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PARTS OF OPERATORS ON HILBERT SPACE

BY

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1. Introduction

In the study of operators on Hilbert spaces it has been found convenient in a number of different situations to introduce a systematic decomposition of every operator into the direct sum of two suboperators, one of which possesses some special property, while the other is "completely free" of that property. The best known of these decompositions is the splitting of each contraction into its *unitary part* and its (complementary) *completely-non-unitary part* used consistently in [13]. Other instances of this type of construction are extractions of a *self-adjoint part* of an operator as in [6], [11], [12], and of the *normal part* as in [2], [6], [14]. (There is also frequent reference in the literature to what might be called the *zero part* of an operator, though not usually by that name.) More general decompositions of this type appear in [5, pp. 177–179].

The primary aim of this paper is to put this kind of construction on a systematic basis. In particular, we give a simple characterization (Theorem 3.3) of those properties of operators that give rise to such a decomposition. It is also shown (Theorem 3.7) that these properties can be defined by certain types of equations. In Section 2 a simple construction is given (Theorem 2.3) for the *part* corresponding to a property defined by a family of equations.

In what follows, H denotes a Hilbert space. If M is any *subspace* (closed linear manifold) of H, then $B(M)$ denotes the set of *operators* (bounded linear transformations) on M, and *dim* M (the dimension of M) denotes the cardinality of an orthonormal basis for M. If $T \in B(M)$, then *size* $T = \dim M$. An operator S is a *suboperator* of an operator T if S is the restriction of T to a nonzero reducing subspace. If T is an operator, then $W^*(T)$ denotes the von Neumann algebra generated by T (i.e., the weakly closed algebra generated by $1, T, T^*$). If $E \subseteq H$, then $\bigvee E$ denotes the subspace spanned by E. A polynomial $p(x, y)$ in the noncommuting variables x, y is a *noncommutative polynomial.*

2. Equationally defined parts

Many of the special properties that an operator can have are defined by equations that the operator is required to satisfy. (Example: T is normal if $T^*T - TT^* = 0$.) A large number of these equations involve noncommutative

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polynomials in an operator and its adjoint. In order to include such properties as hyponormality ($T^*T - TT^* \geq 0$), it is necessary to consider a much larger class of functions.

The motivation for this large class of functions comes from the study of Borel functions of normal operators. Suppose T is a normal operator and ϕ is a complex Borel function. If M is a subspace of H that reduces T, then M reduces $\phi(T)$; furthermore, $\phi(T|M) = \phi(T)|M$. In other words if T can be decomposed into the direct sum of two operators A, B, then $\phi(T)$ can be decomposed into the direct sum of $\phi(A)$, $\phi(B)$. In addition if U is any unitary operator, then $\phi(U^*TU) = U^*\phi(T)U$.

DEFINITION 2.1. A *decomposable function* is a function ϕ on

$$\bigcup \{B(M)\colon M \text{ is a subspace of } H\}$$

such that:

(i) $\phi(B(M)) \subseteq B(M)$ for every subspace M of H.

(ii) If $T \in B(H)$ and M is a reducing subspace of T, then M reduces $\phi(T)$ and $\phi(T|M) = \phi(T)|M$.

(iii) If M, N are subspaces of H, $S \in B(M)$, and $U\colon N \to M$ is unitary, then $\phi(U^*SU) = U^*\phi(S)U$.

A decomposable function ϕ is *continuous* if $\phi\,|\,B(M)$ is continuous for every subspace M of H.

If T is normal and ϕ is a Borel function, then $\phi(T)$ is in the von Neumann algebra $W^*(T)$ generated by T. Since the commutant of $W^*(T)$ is generated by its projections (or by its unitary operators), it follows from (ii) (or from (iii)) and the double commutant theorem that $\phi(T) \in W^*(T)$ whenever T is an operator and ϕ is a decomposable function.

Examples. (A) If $p(x, y)$ is a noncommutative polynomial, then $\phi(T) = p(T, T^*)$ defines a decomposable function ϕ.

(B) If f is an entire function, then $\phi(T) = f(T)$ defines a decomposable function ϕ.

(C) The sum (linear combination, product, composition) of two decomposable functions is a decomposable function.

(D) If f is a complex Borel function and if ψ is a decomposable function such that $\psi(S)$ is normal for every S in $B(H)$, then $\phi(T) = f(\psi(T))$ defines a decomposable function ϕ.

The following definition provides the terminology needed to discuss the class of operators defined by a family of equations involving decomposable functions.

DEFINITION 2.2. If $\mathscr{F}$ is a family of decomposable functions, then an operator T is an *$\mathscr{F}$-operator* provided $\phi(T) = 0$ for each ϕ in $\mathscr{F}$.

THEOREM 2.3. *Suppose $\mathscr{F}$ is a family of decomposable functions and $T \in B(H)$. Let $M = \bigcap \{\ker \phi(T)p(T, T^*)\colon \phi \in \mathscr{F}, p(x, y)$ is a noncommutative polynomial$\}$. Then:*

(1) *$M^\perp = \bigvee \{\text{range } p(T, T^*)\phi(T)^*\colon \phi \in \mathscr{F}, p(x, y)$ is a noncommutative polynomial$\}$.*
(2) *M reduces T.*
(3) *$T|M$ is an $\mathscr{F}$-operator.*
(4) *If N reduces T and $T|N$ is an $\mathscr{F}$-operator, then $N \subseteq M$.*
(5) *M reduces any operator which commutes with T and T^*.*

Proof. (1) This follows from the facts that $(\ker A)^\perp = \text{range } (A^*)^-$ and $(\bigcap M_i)^\perp = \bigvee M_i^\perp$ for any operator A and any collection of subspaces $\{M_i\}$.

(2) It is clear from (1) that $M^\perp$ is left invariant by both T, T^*.

(3) This follows from the fact that $M \subseteq \bigcap \ker \phi(T)$ ($\phi \in \mathscr{F}$) and that $\phi(T|M) = \phi(T)|M$.

(4) Suppose N reduces T and $T|N$ is an $\mathscr{F}$-operator. Since N reduces T, then N reduces each polynomial in T, T^*. Thus

$$\phi(T)p(T, T^*)(N) \subseteq \phi(T)(N) = \phi(T|N)(N) = 0$$

for each ϕ in $\mathscr{F}$ and each polynomial $p(x, y)$.

(5) If A commutes with T and T^*, then A commutes with every operator in $W^*(T)$. Thus A and A^* commute with $\phi(T)p(T, T^*)$ for each ϕ in $\mathscr{F}$ and each polynomial $p(x, y)$. Hence A is reduced by each $\ker \phi(T)p(T, T^*)$. Therefore A is reduced by M.

The subspace M in the preceding theorem is called the *$\mathscr{F}$-subspace* of T.

COROLLARY 2.4. *If $\mathscr{F}$ is a family of decomposable functions, $T \in B(H)$, and $T = A \oplus B$, then the $\mathscr{F}$-subspace of T is the direct sum of the $\mathscr{F}$-subspace of A with the $\mathscr{F}$-subspace of B.*

DEFINITION 2.5. If $\mathscr{F}$ is a family of decomposable functions, $T \in B(H)$, and M is the $\mathscr{F}$-subspace of T, then $T|M$ is the *$\mathscr{F}$-part* of T, and $T|M^\perp$ is the *non-$\mathscr{F}$-part* of T.

The preceding theorem says that, for any given family of decomposable functions, any operator can be decomposed into the direct sum of an $\mathscr{F}$-operator and an operator with no $\mathscr{F}$-part. Part (4) of the theorem says that this decomposition is unique. The corollary says that the $\mathscr{F}$-part of the direct sum of two operators is obtained by taking the direct sum of the $\mathscr{F}$-parts of each summand. In particular, if A and B have no $\mathscr{F}$-parts, then $A \oplus B$ has no $\mathscr{F}$-part.

Examples. (E) Let $\mathscr{F} = \{\phi\}$ where $\phi(T) = T$ for every operator T. The $\mathscr{F}$-part of an operator T is the *zero part* of T, and the $\mathscr{F}$-subspace of T is $\ker T \cap \ker T^*$.

(F) Let $\mathscr{F} = \{\phi\}$ where $\phi(T) = T - T^*$ for every operator T. The $\mathscr{F}$-part of an operator T is the *Hermitian* (self-adjoint) *part* of T.

(G) If $\mathscr{F} = \{\phi, \psi\}$ where $\phi(T) = 1 - T^*T$ and $\psi(T) = 1 - TT^*$ for every operator T, then the $\mathscr{F}$-part of an operator T is the *unitary part* of T.

(H) Hyponormality is defined by an inequality rather than an equation; i.e., T is hyponormal if $T^*T - TT^* \geq 0$. If A is any operator, let $|A| = (A^*A)^{1/2}$. Then $A \geq 0$ if and only if $A = |A|$. If ψ is a decomposable function, then so is $|\psi|$, and if $\phi = \psi - |\psi|$, then $\psi(T) \geq 0$ if and only if $\phi(T) = 0$. Thus inequalities involving decomposable functions can always be rewritten as equations involving decomposable functions. If $\mathscr{F} = \{\phi\}$ where $\phi(T) = T^*T - TT^* - |TT^* - T^*T|$ for every operator T, then the $\mathscr{F}$-part of an operator T is the *hyponormal part* of T.

(I) It was shown by Halmos [8] and Bram [1] that an operator T is subnormal if and only if for each positive integer n, the operator S_n defined on the direct sum of $n + 1$ copies of H by the operator matrix

$$S_n = ((T^*)^j T^i), \quad 0 \leq i, j \leq n,$$

is positive. For each positive integer n, let

$$|S_n| = (\psi_{i,j,n}(T)), \quad 0 \leq i, j \leq n.$$

There is a sequence $\{p_k(x, y)\}$ of noncommutative polynomials such that $p_k(A, A^*) \to |A|$ for every operator A. Since $P_k(S_n, S_n) \to |S_n|$, then each $\psi_{i,j,n}(T)$ is a (norm) limit of noncommutative polynomials in T and T^*. Thus the functions $\psi_{i,j,n}$ are decomposable for $0 \leq i, j \leq n < \infty$. Furthermore, an operator T is subnormal if and only if $\psi_{i,j,n}(T) = (T^*)^j T^i$ for $0 \leq i, j \leq n < \infty$. Therefore if

$$\mathscr{F} = \{\phi_{i,j,n} \colon 0 \leq i, j \leq n < \infty\}$$

where each $\phi_{i,j,n}(T) = \psi_{i,j,n}(T) - (T^*)^j T^i$, then the $\mathscr{F}$-part of an operator T is the *subnormal part* of T.

3. General parts

Although every operator has a normal part and a hyponormal part, operators do not generally have nilpotent parts (consider a direct sum of nilpotent operators, one of each positive order), compact parts or invertible parts (consider a diagonal operator with spectrum [0, 1]). What kinds of properties of operators give rise to parts, and which of these can be defined using decomposable functions? These questions are answered in this section.

For notational convenience we will identify a property $\mathscr{P}$ with the class of all operators having the property. Thus "$T \in \mathscr{P}$" means "T has property $\mathscr{P}$", and a property $\mathscr{P}$ can be defined by a statement like "Let $\mathscr{P}$ be the class of normal operators". In order to be "coordinate-free", a property of operators should be closed under unitary equivalence.

What do we mean when we say that an operator T has a "part" corresponding to a particular property? It should mean that there is a *unique* subspace M that reduces T such that $T|M$ has the prescribed property and $T|M^{\perp}$ has no suboperator with the property. In some sense M should be maximal; that is, M should contain all those subspaces N that reduce T such that $T|N$ has the prescribed property. These facts imply that M equals the span of all those subspaces N that reduce T such that $T|N$ has the prescribed property.

DEFINITION 3.1. If $\mathscr{P}$ is a property of operators and T is an operator, then the *$\mathscr{P}$-subspace* of T, denoted by $\mathscr{P}(T)$, is

$$\bigvee \{N\colon N \text{ reduces } T,\ T|N \in \mathscr{P}\}.$$

Note that $\mathscr{P}(T)$ reduces T. We can now rigorously define what it means for a property of operators to give rise to parts.

DEFINITION 3.2. A property $\mathscr{P}$ of operators is a *part-property* if it is closed under unitary equivalence and, for each operator T,

(i) $T|\mathscr{P}(T) \in \mathscr{P}$, and
(ii) if M reduces T, $T|M \in \mathscr{P}$, and $\mathscr{P}(T|M^{\perp}) = 0$, then $M = \mathscr{P}(T)$.

It may seem that condition (ii) is redundant. To see that it is not, consider the operator $T = A \oplus B$ where A is the unilateral shift operator of multiplicity 1 and B is a normal operator whose spectrum is the closed unit disk. Let $\mathscr{P}$ be the class of all operators that can be written as the sum of a normal operator and a compact operator. It follows from [4] that $\mathscr{P}(T)$ is the entire domain of T and that $T|\mathscr{P}(T)$ is in $\mathscr{P}$. However, (ii) is not satisfied since $B \in \mathscr{P}$ and A has no suboperator in $\mathscr{P}$. (Recall that $A \notin \mathscr{P}$ and A is irreducible.)

The following theorem characterizes those properties that give rise to parts; they are precisely the properties that are closed under direct sums and under restriction to reducing subspaces.

THEOREM 3.3. *Suppose that $\mathscr{P}$ is a property of operators that is closed under unitary equivalence. The following two statements are equivalent:*

(a) *$\mathscr{P}$ is a part-property.*
(b) *A direct sum of operators has property $\mathscr{P}$ if and only if each summand has property $\mathscr{P}$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\mathscr{P}$ is a part-property. Suppose also that T is an operator on a Hilbert space H, and that T is a direct sum of operators in $\mathscr{P}$. It follows that

$$\mathscr{P}(T) = H \quad \text{and} \quad T = T|\mathscr{P}(T) \in \mathscr{P}.$$

Next, suppose that $T \in \mathscr{P}$ and $T = A \oplus B$ relative to $H = M \oplus N$. Consider the operator

$$S = T \oplus T \oplus \cdots = A \oplus B \oplus A \oplus B \cdots.$$

It follows from what was just proved that $S \in \mathscr{P}$. We want to show that $A \in \mathscr{P}$. It will suffice to show that $\mathscr{P}(A) = M$. However, if

$$M_1 = \mathscr{P}(A) \oplus N \oplus M \oplus N \oplus M \oplus N \oplus \cdots,$$

then $S \mid M_1 \in \mathscr{P}$ ($S \mid M_1$ is unitarily equivalent to S) and $S \mid M_1^{\perp}$ has no suboperator in $\mathscr{P}$. Hence $M_1 = \mathscr{P}(S)$. Since S is in $\mathscr{P}$, it follows that $\mathscr{P}(A) = M$.

(b)$\Rightarrow$(a). Suppose (b) is true, and let T be an operator on a Hilbert space H. Using Zorn's lemma, choose a collection $\{M_i : i \in I\}$ of nonzero subspaces of H that is maximal with respect to the following conditions:

(i) The subspaces M_i are pairwise orthogonal.
(ii) Each M_i reduces T.
(iii) $T \mid M_i \in \mathscr{P}$ for every i in I.

Let $M = \bigvee_{i \in I} M_i$. Then M reduces T and, from (b), it follows that $T \mid M \in \mathscr{P}$. By maximality, $T \mid M^{\perp}$ has no suboperator in $\mathscr{P}$. Hence $T \mid M$ and $T \mid M^{\perp}$ have no unitarily equivalent suboperators. It follows from [5, part 3 of Proposition 1.11] that the orthogonal projection Q onto M commutes with every operator that commutes with both T and T^*. Suppose that M_1 reduces T and $T \mid M_1 \in \mathscr{P}$. Since the projection onto M_1 commutes with both T and T^*, then it commutes with Q. Hence

$$M_1 = (M_1 \cap M) \oplus (M_1 \cap M^{\perp}).$$

However, $T \mid M^{\perp}$ has no suboperator in $\mathscr{P}$; hence $M_1 \cap M^{\perp} = 0$. Thus $M_1 \subseteq M$. It follows that $M = \mathscr{P}(T)$ and that $\mathscr{P}$ is a part-property.

DEFINITION 3.4. If $\mathscr{P}$ is a part-property of operators and T is an operator, then $T \mid \mathscr{P}(T)$ is the *$\mathscr{P}$-part* of T and $T \mid \mathscr{P}(T)^{\perp}$ is the *non-$\mathscr{P}$-part* of T.

It is possible that a property of operators may not be a part-property and yet give rise to parts on some particular Hilbert space. (Example: Let H be separable, and let $\mathscr{P}$ be the class of all normal operators with separable range.) It is clear that if a property of operators gives rise to parts on a Hilbert space H, then it gives rise to parts on all Hilbert spaces M with $\dim M \leq \dim H$. Hence, for each infinite cardinal m, we shall consider the properties of operators that give rise to parts on every Hilbert space M with $\dim M < m$.

DEFINITION 3.5. If m is an infinite cardinal, then a property $\mathscr{P}$ of operators is an *m-part-property* if $\mathscr{P}$ is closed under unitary equivalence and, for each operator T with size $T < m$,

(i) $T \mid \mathscr{P}(T) \in \mathscr{P}$, and
(ii) if M reduces T, $T \mid M \in \mathscr{P}$, and $T \mid M^{\perp}$ has no suboperators in $\mathscr{P}$, then $M = \mathscr{P}(T)$.

The proof of Theorem 3.3 is easily adapted to prove the following analogous result.

PROPOSITION 3.6. *Suppose that m is an infinite cardinal and that $\mathscr{P}$ is a property of operators that is closed under unitary equivalence. The following two statements are equivalent:*

(a) *$\mathscr{P}$ is an m-part property.*
(b) *A direct sum T of operators with size $T < m$ has property $\mathscr{P}$ if and only if each summand has property $\mathscr{P}$.*

Since the class of nilpotent (quasinilpotent, compact, invertible) operators is not closed under countable direct sums, this class is not an m-part-property for any cardinal m that is larger than $\aleph_0$. Here are some more examples.

Examples. (J) Let $\mathscr{F}$ be a collection of decomposable functions and let $\mathscr{P}$ be the class of all operators T such that $\phi(T)$ is compact for every ϕ in $\mathscr{F}$. Such operators are *essentially-$\mathscr{F}$-operators.* Since an infinite direct sum of compact operators need not be compact, then $\mathscr{P}$ will generally not be a part-property.
(K) An operator is *block-diagonal* if it is unitarily equivalent to a direct sum of operators having finite size. Since a direct sum of operators is block-diagonal if and only if each summand is block-diagonal (see [5, Corollary 1.9]), then block-diagonality is a part-property. Hence every operator has a *block-diagonal part.* The same holds when the class of block-diagonal operators is replaced by the class of operators that are unitarily equivalent to a direct sum of irreducible operators.
(L) The closure of the set of block-diagonal operators on a Hilbert space is the set of *quasidiagonal* operators [10]. It is clear that a direct sum of quasidiagonal operators is quasidiagonal. However, it was shown by Halmos [9] that the unilateral shift operator is not quasidiagonal, and it was shown by Deddens and Stampfli [4] that this operator is a suboperator of a quasidiagonal operator. Hence the class of quasidiagonal operators is not a part-class.
(M) Let $\mathscr{P}$ be a part-property and let $\mathscr{P}'$ be the class of all operators having no suboperators in $\mathscr{P}$. Then $\mathscr{P}'$ is a part-property and the $\mathscr{P}'$-part of an operator T is the non-$\mathscr{P}$-part of T.

We conclude this section with a determination of the part-properties that can be equationally defined. The surprising (but simple) answer is that they all can. The proof is elementary and is omitted.

THEOREM 3.7. *Suppose that $\mathscr{P}$ is a part-property of operators and ϕ is the function defined by letting $\phi(T)$ be the projection onto $\mathscr{P}(T)^{\perp}$. Then ϕ is a decomposable function and an operator T has property $\mathscr{P}$ precisely when $\phi(T) = 0$.*

4. Remarks and questions

(I) Using Theorem 3.7 we can interpret part (5) of Theorem 2.3 as saying that if $\mathscr{P}$ is a part-property and T is an operator, then the projection onto $\mathscr{P}(T)$

is in the center of $W^*(T)$. Thus if $\mathscr{P}_1$ and $\mathscr{P}_2$ are part-properties and $T \in B(H)$, then we can write

$$H = [\mathscr{P}_1(T) \ominus \mathscr{P}_2(T)] \oplus [\mathscr{P}_1(T) \cap \mathscr{P}_2(T)] \oplus [\mathscr{P}_2(T) \ominus \mathscr{P}_1(T)] \oplus M,$$

where each of the summands reduces T. Hence we can write

$$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$$

where T_1 is a $\mathscr{P}_1$-operator with no $\mathscr{P}_2$-part, T_2 is a $\mathscr{P}_1 \cap \mathscr{P}_2$-operator, T_3 is a $\mathscr{P}_2$-operator with no $\mathscr{P}_1$-part, and T_4 has no suboperator in $\mathscr{P}_1 \cup \mathscr{P}_2$. A similar decomposition holds for a sequence of part-properties.

(II) It is shown in [7] that if $\mathscr{F}$ is a collection of decomposable functions and T is an essentially-$\mathscr{F}$-operator, then the non-$\mathscr{F}$-part of T is unitarily equivalent to a direct sum of irreducible operators. It is also shown in [7] that if $\mathscr{F}$ is a collection of continuous decomposable functions, T is an essentially-$\mathscr{F}$-operator on a separable Hilbert space, and if S is a norm limit of operators that are unitarily equivalent to T, then the non-$\mathscr{F}$-part of S is unitarily equivalent to the non-$\mathscr{F}$-part of T.

(III) Very little is known about decomposable functions. The restriction of a continuous decomposable function to the scalars (operators with size one) yields a continuous complex function. This continuous complex function determines the values of the decomposable function on the diagonal, and hence normal, operators. Similarly, the value of the decomposable function on the quasidiagonal operators is determined by its value on all operators with finite size; once its values are known for quasidiagonal operators, its values are determined for all suboperators of quasidiagonal operators (e.g., the unilateral shift). (It is conjectured that every operator is unitarily equivalent to a sub-operator of a quasidiagonal operator.) Here are a few of the many interesting questions that can be asked about decomposable functions.

(a) Which continuous complex functions in the plane can be extended to continuous decomposable functions?

(b) Which part-properties of operators can be defined by a collection of continuous decomposable functions?

(c) It is easily shown that if ϕ is a decomposable function that is continuous with respect to the weak operator topology, then ϕ has the form $\phi(T) = aT + bT^* + c$. Are there simple characterizations of decomposable functions that are continuous with respect to other operator topologies?

(d) If T is an operator on a separable Hilbert space, then is $\{\phi(T)$: ϕ is a continuous decomposable function$\}$ a C^*-algebra? Is it the C^*-algebra generated by 1 and T? (The answer to both of these questions is "yes" if T is compact or normal.)

(e) If ϕ is a continuous decomposable function and π is a representation of the C^*-algebra generated by 1 and T, then does $\pi(\phi(T)) = \phi(\pi(T))$ define an extension of π to a representation of the C^*-algebra generated by 1, T, and $\phi(T)$?

Added in proof. We have recently learned that some of the results in this paper are similar to those in [15] and [16]. Also [17] contains many interesting properties of decomposable functions, including answers to most of the preceding questions.

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