark.* It is possible to prove Theorem (2.1) in a completely different way, by using the technique of H. Hironaka in his proof of Theorem B, on page 155 of [4]. In fact, it is not very difficult to adapt these methods to prove (2.1) in the case when A is a regular ring; then we use Proposition (1.12) (b) to reduce the general case to that one. The details are rather technical, and we omit them.

3. Equisingular families of curves

(3.1) (a) Let $\mathscr{E}$ denote the category of algebraic schemes over k (algebraically closed, of characteristic zero). Following [4], an algebraic family of plane curve singularities is, by definition, a system (π, X, Y, ε) where $\pi: X \to Y$ is a flat morphism of schemes, $\varepsilon: Y \to X$ is a section of π, and for any geometric point y of Y the fiber X_y is a reduced plane curve (hence $\varepsilon(y)$ is an isolated singular point of $X_{\varepsilon(y)}$). This family is said to be equisingular if for every closed point $y \in Y$ the induced algebroid family

$$\operatorname{Spec}(\hat{\mathscr{O}}_{X,\varepsilon(y)}) \to \operatorname{Spec}(\hat{\mathscr{O}}_{Y,y})$$

is isomorphic to a family

$$\operatorname{Spec}(A[\![x, y]\!]/f) \xrightarrow{p} \operatorname{Spec}(A),$$

where $A = \hat{\mathscr{O}}_{Y,y}$ and f is an equisingular deformation of $f_0 = \operatorname{res}(f)$ (cf. (1.6)), in such a way that ε corresponds to the trivial section of p.

From now on, in our text, "equisingular family" will mean "equisingular family of plane curve singularities."

The pull-back of a family under a morphism $Y' \to Y$ is defined in an obvious way.

(b) An equisingular family has type α (cf. Section 0) if for each closed point $y \in Y$, the plane algebroid curve $\operatorname{Spec}(\hat{\mathscr{O}}_{X_y,\varepsilon(y)})$ has type α.

(c) An equisingular family (π, X, Y, ε) of type α is said to be total if for any algebroid plane curve f_0 of type α there is at least one closed point $y \in Y$ such that $k[\![x, y]\!]/(f_0)$ is isomorphic to $\hat{\mathscr{O}}_{X_y,\varepsilon(y)}$.

(d) An equisingular deformation $f \in A[\![x, y]\!]$ ($A \in \mathscr{A}_c$ of radical M) of a curve f_0 is *equisingular versal* if, given any other equisingular deformation g of f_0, $g \in B[\![x, y]\!]$ ($B \in \mathscr{A}_c$ of radical N), then there is a homomorphism $\rho: A \to B$ such that g is isomorphic to $\rho^*(f)$ (cf. (1.8)), by an isomorphism reducing to the identity modulo $\mathscr{N}$. If, in addition, for any such g the induced homomorphism $M/M^2 \to N/N^2$ is unique, then f is said to be *semiuniversal.*

(3.2) Before we present the main result of this section, we review some well-known facts that will be used in its proof.

Let f_0 be a plane algebroid curve, of equisingular type α, having r branches $f_0^{(1)}, \ldots, f_0^{(r)}$. Let B_i be the local ring of $f_0^{(i)}$, $\bar{B}_i$ its normalization. Let $\delta_i = \dim_k \bar{B}_i/B_i$, $d_i = \dim_k \bar{B}/\mathscr{C}$, where $\mathscr{C}$ is the conductor of $\bar{B}_i$ in B_i. It is known [16, p. 10] that $2\delta_i = d_i$, and that these numbers depend on the type of $f_0^{(i)}$ only. If μ is the Milnor number of f_0 then

$$\text{(3.2.1)} \qquad \mu = 2\left(\sum_{i=1}^{r} \delta_i\right) + 2\sum_{i<j} (f_0^{(i)} \cdot f_{(0)}^{(j)}) - r + 1$$

(cf. [7]). Let n_i be the multiplicity of $f_0^{(i)}$. In the course of the proof of the main theorem, we shall use the integers $M_i = n_i(2\mu + 1)$. From (3.2.1) we get

$$\text{(3.2.2)} \qquad M_i \geq \max(n_i, d_i), \quad i = 1, \ldots, r$$

Our main result is the following theorem.

(3.3) THEOREM. *Fix an equisingular type* α. *Then, there is a total equisingular family of type* α, $\mathfrak{F} = (\pi, X, V, \varepsilon)$, *with* V *smooth, and satisfying the following property: for each closed point* $y \in V$, *the induced family* π_y: Spec $(\mathcal{O}_{X,\varepsilon(y)}) \to$ Spec $(\mathcal{O}_{v,y})$ *is an equisingular versal deformation of* Spec $(\mathcal{O}_{X_y,\varepsilon(y)})$.

(3.4) We begin the proof of (3.3). Any curve of a fixed type α will have the same number r of branches and (after reordering the branches, if necessary) its ith branch will have a certain fixed characteristic $\mathbf{c}_i = (n_i; \beta_{i1}, \ldots, \beta_{ig_i})$; moreover the intersection number of the ith and jth branch will be a fixed number $d(i, j)$.

For reasons that will be clear in the course of this proof, we fix the number $\tau = 2\mu + 1$, where μ is the Milnor number of the type α (cf. (2.8)) and the r-tuple of integers

$$M = (M_1, \ldots, M_r), \quad M_i = n_i \tau. \tag{3.4.1}$$

Given two r-tuples of integers L and L', $L \geq L'$ means $L_i \geq L'_i$, $i = 1, \ldots, r$. Fix any $L \in \mathbf{Z}^r$, such that $L \geq M$.

Using notations as in (3.2) for any branch of characteristic $\mathbf{c}_i$, the number d_i will be the same, and $L_i \geq M_i \geq \max(n_i, d_i)$, $i = 1, \ldots, r$ (cf. (3.2.2)). Let $e_{il} = \text{G.C.D.}(\beta_{il}, e_{i,l-1})$ (with $e_{i0} = n_i$), $l = 1, \ldots, g_i$, and $T_i = \{j/\beta_{i1} < j < d_i, j \neq \beta_{il}, l = 1, \ldots, g_i$, and if $\beta_{i,l} < j < \beta_{i,l+1}$, then $j \not\equiv 0 \pmod{e_{il}}, j = 1, \ldots, g_i\}$. Let

$$W_i(L_i) = \{j/n_i \leq j < L_i, j \notin T_i\}, \qquad L_i^{(0)} = \text{card } W_i(L_i). \tag{3.4.1}$$

(3.5) Consider polynomials, with formal coefficients,

$$x = t^{n_i}, y = \sum_{l \in W_i} \lambda_l^{(i)} t^l, \quad W_i = W_i(L_i), i = 1, \ldots, r. \tag{3.5.1}$$

(That is, if we specialize the variables $\lambda_j^{(i)}$, with the condition $\lambda_j^{(i)} \neq 0$ for $j = \beta_{il}$, $l = 1, \ldots, g_i$, then (3.4.2) is a parametrization of a branch of characteristic $\mathbf{c}_i$.)

Let ω_i, $i = 1, \ldots, r$, be a primitive n_ith root of unity, and let

$$f_{L_i}^{(i)}(x, y) = \prod_{j=1}^{r} \left(y - \sum_{l \in W_i} \lambda_l^{(i)} (\omega_i^j x^{1/n_i})^l \right). \tag{3.5.2}$$

Clearly, this is a polynomial in x, y, say,

$$f_{L_i}^{(i)} = y^{n_i} + \sum_{j=1}^{n_i} \left(\sum_{l} a_{jl}^{(i)} x^l \right) y^{n_i - j} \tag{3.5.3}$$

of degree $\leq L_i$ and order n_i; moreover, the coefficients $a_{jl}^{(i)}$ are polynomials in $\{\lambda_j^{(i)}\}$, $j \in W_i$. These latter polynomials define a morphism

$$\phi_{L_i}\colon \mathbf{A}^{L_i^{(0)}} \to \mathbf{A}^{L_i'}, \tag{3.5.4}$$

where L_i' is the number of coefficients $a_{jl}^{(i)}$ (clearly this depends on L_i).

Next, note that there is a morphism

$$\Lambda_{(L)}\colon \mathbf{A}^{L'} \to \mathbf{A}^N \tag{3.5.5}$$

(where $L' = L'_1 + \cdots + L_r$) induced by multiplication of polynomials. In other words write $\prod_{i=1}^{r} (y^{n_i} + \sum_j a_j(x)y^{n_i-j}) = y^n + \sum_{p=1}^{n} (b_{pq}x^q)y^{n-p}$, where b_{pq} is a polynomial in $\{a_{jl}^{(i)}\}$; these define $\Lambda_{(L)}$. Clearly, $N = N(L_1, \ldots, L_r)$ depends on $L_1, \ldots, L_r$. Let

$$\Phi_{(L)} = \Lambda_{(L)} \circ \left(\prod_{i=1}^{r} \phi_{L_i}\right): \prod_{i=1}^{r} \mathbf{A}^{L_i(0)} \to \mathbf{A}^{N(L)}. \tag{3.5.6}$$

We shall need the following lemma.

(3.6) LEMMA. *For any $(L) = (L_1, \ldots, L_r)$, the morphism $\Phi_{(L)}$ is finite.*

Proof. Since clearly $\Phi_{(L)}$ is of finite type, it suffices to show that it is integral. We claim that the morphisms ϕ_{L_i} and Λ are integral. This is an easy consequence of the following classical result. Let B and D, $B \subset D$, be integral domains, $\bar{B}$ the integral closure of B in D.

(a) If $f_i = \sum_{j=1}^{m_i} a_j^{(i)} z^j$ are monic polynomials in $B[z]$, $i = 1, 2$, $\bar{B}$ is the integral closure of B in D and $f_1 f_2 \in \bar{B}[z]$, then $a_j^{(i)} \in \bar{B}$, $j = 1, \ldots, m_i$, $i = 1, 2$.

(b) If $h(z) \in D[z]$ is integral over $B[z]$, then all its coefficients are in $\bar{B}$ (cf. [2, Chapter V, Exercises 8 and 9]).

Since the product and composition of integral morphisms are integral, the lemma follows.

Note that in particular $\Phi_{(L)}$ is a closed morphism, for all (L).

(3.7) We continue the proof of (3.3). We have, for any (L), an algebraic family of curves with parameter space $\mathbf{A}^{N(L)}$, with coordinates $\{a_{jl}^{(i)}\}$ (cf. (3.5.5)), defined by

$$f(x, y) = y^n + \sum_{p=1}^{n} \left(\sum_q b_{pq} x^q\right) y^{n-p} = 0. \tag{3.7.1}$$

There is a trivial section defined by $x = y = 0$. We shall define, for $(L) \geq M$, a locally closed subscheme $U(L)$ of $\mathbf{A}^{(L)} = \prod_{i=1}^{r} \mathbf{A}^{L_i}$, such that if $V(L) = \Phi_{(L)}(U(L))$ (scheme-theoretic image), then the family (3.7.1) (restricted to $V(L)$) is equisingular of type α.

First of all, to get the "right characteristic" for the different branches, consider the condition (in $\mathbf{A}^{(L)}$)

$$\prod \lambda_{\beta_{ij}} \neq 0, \quad j = 1, \ldots, n_i, \; i = 1, \ldots, r. \tag{3.7.2}$$

To get the "right intersection numbers," consider the series $f_{L_i}^{(i)}$ (cf. (3.5.3)) and then the series

$$f_L^{(i)}\left(t^{n_j}, \sum_{p \in W_j} \lambda_l^{(j)} t^l\right), \tag{3.7.3}$$

for each pair $i \neq j$. The coefficient of t^u in this series is a polynomial $p_{ij}^{(u)}$ in

$\lambda = (\lambda_j^{(i)})$. We want (cf. condition (1.2.1))

$$p_{ij}^{(u)}(\lambda) = 0, \quad u < d(i, j), \ 1 \le i, j \le r, \ i \ne j, \tag{3.7.4}$$

$$p_{ij}^{(d(i,j))}(\lambda) \ne 0, \quad 1 \le i < j \le r. \tag{3.7.5}$$

Note that by the choice (3.4.1) of L_i, these conditions are not vacuous for any pair (i, j).

Let $Z(L)$ be the closed subscheme of $\mathbf{A}^{(L)}$ defined by the equations (3.7.4), $U(L)$ its open subscheme defined by (3.7.2) and (3.7.5). Since $\Phi_{(L)}$ is finite, the induced continuous map of supports $|Z(L)| \to |\Phi_{(L)}(Z(L))|$ is surjective. Let F be the closed set of $\mathbf{A}^{(L)}$ defined by $(\prod \lambda_{\beta_{il}})(\prod_{i \ne j} p_{ij}^{d(i,j)}) = 0$, and $V(L) = \Phi_{(L)}(Z(L)) - \Phi_{(L)}(F))$ (an open subscheme of $\Phi_{(L)}(Z(L))$. It is easy to verify that $V(L) = \Phi_{(L)}(U(L))$ (scheme-theoretic image). The restriction of the family (3.7.1) to $V(L)$ will be denoted $\mathfrak{F}(L)$.

Note the following.

(a) The morphism $\Phi_{(L)}^0$: $U(L) \to V(L)$ induced by $\Phi_{(L)}$ is finite.

(b) The polynomials (3.7.2), (3.7.4) and (3.7.5) are the same for any r-tuple $L \ge M$.

Both statements are easily verified.

So far, we have seen that for $L \ge M$ the closed fibers of the family $\mathfrak{F}(L)$ have singularities of type α at the origin. Next we shall show that, moreover, $\mathfrak{F}(L)$ is equisingular.

Note that there are certain interesting automorphisms of $\mathbf{A}^{(L)}$ (for any (L)): those induced by "changing the parameter" in one of the branches (3.4.2) and those induced by interchanging two isomorphic branches (whenever this is possible). In particular, we mean

$$\begin{aligned} &\lambda_j^{(i)} \to \lambda_j^{(i)}, \quad i \ne l, \ j \in W_i(L_i), \\ &\lambda_j^{(l)} \to \omega^j \lambda_j^{(l)}, \quad j \in W_l(L_l), \ \omega \text{ an } n_l\text{th root of } 1, \end{aligned} \tag{3.7.6}$$

and, if $L_i = L_j$ and $W_i(L_i) = W_j(L_j)$ (cf. (3.4.1)), we may also define

$$\begin{aligned} &\lambda_p^{(l)} \to \lambda_p^{(l)}, \quad l \ne i, j, \ p \in W_l(L_l), \\ &\lambda_p^{(i)} \to \lambda_p^{(j)}, \ p = 1, \ldots, n_i, \quad \text{and} \quad \lambda_p^{(j)} \to \lambda_p^{(i)}, \ p \in W_i(L_i). \end{aligned} \tag{3.7.7}$$

These automorphisms clearly commute with $\Phi_{(L)}$.

Moreover, it is easy to see, using the unique factorization property in $k[x, y]$ and the "essential uniqueness" of the parametrization of a branch (see for instance [6, Proposition 1.5]), that if λ, λ' are two closed points of $A^{(L)}$, then $\Phi_{(L)}(\lambda) = \Phi_{(L)}(\lambda')$ if and only if there is a finite sequence of automorphisms $\sigma_1 \cdots \sigma_s = \sigma$ such that $\sigma(\lambda) = \lambda'$ and σ_i $(i = 1, \ldots, s)$ is either of type (3.7.6) or (3.7.7). Note that σ commutes with $\Phi_{(L)}$. Using this fact and the finiteness of $\Phi_{(L)}$, a standard argument shows that, if $\Phi_{(L)}(\lambda_0) = b_0$, the induced homomorphism

(3.7.8) $$\hat{\mathcal{O}}_{V,b_0} \to \hat{\mathcal{O}}_{U,\lambda_0}, \quad V = V(L),\ U = U(L),$$

is injective.

To show that $\mathfrak{F}(L)$ is equisingular, we must show that the deformation $f \in \hat{\mathcal{O}}_{V,b_0}[\![x, y]\!]$ (cf. (3.7.1)) is equisingular. By construction, the pull-back of (3.7.1) to $U(L)$ is equisingular. Thus, the series $f \in \hat{\mathcal{O}}_{U,\lambda_0}[\![x, y]\!]$ is an equisingular deformation of res (f) over $\hat{\mathcal{O}}_{U,\lambda_0}$. To deduce the equisingularity of f as a deformation over $\hat{\mathcal{O}}_{V,b_0}$, we use the following:

(3.8) LEMMA. *Let $A \subset B$ be rings in $\mathscr{A}_c$, $g \in A[\![x, y]\!]$ a deformation of a curve $g_0 \in k[\![x, y]\!]$ (satisfying (1.5)). Assume $g \in B[\![x, y]\!]$ is an equisingular deformation of g_0 over B. Then, g is equisingular as a deformation of g_0 over A.*

Proof. Since $g \in B[\![x, y]\!]$ is equisingular, we may assume that

$$g = y^n + \sum_{i=1}^{n} a_i(x)y^{n-i}, \quad a_i(x) \in A[\![x]\!]$$

and g is equimultiple (cf. [8, (1.6)]). Write g_0 as $g_0^{(1)} \cdots g_0^{(s)}$, the product of its tangential components. It is known (cf. [10, (1.10)]) that the equimultiplicity of g implies that the ideal $(g)A[\![x, y]\!]$ can be uniquely written as a product of ideals $(g^{(1)}) \cdots (g^{(1)})$, such that res $(g^{(i)}) = g_0^{(i)}$. If we choose $g^{(i)} = g_{n_i}^{(i)} + g_{n_i+1}^{(i)} + \cdots$ ($g_j^{(i)}$ homogeneous of degree j) such that the coefficient of y^{n_i} is 1, $i = 1, \ldots, s$, then $g_{n_i}^{(i)}$ is uniquely determined, for $i = 1, \ldots, s$. The same is true over B. But over B, g is equisingular. By (1.9), $g_{n_i}^{(i)} = (y - \alpha_i x)^{n_i}$, $\alpha_i \in B$. Since $g_{h_i}^{(i)} \in A[x, y]$, it follows that $\alpha_i \in A$. Hence, the quadratic transform of $g \in A[\![x, y]\!]$ will have s connected components, of centers $(0, \alpha_i)$, $i = 1, \ldots, s$. It is easily checked that if $g_i'(x_{(i)}, y_{(i)}) = 0$ is the equation of the component of center $(0, \alpha_i)$, then the assumptions of Lemma (3.8) still hold. Now, by using arguments as in (1.11) and (1.12), the proof is easily completed by induction on $\sigma(g_0)$.

(3.9) *Remark.* Denoting with a bar the image of an element of $k[b]$ (respectively $k[\lambda]$) in $\hat{\mathcal{O}}_{V,b_0}$ (respectively $\hat{\mathcal{O}}_{U,\lambda_0}$), we have

$$\hat{\mathcal{O}}_{V,b_0} = k[\![\{\bar{b}_{pq}\}]\!], \qquad \hat{\mathcal{O}}_{U,\lambda_0} = k[\![\{\bar{\lambda}_j^{(1)}\}]\!].$$

The deformation f admits, over $\hat{\mathcal{O}}_{U,\lambda_0}$, equisingular parametrizations with coefficients $\{\bar{\lambda}_j^{(i)}\}$. By (3.8), and the uniqueness of parametrizations (cf. [6, Proposition 1.11]), the elements $\bar{\lambda}_j^{(i)}$ must be in $\hat{\mathcal{O}}_{V,b_0}$. This shows that (3.7.8) is also surjective, i.e., an isomorphism. It follows that $\Phi_{(L)}^0$: $U(L) \to V(L)$ (L large enough) is an etale morphism.

(3.10) Continuing the proof of (3.3), now we check that for $(L) \geq M$ the family $\mathfrak{F}(L)$, defined by (3.7.1) over $V(L)$, is total. Take any curve of type α; we may assume that it has an equation $h = \prod_{i=1}^{r} h_i$, where its ith irreducible component h_i has characteristic $\mathbf{c}_i$ (cf. (3.4)) and an equation

$$h_i = y^{n_i} + \sum_{i=1}^{n_i-1} a_j^{(i)}(x)y^{n-j}, \quad O(a_j^{(i)}) \geq i.$$

Let h_i be parametrized by $x = t^{n_i}$, $y = \phi^{(i)} = \sum \lambda_l^{(i)}t^l$. Let $\phi_{L_i}^{(i)} = \sum_{j<L_i} \lambda_l^{(i)}t^j$. Let $f_{L_i}^{(i)}$ be defined by (3.5.2). Since $L_i \geq M_i \geq n_i(2\mu + 1)$, it follows that

$$h_i \equiv f_{L_i}^{(i)} \bmod (x, y)^\tau, \quad \tau = 2\mu + 1,$$

hence $h \equiv \prod f_{L_i}^{(i)} = f_L \bmod (x, y)^{2\mu+1}$. By Remark (2.10), h and f_L are isomorphic. Since clearly f_L is a member of the family $\mathfrak{F}(L)$, this proves that $\mathfrak{F}(L)$ is total.

In the following, we use these notations. Fix a closed point $y \in V(L)$ and write $B = \hat{\mathcal{O}}_{V,y}$, let the coordinate b_{pq} (respectively $\lambda_j^{(i)}$) induce an element $\bar{b}_{pq}$ (respectively $\bar{\lambda}_j^{(i)}$) of B (cf. Remark (3.9)). Let $\bar{f}$ be the deformation (of its special fiber f_0) of equation

(3.10.1) $$\bar{f} = y^n + \sum (\sum \bar{b}_{pq}x^q)y^{n-p} \in B[\![x, y]\!],$$

i.e., the one induced by (3.7.1). Actually, we should write $B(L)$, $\bar{f}_{(L)}$, etc., since these depend on (L). We omitted (L) to simplify the notation.

(3.11) LEMMA. *Let $L \geq M$. Let $\rho: A' \to A$, $\chi: B \to A$ $(B = B(L))$ be homomorphisms in $\mathscr{A}$ and $\mathscr{A}_c$, respectively, where ρ is surjective. Let $\bar{f} = \bar{f}_{(L)}$, $g = \chi^*(\bar{f})$ and let g' be any equisingular lifting of g to A'. Then, there is a homomorphism $\chi': B \to A'$ such that $\chi = \rho\chi'$ and*

$$A'[\![x, y]\!]/(g') \approx A'[\![x, y]\!]/(g_1)$$

(isomorphism of A'-algebras), where $g_1 = \chi'^*(\bar{f})$.

Proof. Note that $\bar{f}$ is given by (3.10.1), and its ith component has an equisingular parametrization $x = t^{n_i}$, $y = \sum_{j \in W_i} \bar{\lambda}_j^{(i)}t^j$, $\bar{\lambda}_j^{(i)} \in B$ (cf. (3.4.1)). Hence, the ith component of g has parametrization

$$x = t^{n_i}, \quad y = \sum_{j \in W_i} \mu_j^{(i)}t^j, \; \mu_j^{(i)} = \chi(\bar{\lambda}_j^{(i)}).$$

The equisingular deformation $g' = \prod_{i=1}^r g'^{(i)}$ will have parametrizations $x = t^{n_i}$, $y = \sum_j \mu_j'^{(i)}t^j$. Since g' is equisingular and g' reduces to g in A, it follows that (after replacing t by ωt, with ω an n_ith root of 1, if necessary) $\rho(\mu_j'^{(i)}) = \mu_j^{(i)}$, $j \in W_i$, $i = 1, \ldots, r$. Consider parametrizations $x = t^{n_i}$, $y = \sum_{i \in W_i} \mu_j'^{(i)}t^j$, and the deformation they define, i.e., that defined by

$$g_1 = \prod_{i=1}^r \left(\prod_{j=1}^{n_i} y - \sum_{l \in W_i} \mu_l'^{(i)}(\omega_i^j x^{1/n_i})^l) \right)$$

where ω_i is a primitive n_ith root of 1. By the choice of (L), the following facts are easily verified.

(a) g_1 is equisingular (the only problem is in the intersection numbers—use (3.7.4) and (3.7.5), and the fact that $L_i \geq M_i$).

(b) $g_1 \equiv ug' \bmod (x, y)^{\tau(\alpha)}$, where u is a suitable unit in $A'[\![x, y]\!]$.

Then, Theorem (2.1) implies $A'[\![x, y]\!]/(g_1) \approx A'[\![x, y]\!]/(g')$. On the other hand, there is a homomorphism χ': $B \to A'$ such that $\chi'(\bar{\lambda}_j^{(i)}) = \mu_j'^{(i)}$, for all possible i, j. In fact, the relations (3.7.4) are satisfied by the elements $\mu_j'^{(i)}$ (by the equisingularity of g_1), hence such a well-defined χ' exists. Clearly the homomorphism χ' satisfies the required conditions, and the lemma is proved.

(3.12) Lemma (3.11) implies the smoothness of $V(L)$ (for $L \geq M$). In fact, it suffices to show that the k-algebra $\hat{\mathcal{O}}_{V,y}$ is smooth, for any closed point $y \in V(L)$. Recall that this means that for any surjection $A' \to A$ in $\mathscr{A}_c$ (cf. (1.1)) the canonical map Hom $(\hat{\mathcal{O}}_{V,y}, A') \to$ Hom $(\hat{\mathcal{O}}_{V,y}, A)$ is surjective. But, by (1.12) any equisingular deformation over A can be lifted to an equisingular deformation over A' (notations as in (3.10) and (3.11)). So, we may apply (3.11) to get the desired map. The proof of Theorem (3.3) is complete.

(3.15) *Examples.* In the following examples, we use the notations of the proof of Theorem (3.3).

(a) If α is the type of an irreducible plane curve, then the set of equations (3.7.4) is empty. Consequently, the only restrictions are the inequalities (3.7.2); and $U(L)$, $L \geq L^{(0)}$ is an open set in $A^{L'}$, hence $U(L)$ and its image $V(L)$ are irreducible. For an expression of $\mu = 2\delta - r + 1$ in terms of the characteristic pairs, see [16, Chapter II, Section 3].

(b) Let α be the type of a curve, such that all its irreducible components are nonsingular. In this case, the equalities (3.7.3) become $Q_{ij}(x) = (\sum_p \lambda_p^{(i)} x^p) - (\sum_p \lambda_p^{(j)} x^p)$, where the terms on the right hand side describe the ith and jth component, respectively. Hence, the equalities (3.7.4) are linear equalities of the form $\lambda_p^{(i)} - \lambda_p^{(j)} = 0$. Hence, $U(L)$ is an open subvariety of a linear variety; consequently $U(L)$ and $V(L)$ are irreducible.

(c) Let α be the type of a curve C of multiplicity 3. These are the possibilities: (i) C is irreducible, (ii) C has three linear branches, (iii) C has a linear and a quadratic branch. The only case not discussed yet is (iii). In this case, we get two relations (3.7.3), and the ideal of the polynomials (3.7.4) can be generated by the linear forms

$$\lambda_{2m}^{(2)} - \lambda_m^{(1)}, \quad 2m < \gamma,$$

$$\lambda_{2m+1}^{(2)}, \qquad 2m + 1 < \gamma,$$

where we assume that the linear branch is parametrized by $y = \sum \lambda_i^{(1)} t^i$, and γ is the intersection number (this is an elementary calculation—cf. example (d)). Again, $U(L)$ is an open subvariety of a linear variety, hence $U(L)$ and $V(L)$ are irreducible.

(d) Let α be the type of a curve, C, consisting of two branches, each isomorphic to $y^2 - x^3 = 0$, with intersection number 7. To simplify the notations, let $\lambda_i^{(1)} = \lambda_i$, $\lambda_j^{(2)} = \nu_j$ (cf. (3.5.1)). Here, $\mu = 17$ and $M = (70, 70)$. Fix $L \geq M$. The series (3.5.3) is

$$f = f_{L_1}^{(1)} = (y - \lambda_2 x)^2 - 2\lambda_4 x^2 y + (2\lambda_2\lambda_4 - \lambda_3^2)x^3 - 2\lambda_6 x^3 y + \cdots,$$

and (3.7.3) becomes

$$(\lambda_2 - \nu_2)^2 t^4 + 2(\nu_2 - \lambda_2)\mu_3 t^5 + [(\nu_3^2 - \lambda_3^2) + 2(\nu_2 - \lambda_2)(\nu_4 + \nu_4)]t^6 + [\nu_2\nu_5 + \nu_3\nu_4 - \lambda_2\lambda_5 - \lambda_3\lambda_4]t^7 + \cdots.$$

Using the fact that $\mu_3 \neq 0$ (cf. (3.7.2)), it is clear that to get a series of order 7 it is necessary and sufficient to have:

$$\lambda_2 - \nu_2 = 0, \qquad \lambda_3^2 - \nu_3^2 = 0, \tag{3.15.1}$$

$$\nu_2\nu_5 + \nu_3\nu_4 \neq \lambda_2\lambda_5 + \lambda_3\lambda_4. \tag{3.15.2}$$

If we consider instead $f_{L_2}^{(2)}(t^2, \sum \lambda_i t^i)$, we get exactly the same conditions. Hence, $U(L)$ is defined in A^{2Q}, with coordinates $(\lambda_2, \ldots, \lambda_Q, \nu_2, \ldots, \nu_Q)$ (where $Q = L_1^{(0)} = L_2^{(0)}$—cf. (3.4.1)) by (3.15.1), (3.15.2), and the inequality $\lambda_3\nu_3 \neq 0$. Note that (3.15.1) defines a reducible algebraic variety (a union of two linear varieties S_1 and S_2, each of codimension 2). After removing the variety $\lambda_3\nu_3 = 0$ (which contains $S_1 \cap S_2$), we get a smooth variety, having two connected components U_1' and U_2'. The condition (3.15.2) gives nonempty open sets $U_i \subset U_i'$, and $U = U(L) - U_1 \cup U_2$. So, in this case, $U(L)$ is not irreducible. However, in this case, $V = V(L)$ is irreducible. In fact, we claim that given a closed point $w \in U_1$, there is an automorphism ϕ of U, commuting with the projection $U \to V$, and a point $w' \in U_1$ such that $\phi(w') = w$. Clearly, this implies $V = \phi(U_2)$, hence V is irreducible. To see it, note that a typical point of U has coordinates

$$w = (u, v, \lambda_4, \ldots, \lambda_Q, u, \pm v, \nu_4, \ldots, \lambda_Q),$$

subject to the condition (3.15.2), and $v \neq 0$. If the sign of v is "+," $w \in U_1$; if "−," $w \in U_2$. So, if $w \in U_1$, let ϕ be one of the automorphisms (3.7.6), with $\lambda_j \to \lambda_j$, $\nu_j \to -\nu_j$. Then $\phi(w') = w$, where

$$w' = (u, v, \lambda_4, \ldots, \lambda_Q, u, -v, \ldots, (-1)^i \nu_i, \ldots),$$

and our assertion is proved.

We do not know, in general, whether $V(L)$ is irreducible or not.

(3.16) In these final paragraphs, we briefly discuss the complex analytic case. We shall not give many details, since the methods are similar to those used in [6, Sections 4 and 5].

The results of Section 1 extend to the convergent case with no difficulties.

The analytic version of Theorem (2.1) can be proved, for instance, by reducing to the formal case (treated in Section 2) and by using Artin's analytic approximation lemma (cf. [1]).

Regarding Section 3, note that the algebraic family $\mathfrak{F}(L)$ can be considered as an analytic family in a natural way. Moreover, there is an analytic version of Lemma (3.11), in which B is replaced by $\mathcal{O}_{V^h,y}$ (V^h is the analytic variety associated to V) and $\rho: A' \to A$ by a surjection of analytic rings. The proof is essentially the same as in (3.11).

Finally, as in [6, Section 5], we can deduce the existence of an analytic versal equisingular deformation for a complex analytic germ of a plane curve (cf. [6, Definition 5.1]). The proof is like the proof of Theorem 5.7 of [6], with the following differences (we use the notations of [6, Section 5]): (i) Assume f_0 to be given by Equation (3.7.1) (with coefficients b_{pq} corresponding to suitable $\{\lambda_l^{(i)}\}$—cf. (3.10)); (ii) Replace Lemma (5.3) by the analytic version of Lemma (3.11); (iii) use the analytic version of Lemma (3.8) of this paper instead of Lemma (5.5) of [6].

The proof of the analytic version of (3.8) is like that of the formal one, except that the formal power series $g^{(i)}$ that occur there must be replaced by suitable convergent series. This can be done by using Artin's approximation lemma (cf. [1]). Theorem 5.9 of [6] (and its preceding remark) extend to our situation without difficulties.

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