

# CHARACTER CORRESPONDENCES IN SOLVABLE GROUPS

BY

THOMAS R. WOLF

## 1. Introduction

All groups considered here are finite, unless otherwise specified. By  $\text{Ch}(G)$ , we denote all complex characters of  $G$ ; and by  $\text{IRR}(G)$ , we denote the set of those  $\psi \in \text{Ch}(G)$  that are irreducible. (On occasion, where it involves no loss of generality to the specific argument, we may say  $\Lambda \in \text{Ch}(G)$  allowing the possibility  $\Lambda = 0$ ). If a group  $A$  acts on  $G$  by automorphisms and if  $a \in A$ ; then  $\chi^a$  defined by  $\chi^a(h^a) = \chi(h)$  is an irreducible character of  $G$  whenever  $\chi$  is. If  $A$  is cyclic, the actions of  $A$  on  $\text{IRR}(G)$  and on the conjugacy classes of  $G$  are permutation isomorphic. Counterexamples exist for noncyclic  $A$ . We write  $\text{IRR}_A(G)$  to denote the  $A$ -fixed irreducible characters of  $G$ .

Now assume  $A$  acts on  $G$  by automorphisms and  $(|G|, |A|) = 1$ . Should  $A$  be solvable, G. Glauberman has defined a "natural" one-to-one correspondence between  $\text{IRR}_A(G)$  and  $\text{IRR}(C)$  [5], if  $C = C_G(A)$ . When  $|G|$  is odd, I. M. Isaacs has described a "natural" correspondence between  $\text{IRR}_A(G)$  and  $\text{IRR}(C)$  [6]. By "natural" we mean a map uniquely determined by the action of  $A$  on  $G$  and thus independent of choices made in an algorithm. The Odd-Order Theorem implies one of these correspondences occurs. One corollary of these correspondences is that  $A$  acts isomorphically on  $\text{IRR}(G)$  and the conjugacy classes of  $G$  (see comments preceding Theorem 5.5). Both correspondences exist precisely when  $|G|$  is odd and  $A$  is solvable; and we show in this paper that the two are then identical.

Let  $N \trianglelefteq G$ ,  $\chi \in \text{IRR}(G)$ ,  $\theta \in \text{IRR}(N)$ , and  $T = I_G(\theta)$  (i.e.,  $T = \{g \in G \mid \theta^g = \theta\}$ ). We say  $\chi \in \text{IRR}(G \mid \theta)$  if  $[\chi_N, \theta] \neq 0$ . If  $\chi \in \text{IRR}(G \mid \theta)$ ,  $\chi_T$  has a unique irreducible constituent  $\mu \in \text{IRR}(T \mid \theta)$ . Also,  $\mu^G = \chi$ .

Assume  $T = G$  and  $\chi \in \text{IRR}(G \mid \theta)$ . So  $\chi_N = f\theta$  for some  $f \in Z$ . If  $f = 1$ ; the constituents of  $\theta^G$  are precisely the characters  $\beta\chi$  for  $\beta \in \text{IRR}(G/N)$  and are distinct for distinct  $\beta$ . This will occur whenever  $G/N$  is cyclic. If, on the other hand,  $f^2 = |G:N|$ ; we say  $\chi$  or  $\theta$  is fully ramified with respect to  $G/N$ . This will occur if  $I_G(\theta) = G$  and either  $\chi$  vanishes on  $G - N$  or  $\chi$  is the unique constituent of  $\theta^G$ .

If  $K/L$  is an abelian chief factor of  $G$  and  $\phi \in \text{IRR}(K)$  is invariant in  $G$ ; then  $\phi_L \in \text{IRR}(L)$ ,  $\phi$  is fully ramified with respect to  $K/L$ , or  $\phi_L$  is the sum of  $|K:L|$  distinct irreducible characters of  $L$ . The results of these last few paragraphs are well known (see Chapter 6 of [7]) and will be used without reference.

Section 2 basically deals with preliminaries. Sections 3 and 4 define and investigate the correspondences of Glauberman and Isaacs, respectively. Via

his theory of Clifford systems, E. C. Dade (see [1], [2], and [3]) has done work inclusive of that of Isaacs'. In Chapter 13 of his book [7], Isaacs presents and develops some properties of the Glauberman correspondence. Section 5 proves the equivalence of the two correspondences.

If  $N \trianglelefteq G$  and  $\theta \in \text{IRR}(N)$  is invariant in  $G$ , we say  $(G, N, \theta)$  is a character triple. We will use some facts about character triple isomorphisms. (See Section 8 of [6] or Chapter 11 of [7].) Otherwise, everything should be self-explanatory.

## 2. Preliminaries

In this section, we prove some properties of coprime actions and of characters. The first lemma is quite useful and is easily proved by Frobenius reciprocity and counting character degrees.

**2.1 LEMMA.** *Let  $N \trianglelefteq G$ ,  $NH = G$ , and let  $\theta \in \text{IRR}(N)$  be invariant in  $G$ . Let  $M = N \cap H$  and assume  $\theta_M \in \text{IRR}(M)$ . Then  $\chi \leftrightarrow \chi_H$  defines a one-to-one correspondence between  $\text{IRR}(G|\theta)$  and  $\text{IRR}(H|\theta_M)$ .*

*Proof.* See Lemma 10.5 of [6].

**2.2 LEMMA.** *Let  $(G, N, \theta)$  be a character triple with  $G/N$  abelian. Then:*

- (a) *There is a unique  $U \leq G$  maximal with respect to  $\theta$  having a  $G$ -invariant extension to  $U$ .*
- (b) *Every extension of  $\theta$  to  $U$  is fully ramified with respect to  $G/U$ .*
- (c) *If furthermore  $G \leq \Gamma$  with  $G, N$ , and  $\theta$  invariant in  $\Gamma$ , then  $U \trianglelefteq \Gamma$ .*

*Proof.* Let  $\alpha$  and  $\beta$  be  $G$ -invariant extensions of  $\theta$  to  $V$  and  $W$ . Then  $(\delta\alpha)_V \cap W = \beta_{V \cap W}$  for some linear  $\delta \in \text{IRR}(V/N)$ . By Lemma 2.1,  $\beta = \psi W$  for a unique  $\psi \in \text{IRR}(VW|\delta\lambda)$ . Since  $I_G(\delta) = G$  and  $\psi$  is unique,  $\psi$  is  $G$ -invariant, proving (a).

Let  $G \leq \Gamma$  and assume  $G, N$ , and  $\theta$  are  $\Gamma$ -invariant. Let  $\alpha$  be a  $G$ -invariant extension of  $\theta$  to  $U$ . For  $y \in \Gamma$ ,  $\alpha^y$  extends  $\theta$  to  $U^y$  and  $\alpha^y$  is  $G$ -invariant. Part (c) follows by the uniqueness of  $U$ .

As all extensions of  $\theta$  to  $U$  are  $G$ -invariant, part (b) will be proved if we assume  $U = N$  and show  $\theta$  is fully ramified with respect to  $G/N$ . Let  $\chi \in \text{IRR}(G|\theta)$  so that it suffices to show  $\chi$  vanishes off  $N$ . Let  $N \leq T \leq G$  with  $T/N$  cyclic, and let  $\phi_1 \in \text{IRR}(T)$  extend  $\theta$  such that  $[\chi_T, \phi_1] \neq 0$ . Now  $\chi_T = t \sum_{\lambda \in S} \lambda \phi_1$  for some subset  $S$  of  $\text{IRR}(T/N)$ ,  $t \in \mathbb{Z}$ . For  $\lambda, \beta \in S$ , we have  $(\phi_1)^x = \lambda \phi_1$  and  $(\phi_1)^v = \beta \phi_1$  for some  $x, v \in G$ . Then  $(\phi_1)^{xv} = (\lambda \phi_1)^v = (\lambda \beta) \phi_1$ , and  $S$  is a subgroup of the cyclic group  $\text{IRR}(T/N)$ . Since  $U = N$ , a generator of  $S$  is faithful. Hence,  $S = \text{IRR}(T/N)$  and  $\chi$  vanishes on  $T - N$ . Thus  $\chi$  vanishes on  $G - N$ . The proof is complete.

The following lemma, which can be proved via the Schur-Zassenhaus Theorem, is quite useful when looking at coprime actions. It is due to G. Glauberman [4]. Also, a proof can be found in 13.8 and 13.9 of [7].

2.3 LEMMA. Assume  $A$  acts on  $G$  by automorphisms and  $(|A|, |G|) = 1$ . Suppose  $A$  and  $G$  act on a set of  $T$  such that  $G$  is transitive on  $T$  and  $(t \cdot g) \cdot a = (t \cdot a) \cdot g^a$  for  $t \in T, g \in G,$  and  $a \in A$ . Then (a)  $A$  has fixed points in  $T$ , and (b)  $C_G(A)$  acts transitively on the fixed points of  $A$ .

2.4 COROLLARY. Assume  $A$  acts on  $G, N \trianglelefteq G$  is  $A$ -invariant, and  $(|G:N|, |A|) = 1$ . Let  $\chi \in \text{IRR}_A(G)$ . Then:

- (a)  $[\chi_N, \theta] \neq 0$  for some  $\theta \in \text{IRR}_A(G)$ .
- (b) If  $C_{G/N}(A) = 1, \theta$  is unique.
- (c) If  $C_{G/N}(A) = G/N,$  every constituent of  $\chi_N$  is  $A$ -invariant.

*Proof.* Now  $G/N$  acts transitively on the set  $S$  of irreducible constituents of  $\chi_N,$  and  $A$  acts on  $S$  as  $\chi$  is  $A$ -invariant. Application of Lemma 2.3 yields the result.

If  $(|A|, |G|) = 1$  in Corollary 2.4, then  $C_{G/N}(A) = NC_G(A)/N$ . So the conditions in (b) and (c) may be stated as  $C_G(A) \leq N$  and  $NC_G(A) = G,$  respectively.

2.5 LEMMA. Assume  $A$  acts on  $G$  by automorphisms,  $N \trianglelefteq G$  is  $A$ -invariant,  $(|G:N|, |A|) = 1,$  and  $G = NC_G(A)$ . Let  $\chi \in \text{IRR}(G)$  and  $\theta \in \text{IRR}(N)$  such that  $[\chi_N, \theta] \neq 0$ . Then  $\chi \in \text{IRR}_A(G)$  if and only if  $\theta \in \text{IRR}_A(N)$ .

*Proof.* One direction is Corollary 2.4(c). So assume  $\theta \in \text{IRR}_A(N)$  and induct on  $|G|$ . If  $N \leq U \leq G,$  the hypotheses imply  $U$  is  $A$ -invariant. We need just show  $\chi^a(g) = \chi(g)$  for  $a \in A, g \in G$ . Hence, we assume by induction that  $G/N$  is cyclic. Choose  $N \leq W \trianglelefteq G$  with  $|G:W|$  prime. By induction, choose  $\alpha \in \text{IRR}_A(W|\theta)$  such that  $[\chi_W, \alpha] \neq 0$ . Now  $\alpha^G$  is  $A$ -invariant and  $A$  acts on the irreducible constituents of  $\alpha^G$ . If  $\chi = \alpha^G,$  we're done. So, we may assume  $\chi$  extends  $\alpha$ . Now  $\text{IRR}(G/W)$  and  $A$  act on the irreducible constituents of  $\alpha^G$ . The result follows from Lemma 2.3.

### 3. Glauberman correspondence

The intent of this section is to expose the character correspondence for coprime action developed by G. Glauberman. The first theorem states and characterizes the correspondence.

THEOREM 3.1. For each pair of groups  $G$  and  $A$  where  $(|G|, |A|) = 1, A$  is solvable, and  $A$  acts on  $G$  via automorphisms, there is a uniquely defined map

$$\pi_1(G, A): \text{IRR}_A(G) \rightarrow \text{IRR}(C) \quad \text{with } C = C_G(A)$$

satisfying:

- (a)  $\pi_1(G, A)$  is one-to-one and onto.
- (b) If  $A$  is a  $p$ -group and  $\chi \in \text{IRR}_A(G),$  then  $\chi\pi_1(G, A)$  is the unique  $\alpha \in \text{IRR}(C)$  such that  $p \nmid [\chi_C, \alpha]$ .
- (c) If  $T \trianglelefteq A$  and  $B = C_G(T),$  then  $\pi_1(G, A) = \pi_1(G, T)\pi_1(B, A/T)$ .

*Proof.* See [5] or Chapter 13 of [7].

By “uniquely defined”, we imply that the map is independent of choices made in any algorithm, i.e., the map is completely determined by the action of  $A$  on  $G$ . Note that conditions (b) and (c) imply that  $\chi\pi_1(G, A)$  is a constituent of  $\chi_C$ .

**3.2 HYPOTHESES.** Let  $A$  act on  $G$  such that  $(|A|, |G|) = 1$ . Let  $C = C_G(A)$  and let  $\Gamma$  be the semi-direct product  $GA$ .

**3.3 COROLLARY.** Assume Hypotheses 3.2 and assume that  $A$  is a  $p$ -group. Let  $C \leq V \leq G$  with  $V$  being  $A$ -invariant. If  $\chi \in \text{IRR}_A(G)$ , there is a unique  $\psi \in \text{IRR}_A(V)$  such that  $p \nmid [\chi_V, \psi]$ . Furthermore  $\chi\pi_1(G, A) = \psi\pi_1(V, A)$ .

*Proof.* For  $\alpha, \beta \in \text{IRR}(V)$  with  $\alpha^a = \beta$  for some  $a \in A$ , we have  $[\chi_V, \alpha] = [\chi_V, \beta]$  and  $\alpha_C = \beta_C$ . Write  $\chi_V = \sum a_i \psi_i + \sum b_j \Delta_j$  where  $\psi_i \in \text{IRR}_A(V)$  and each  $\Delta_j$  is the sum of a nontrivial  $A$  orbit in  $\text{IRR}(V)$ . Let  $\varepsilon_i = \psi_i\pi_1(V, A)$ . Now  $[\chi_C, v] \equiv \sum_i a_i [(\psi_i)_C, v] \pmod{p}$  for each  $v \in \text{IRR}(C)$ . So  $[\chi_C, v] \equiv 0 \pmod{p}$  if  $v$  is not one of the  $\varepsilon_i$ , and  $[\chi_C, \varepsilon_i] \equiv a_i [(\psi_i)_C, \varepsilon_i]$  for each  $i$ . By Theorem 3.1, it follows that  $a_i \not\equiv 0 \pmod{p}$  for exactly one  $i$ . This gives our result.

The above result will be used to show the equivalence of the two correspondences. It can also be used with Lemma 2.5 for a straightforward proof of Lemma 3.4. A similar proof appears in Chapter 13 of [7], where Lemma 3.5 appears as an exercise.

**3.4 LEMMA.** Assume Hypotheses 3.2 with  $A$  solvable. Suppose  $N \trianglelefteq \Gamma$  with  $N \leq G$ . Let  $\chi \in \text{IRR}_A(G)$ ,  $\theta \in \text{IRR}_A(N)$ ,  $\varepsilon = \chi\pi_1(G, A)$ , and  $\delta = \theta\pi_1(N, A)$ . Then  $[\chi_N, \theta] \neq 0$  if and only if  $[\varepsilon_{N \cap C}, \delta] \neq 0$ .

*Proof.* See Theorem 13.29 of [7].

**3.5 LEMMA.** Assume Hypotheses 3.2,  $A$  is solvable,  $N \trianglelefteq \Gamma$ ,  $NC = G$ ,  $\theta \in \text{IRR}_A(N)$ , and  $\delta = \theta\pi_1(N, A)$ . If  $T = I_G(\theta)$ , then

- (a)  $T \cap C = I_C(\delta)$ , and
- (b) whenever  $\psi \in \text{IRR}(T|\theta)$ ,  $\psi^G\pi_1(G, A) = (\psi\pi_1(T, A))^C$ .

*Proof.* As  $NC = G$ ; we may use Lemma 2.5 to see that  $T, \psi$ , and  $\psi^G$  are  $A$ -invariant. So (b) is meaningful. If  $B \leq A$ ,  $NC_G(B) = G$ . By induction and Theorem 3.1, we assume  $|A| = p$ , a prime.

Write  $\theta_{N \cap C} = s\delta + p\Lambda$  with  $\Lambda \in \text{Ch}(N \cap C)$  and  $p \nmid s$ . For  $x \in C$ ,  $\theta^x \in \text{IRR}_A(N)$  and  $(\theta^x)_{N \cap C} = s\delta^x + p\Lambda^x$ . So  $x \in T$  if and only if  $x \in I_C(\delta)$ , proving (a). Write  $\psi_{T \cap C} = t\alpha + p\Xi$  with  $\Xi \in \text{Ch}(T \cap C)$  and  $p \nmid t$ . As  $[\alpha_{N \cap C}, \delta] \neq 0$  by Lemma 3.4, it follows from (a) that  $\alpha^C \in \text{IRR}(C)$ . Now  $(\psi^G)_C = (\psi_{T \cap C})^C = t\alpha^C + p\Xi^C$ . Now, application of Theorem 3.1 completes the proof.

4. Isaacs correspondence

Here we describe the character correspondence developed by I. M. Isaacs [5], and then proceed to develop some properties of it. For convenience, we first establish some notation. Suppose  $A$  acts on  $G$ ,  $H \leq G$  is  $A$  invariant, and  $\chi \in \text{IRR}_A(G)$ . If there is a unique  $\psi \in \text{IRR}_A(H)$  such that  $[\chi_H, \psi]$  is odd, we write  $\psi = \chi\sigma(G, H, A)$ .

We will need the following two results of Isaacs' paper. Theorem 4.1 combines Theorems 9.1 and 6.3 and Corollary 6.4 of [7]. This, in turn, is used to prove Theorem 4.2 (Corollary 10.7 of [7]).

4.1 THEOREM. Assume  $K, L \trianglelefteq G, K' \leq L \leq K; |K:L|$  or  $|G:K|$  is odd;  $\theta \in \text{IRR}(K)$  and  $\phi \in \text{IRR}(L)$  are invariant in  $G; [\theta_L, \phi] \neq 0$ ; and  $\phi$  is fully ramified with respect to  $K/L$ . Then there is a  $\psi \in \text{Ch}(G/K)$  and a conjugacy class  $F$  of subgroups  $U \leq G$  such that:

- (a)  $\psi^2(x) = \pm |C_{K/L}(x)|$  for  $x \in G$ .
- (b)  $UK = G$  and  $U \cap K = L$ .
- (c)  $U^a \in F$  for every  $a \in \text{Aut}(G)$  which stabilizes  $K, L, \theta$ , and  $\phi$ .
- (d) For  $\chi \in \text{IRR}(G|\theta), \chi_U = \psi_U \alpha$  for a unique  $\alpha \in \text{IRR}(U)$ .
- (e)  $\chi_U = \psi_U \alpha$  defines a one-to-one correspondence between  $\chi \in \text{IRR}(G|\theta)$  and  $\alpha \in \text{IRR}(U|\phi)$ .
- (f) If  $|G:L|$  is odd,  $\chi \in \text{IRR}(G|\theta)$ , and  $\alpha \in \text{IRR}(U)$  then  $\chi_U = \psi_U \alpha$  if and only if  $[\chi_U, \alpha]$  is odd.

Note if  $A$  acts on  $G$  in the above situation and  $A$  stabilizes  $\chi$  and some  $U \in F$ ; then  $\alpha = \chi\sigma(G, U, A)$  in (f) above.

4.2 THEOREM. Assume  $A$  acts on  $G, K$  and  $L$  are  $A$ -invariant normal subgroups of  $G, K' \leq L \leq K$ , and  $(|G:L|, 2|A|) = 1$ . Let  $H/L = C_{G/L}(A)$ , and assume  $HK = G$  and  $H \cap K = L$ . Then  $\chi\sigma(G, H, A)$  exists for all  $\chi \in \text{IRR}_A(G)$ . Furthermore,  $\chi \leftrightarrow \chi\sigma(G, H, A)$  defines a one-to-one correspondence between  $\text{IRR}_A(G)$  and  $\text{IRR}_A(H)$ .

4.3 COROLLARY. Assume Hypotheses 3.2 with  $|G|$  odd. Let  $[G, A]C \leq H \leq G$  such that  $H$  is  $A$ -invariant. Then  $\chi\sigma(G, H, A)$  exists for all  $\chi \in \text{IRR}_A(G)$ , and  $\sigma(G, H, A)$  is a one-to-one map from  $\text{IRR}_A(G)$  onto  $\text{IRR}_A(H)$ .

*Proof.* Let  $K = [G, A]$  and  $L = K \cap H$ . Note that  $K' \leq L \trianglelefteq G$ . As  $C \leq H, HK = G$ . Also,  $[H, A] \leq K \cap H = L$ . By properties of coprime action,  $H/L \leq C_{G/L}(A) = LC/L \leq H/L$ . Apply Theorem 4.2 to finish the proof.

Assume Hypotheses 3.2 with  $|G|$  odd. Now  $G$  is solvable by the Odd-Order

Theorem. If  $C < G$ ,  $[G, A]' < [G, A]$  and  $[G, A]'C < G$ . We define a map  $\pi_2(G, A): \text{IRR}_A(G) \rightarrow \text{IRR}(C)$  inductively as follows. If  $\chi \in \text{IRR}_A(G)$ , let

$$\chi\pi_2(G, A) = \chi\sigma(G, G_1, A)\pi_2(G_1, A)$$

if  $G_1 = [G, A]'C$  is proper in  $G$ . Otherwise, let  $\chi\pi_2(G, A) = \chi$ . Induction and Corollary 4.3 show that  $\pi_2(G, A)$  is one-to-one and onto, and they also imply that  $\pi_2(G, A)$  is completely determined by the action of  $A$  on  $G$ . Now  $\pi_2(G, A)$  is the character correspondence described by Isaacs [7]. We need some of its properties.

4.4 LEMMA. Assume Hypotheses 3.2 with  $|G|$  odd. Suppose  $NC = G$  with  $N \trianglelefteq \Gamma$ . Assume that  $[G, A]'C \leq H \leq G$  and that  $H$  is  $A$ -invariant. Let  $L = N \cap H$ ,  $\theta \in \text{IRR}_A(N)$ ,  $T = I_G(\theta)$ ,  $\psi \in \text{IRR}(T|\theta)$ ,  $\phi = \theta\sigma(N, L, A)$ , and  $\beta = \psi\sigma(T, T \cap H, A)$ . Then

- (a)  $H \cap T = I_H(\phi)$ ,
- (b)  $C \cap T = I_C(\theta\pi_2(N, A))$ ,
- (c)  $\beta^H = \psi^G\sigma(G, H, A)$ , and
- (d)  $\psi^G\pi_2(G, A) = (\psi\pi_2(T, A))^C$ .

*Proof.* As  $NC = G$ ,  $T$  is  $A$ -invariant, and, by Lemma 2.5,  $\psi \in \text{IRR}_A(T)$ . As

$$[T, A]'(T \cap C) \leq [G, A]' \cdot (T \cap C) \leq H \cap T$$

and as similarly  $[N, A]' \cdot (N \cap C) \leq L$ ,  $\psi\sigma(T, T \cap H, A)$  and  $\theta\sigma(N, L, A)$  are defined. If  $C < G$ ,  $[G, A]'C < G$ . So parts (a) and (c) with  $H = [G, A]'C$  imply parts (b) and (d) by induction.

So we prove (a) and (c). Write  $\theta_L = \phi + 2\lambda + \Xi$  and  $\psi_{T \cap H} = \beta + 2\mu + \Lambda$  for (possibly zero) characters  $\lambda, \Xi$  of  $L$  and  $\mu, \Lambda$  of  $T \cap H$  such that  $\Xi$  and  $\Lambda$  have no  $A$ -invariant irreducible constituents. For a  $\Gamma$ -invariant subgroup  $W$  of  $G$ ,  $C$  permutes  $\text{IRR}_A(W)$  by conjugation. As  $(\theta^c)_L = \phi^c + 2\lambda^c + \Xi^c$  for  $c \in C$ , part (a) follows.

As  $LC = H$ ,  $[\Lambda_L, \phi] = 0$  by Lemma 2.5. So  $[\beta_L, \phi] \neq 0$ . Note that

$$\psi^G \in \text{IRR}_A(G) \quad \text{and} \quad \beta^H \in \text{IRR}_A(H).$$

It suffices to show that  $[(\psi^G)_H, \beta^H] = [(\psi_{T \cap H})^H, \beta^H]$  is odd. Now

$$(\psi_{T \cap H})^H = \beta^H + 2\mu^H + \Lambda^H.$$

We are done if  $[\Lambda^H, \beta^H] = [(\beta^H)_{T \cap H}, \Lambda]$  is zero. As  $LC = H$ , Lemma 2.5 implies every constituent of  $(\beta^H)_{T \cap H}$  is  $A$ -invariant. So  $[(\beta^H)_{T \cap H}, \Lambda] = 0$ , completing the proof.

4.5 LEMMA. Let  $K \trianglelefteq G$  and  $K' \leq L = \mathbf{Z}(K) \leq \mathbf{Z}(G)$  with  $|K:L|$  odd. Assume each coset of  $L$  in  $K$  contains an element  $k$  such that  $L \cap \langle k \rangle = 1$ . Then there is a conjugacy class of subgroups  $U \leq G$  and there exist involutions  $\rho \in \text{Aut}(G)$  such that  $UK = G$ ,  $U \cap K = L$ ,  $U = C_G(\rho)$ , and  $\rho$  inverts  $K/L$ .

*Proof.* See Lemma 4.4 of [7].

The above is used to prove the next theorem. One would expect the next theorem to be true, but it does take some work. Also, see the comments following Corollary 4.7. This theorem allows us to prove statements analogous to 3.4 and 3.1(c) for  $\pi_2(G, A)$  (Lemmas 4.8 and 4.9), allows more flexibility for the algorithm for  $\pi_2(G, A)$  (Corollary 4.7), and helps prove the equivalence of the correspondences.

**4.6 THEOREM.** *Assume Hypotheses 3.2 with  $|G|$  odd. Let  $K = [G, A]$ ,  $L = K'$ , and  $S = LC$ . Assume that  $S \leq H \leq G$  is  $A$ -invariant. Let  $\chi \in \text{IRR}_A(G)$  and  $\psi = \chi\sigma(G, H, A)$ . Then*

- (a)  $\chi\sigma(G, S, A) = \psi\sigma(H, S, A)$ , and
- (b)  $\chi\pi_2(G, A) = \psi\pi_2(H, A)$ .

*Proof.* Of course, both parts are true when  $H = G$ . Note that  $[H, A]C \leq S$ . So, inductive arguments show parts (a) and (b) are equivalent.

Among all possible counterexamples with  $|G|$  minimal, choose one with  $|G:H|$  minimal. We may assume that  $H < G$  and hence that  $C < G$ . Let  $\beta = \chi\sigma(G, S, A)$ . It suffices to show that  $[\psi_S, \beta]$  is odd, by induction.

By Lemma 2.4, we can choose  $\theta \in \text{IRR}_A(K)$  such that  $[\chi_K, \theta] \neq 0$ . As

$$[I_G(\theta), A](I_G(\theta) \cap C) \leq I_G(\theta) \cap S,$$

induction and Lemma 4.4 permit us to assume  $I_G(\theta) = G$ .

Let  $N = H \cap K$ . As  $L \leq N$ ,  $N \triangleleft \langle H, K, A \rangle = \Gamma$ . As  $K = [G, A]$  and  $(|G|, |A|) = 1$ , Fitting's Lemma implies  $C_{K/L}(A) = 1$  and  $C_{K/N}(A) = 1$ . By Corollary 2.4(b), we can let  $\phi$  be the unique  $A$ -invariant irreducible constituent of  $\theta_N$ . As  $\chi_K$  is a multiple of  $\theta$ ,  $\phi$  is the unique  $A$ -invariant constituent of  $\chi_N$  and of  $\psi_N$ . Let  $T = I_G(\phi)$ , so that  $T$  is  $A$ -invariant and  $H \leq T$  by Lemma 4.4. Choose the unique  $\delta \in \text{IRR}(T)$  such that  $[\chi_T, \delta] \neq 0$  and  $[\delta_N, \phi] \neq 0$ . The uniqueness implies  $\delta \in \text{IRR}_A(T)$ . Now  $\delta^G = \chi$  and  $\chi_T = \delta + \Xi$  where  $[\Xi, \phi] = 0$ . So  $[\Xi_H, \psi] = 0$  and  $[\chi_H, \psi] = [\delta_H, \psi]$  is odd. If  $H < T < G$ , we have by induction on  $|G|$  and on  $|G:H|$  that  $\chi\pi_2(G, A) = \delta\pi_2(T, A) = \psi\pi_2(H, A)$ . If  $T = G$ ,  $\chi_N$  is a multiple of  $\phi$ , and, since  $NC = H$ , Lemma 2.5 implies every irreducible constituent of  $\chi_H$  is  $A$ -invariant. Then  $\chi_H = \psi + 2\lambda$  for some  $\lambda \in \text{Ch}(H)$ , and hence  $1 \equiv [\chi_S, \beta] \equiv [\psi_S, \beta] \pmod{2}$ . So we assume  $I_G(\phi) = H$  and  $\psi^G = \chi$ .

If  $H < F < G$  with  $F$   $A$ -invariant, we have, by induction,

$$\chi\pi_2(G, A) = \psi^F\pi_2(F, A) = \psi\pi_2(H, A).$$

So we assume  $H$  is a maximal  $A$ -invariant subgroup of  $G$ . Note  $K/N$  is an elementary abelian chief factor of  $\Gamma$  and  $\phi^K = \theta$ .

Via Corollary 2.4(b), we can let  $\varepsilon$  be the unique  $A$ -invariant constituent of  $\theta_L$  (and, hence of  $\phi_L$  and of  $\chi_L$ ). Let  $D = I_H(\varepsilon)$  and  $J = I_G(\varepsilon)$ . By Lemma 4.4,

$S \leq D \leq J \leq G$ . Note that  $\beta_L$  is a multiple of  $\varepsilon$ . Choose the unique  $\tau \in \text{IRR}(D|\varepsilon)$  such that  $[\psi_D, \tau] \neq 0$ . Note that  $\tau^H = \psi$  and  $\tau^G = \chi$ . Now  $\tau^J \in \text{IRR}(J|\varepsilon)$  is unique such that  $[\chi_J, \tau^J] \neq 0$ . Hence  $[\chi_S, \beta] = [(\tau^J)_S, \beta]$  is odd. Of course,  $D, J, \tau$ , and  $\tau^J$  are fixed by  $A$ . If  $J < G$ , we have, by induction,

$$\psi\pi_2(H, A) = \tau\pi_2(D, A) = \tau^J\pi_2(J, A) = \beta\pi_2(S, A) = \chi\pi_2(G, A).$$

We assume  $I_G(\varepsilon) = G$ .

By Lemma 2.2, we can choose  $L \leq U \leq K$  with  $U \triangleleft \Gamma$  such that  $\varepsilon$  extends to  $U$  and every extension is fully ramified with respect to  $K/U$ . Now both  $A$  and the group  $B$  of linear characters of  $U/L$  act on the extensions of  $\varepsilon$  to  $U$ . Note  $B$  acts transitively and  $C_B(A) = 1$ . By Lemma 2.3,  $\varepsilon$  has a unique extension  $\eta \in \text{IRR}_A(U)$ . Note that  $\eta$  is fully ramified with respect to  $K/U$ , is the unique irreducible constituent of  $\theta_U$  and of  $\chi_U$ , and is invariant in  $\Gamma$ . Note that  $\eta_{U \cap N} \in \text{IRR}(U \cap N)$ . If  $UN = K$ ,  $\theta_N = \phi$  by Lemma 2.1. This contradiction implies  $U \leq N$ . Let  $M = UC = US \leq H$ .

As  $\chi_U$  is a multiple of  $\eta$  and as  $UC = M$ , Lemma 2.5 implies that every irreducible constituent of  $\chi_M$  is  $A$ -invariant. So  $\chi_M = \omega + 2\Delta$  for some  $\omega \in \text{IRR}_A(M)$  and  $\Delta \in \text{Ch}(M)$ . Now

$$1 \equiv [\chi_S, \beta] \equiv [\omega_S, \beta] \pmod{2},$$

and, by induction,  $\omega\pi_2(M, A) = \beta\pi_2(S, A) = \chi\pi_2(G, A)$ .

By induction, it suffices to show  $[\psi_M, \omega]$  is odd. To do this, we choose a character triple  $(\Gamma^*, U^*, \eta^*)$  isomorphic to  $(\Gamma, U, \eta)$  such that  $\eta^*$  is faithful and linear and such that each coset of  $U^*$  contains an element  $x$  such that  $\langle x \rangle \cap U^* = 1$  (see Theorem 8.2 of [7]). (Note that we allow  $(|A|, |U^*|) \neq 1$ .) We use  $*$  to denote appropriate images. Note that  $\Gamma^*/U^*$  is isomorphic to  $\Gamma/U$ , and note that  $A$ -fixed characters lying over  $\eta$  are mapped by  $*$  to  $(UA)^*$ -fixed characters lying over  $\eta^*$  since inertia groups are preserved by the character triple isomorphism.

Since  $U^* \leq Z(G^*)$  and  $\theta^*$  is fully ramified with respect to  $K^*/U^*$ ,  $U^* = Z(K^*)$ .

By the Schur-Zassenhaus Theorem, all complements for  $G^*/U^*$  in  $\Gamma^*/U^*$  are conjugate in  $\Gamma^*$ . Now Lemma 4.5 implies that there exist  $W$  so that  $(UA)^* \leq W \leq \Gamma^*$  and involution  $\rho \in \text{Aut}(\Gamma^*)$  such that  $W$  is the centralizer in  $\Gamma^*$  of  $\rho$ ,  $WK^* = \Gamma^*$ ,  $W \cap K^* = U^*$ , and  $\rho$  inverts  $K^*/U^*$ . Let  $V = W \cap G^* = W$ . Now

$$[V, (UA)^*] \leq [G^*, (UA)^*] \cap V = U^*.$$

So  $V$  centralizes  $(UA)^*/U^*$ . So  $V = M^*$  and  $(MA)^* = W$  is the centralizer in  $\Gamma^*$  of  $\rho$ .

Of course,  $\eta^*$  and  $\omega^*$  are  $\rho$ -invariant. As  $\theta^*$  vanishes off  $U^*$ ,  $\theta^*$  is fixed by  $\rho$ . As  $K^*M^* = G^*$ , Lemma 2.5 implies that  $\chi^*$  is  $\rho$ -invariant. As  $\rho$  inverts  $K^*/U^*$ ,  $\rho$  fixes  $N^*$  and  $N^*M^* = H^*$ . As  $(UA)^*$  centralizes  $\rho$ ,  $(\psi^*)^\rho$  is fixed by  $(UA)^*$ . The uniqueness of  $\psi$  (and of  $\psi^*$ ) implies  $(\psi^*)^\rho = \psi^*$ . Theorem 4.2 implies



$\psi^* = \chi^* \sigma(G^*, H^*, \langle \rho \rangle)$ . Hence,  $\chi^*$  restricted to  $H^*$  is  $\psi^* + 2\gamma + \sum_i (v_i + v_i^?)$  where  $\gamma \in \text{Ch}(H^*)$ ,  $v_i \in \text{IRR}(H^*)$ , and  $v_i \neq v_i^?$ . As  $\omega^*$  is  $\rho$ -invariant and as  $[\chi_M, \omega]$  is odd, the character triple isomorphism yields  $[\chi_M, \omega] \equiv [\psi_M, \omega] \equiv 1 \pmod{2}$ . The result follows from above comments.

4.7 COROLLARY. Assume Hypotheses 3.2 with  $|G|$  odd. Assume

$$C = G_n \leq G_{n-1} \cdots \leq G_0 = G$$

for  $A$ -invariant  $G_i$ , and  $[G_i, A]C \leq G_{i+1}$ . Let  $\chi_0 \in \text{IRR}_A(G)$  and define  $\chi_i \in \text{IRR}_A(G_i)$  inductively by

$$\chi_i = \chi_{i-1} \sigma(G_{i-1}, G_i, A).$$

Then  $\chi_n = \chi_0 \pi_2(G, A)$ .

*Proof.* Induct on  $|G|$ . Without loss,  $C < G$  and  $G_1 < G$ . By induction and Theorem 4.6,  $\chi_0 \pi_2(G, A) = \chi_1 \pi_2(G_1, A) = \chi_n$ , completing the proof.

Note that the condition  $[G_i, A]C \leq G_{i+1}$  is equivalent to saying there exist  $K, L \leq G_i$  such that  $KG_{i+1} = G_i$ ,  $K \cap G_{i+1} = L$ , and  $G_{i+1}/L$  is the centralizer of  $A$  in  $G_{i+1}/L$ .

We stress that the step by step algorithm in Corollary 4.7 is necessary. We mention an example and leave the details to the reader. We construct  $G$  by taking the semi-direct product  $EB$  where  $|E| = 23^5$  and  $|B| = 11^5$ . We let  $A \leq A_5$  be transitive (e.g.,  $A$  could be cyclic of order 5 or simple of order 60), and then define an appropriate action of  $A$  of  $G$ . We may choose a  $\chi \in \text{IRR}_A(G)$  such that there is a unique  $\alpha \in \text{IRR}(C)$  with  $[\chi_C, \alpha]$  odd. However,  $\chi \pi_2(G, A) \neq \alpha$  even though  $C = G_2$  in the notation of Corollary 4.7.

4.8 LEMMA. Assume Hypotheses 3.2,  $|G|$  odd,  $N \trianglelefteq \Gamma$ , and  $N \leq G$ . Let  $\chi \in \text{IRR}_A(G)$ ,  $\theta \in \text{IRR}_A(N)$ ,  $\varepsilon = \chi \pi_2(G, A)$ , and  $\delta = \theta \pi_2(N, A)$ . Then  $[\chi_N, \theta] \neq 0$  if and only if  $[\varepsilon_{N \cap C}, \delta] \neq 0$ .

*Proof.* Induct on  $|G|$ . With no loss,  $C < G$  and  $N < G$ . Let  $K = [G, A]$ ,  $L = K'$ , and  $H = LC < G$ . Choose  $N \leq M \triangleleft G$  maximal such that  $M \triangleleft \Gamma$ . Now  $L \leq G' \leq M$  and  $M \cap H = C_{M/L}(A)$ . Now  $[M, A] \leq M \cap K$  and hence  $M = (M \cap K)(M \cap H)$ . As  $MK$  is either  $M$  or  $G$ , then either  $K \leq M$  or  $H \leq M$ .

Consider the case  $H \leq M$ . Choose the unique  $\psi \in \text{IRR}_A(M)$  such that  $[\chi_M, \psi] \neq 0$ , by application of Corollary 2.4(b). By induction and Lemma 4.6,  $[\psi_N, \theta] \neq 0$  if and only if  $[\varepsilon_{N \cap C}, \delta] \neq 0$ . Then  $[\varepsilon_{N \cap C}, \delta] \neq 0$  implies  $[\chi_N, \theta] \neq 0$ . Assume  $[\chi_N, \theta] \neq 0$ . Now  $\psi_N$  has an irreducible  $A$ -invariant constituent, namely  $\theta^g$  for some  $g \in G$ . Lemma 2.3 implies  $\theta^g = \theta^c$  for some  $c \in C$ . Now  $[\psi_N, \theta] = [\psi_N, \theta^c] \neq 0$  as  $c \in C \leq M$ . Hence  $[\varepsilon_{N \cap C}, \delta] \neq 0$ . We are done if  $H \leq M$ .

So, assume  $K \leq M$ . Now  $|G:M| = p$ , a prime. Now  $\chi_M = \sum_{i=1}^t \phi_i$  where  $t$  is 1 or  $p$  and where each  $\phi_i \in \text{IRR}_A(M)$ , by Corollary 2.4(c). Let

$\alpha_i = \phi_i \pi_2(M, A)$ . It suffices, via induction, to show that the irreducible constituents of  $\varepsilon_{M \cap C}$  are precisely the  $\alpha_i$ ,  $1 \leq i \leq t$ . If  $t = p$ , this follows from Lemma 4.4. So assume  $\chi_M = \phi_1$ . Let  $\beta = \chi\sigma(G, H, A)$  and  $\gamma = \phi_1 \sigma(M, M \cap H, A)$ . As  $[\chi_{H \cap M}, \gamma]$  is odd and as  $(H \cap M)C = H$ , Lemma 2.5 and Theorem 4.2 imply  $[\beta_{H \cap M}, \gamma] \neq 0$ . Now Lemma 4.4 implies  $\beta_{H \cap M} = \gamma$ . Theorem 4.6 and induction yield  $\varepsilon_{M \cap C} = \alpha_1$ . This completes the proof.

The following, analogous to Theorem 3.1(c), is useful for inductive arguments.

4.9 LEMMA. *Assume Hypotheses 3.2 with  $|G|$  odd. Let  $T \trianglelefteq A$  and  $B = C_G(T)$ . Then*

- (a)  $\chi\pi_2(G, T) \in \text{IRR}_A(B)$  whenever  $\chi \in \text{IRR}_A(G)$ , and
- (b)  $\pi_2(G, A) = \pi_2(G, T)\pi_2(B, A/T)$ .

*Proof.* As  $T \trianglelefteq A$ , it is routine to check that  $[G, T]$  and  $B$  are  $A$ -invariant. In particular,  $M = [G, T]'B$  is  $A$ -invariant.

To prove (a), we let  $\chi_1 = \chi\sigma(G, M, T)$ . Note that  $A$  permutes the  $T$ -invariant irreducible constituents of  $\chi_M$ . Also  $[\chi, \chi_1^a]$  is odd for  $a \in A$ . As either  $B = G$  or  $M < G$ , part (a) follows by induction.

To prove (b), we induct on  $|G|$ . Without loss of generality,  $M < G$ . Let  $K = [G, A]$ ,  $L = K'$ , and  $H = LC < G$ . As  $C \leq B$ ,  $LM = HM$ . Let  $\chi \in \text{IRR}_A(G)$ . We show  $\chi\pi_2(G, A) = \chi\pi_2(G, T)\pi_2(B, A/T)$ .

If  $HM < G$ ,  $\chi\sigma(G, HM, A) = \chi\sigma(G, HM, T)$ . Then, induction and Theorem 4.6 imply

$$\begin{aligned} \chi\pi_2(G, A) &= \chi\sigma(G, HM, A)\pi_2(HM, A) \\ &= \chi\sigma(G, HM, T)\pi_2(HM, T)\pi_2(B, A/T) \\ &= \chi\pi_2(G, T)\pi_2(B, A/T). \end{aligned}$$

So, we assume  $HM = LM = G$ .

Now  $G/L = ML/L \cong M/M \cap L$ . As  $G, L$ , and  $M$  are fixed by  $A$ , the natural isomorphism between  $G/L$  and  $M/M \cap L$  is an  $A$ -isomorphism. In particular,  $H \cap M/L \cap M = C_{M/L \cap M}(A)$ . Now  $M = [G, T]'B$ ,  $ML = G$ , and  $[G, T]' \leq L$ . So  $LB = G$  and  $[G, T] \leq L$ . Also  $[H, T] = [G, T]$  and  $[H, T]'(B \cap H) \leq M \cap H$ .

Let  $\beta = \chi\sigma(G, H, A)$ ,  $\varepsilon = \chi\sigma(G, M, T)$ , and  $\delta = \beta\sigma(H, H \cap M, T)$ . The argument in (a) shows that  $\varepsilon$  and  $\delta$  are fixed by  $A$ . By induction and Theorem 4.6,

$$\begin{aligned} \chi\pi_2(G, A) &= \beta\pi_2(H, A) \\ &= \beta\pi_2(H, T)\pi_2(B \cap H, A/T) \\ &= \delta\pi_2(H \cap M, T)\pi_2(B \cap H, A/T) \\ &= \delta\pi_2(H \cap M, A) \end{aligned}$$

and

$$\chi\pi_2(G, T)\pi_2(B, A/T) = \varepsilon\pi_2(M, T)\pi_2(B, A/T) = \varepsilon\pi_2(M, A).$$

So it suffices to show  $\varepsilon\pi_2(M, A) = \delta\pi_2(M \cap H, A)$ . As  $[M, A]C \leq (K \cap M)C \leq (L \cap M)C = H \cap M$ , we need just show that  $[\varepsilon_{H \cap M}, \delta]$  is odd.

As  $LB = G$ , Lemma 2.5 implies that every constituent of  $\chi_H$  is  $T$ -invariant. As  $\beta \in \text{IRR}_T(H)$  is unique such that  $[\beta, \delta^H]$  is odd,  $[\chi_H, \delta^H] = [\chi_{H \cap M}, \delta]$  is odd. Write  $\chi_M = \varepsilon + 2\phi + \omega$  for  $\phi, \omega \in \text{Ch}(M)$  where no irreducible constituent of  $\omega$  is fixed by  $T$ . As  $(L \cap M)B = M$ ,  $[\omega_{H \cap M}, \delta] = 0$  by Corollary 2.5. So  $1 \equiv [\chi_{H \cap M}, \delta] \equiv [\varepsilon_{H \cap M}, \delta] \pmod{2}$  and the result follows from above comments.

### 5. Equivalence

Here we prove the equivalence of the character correspondences when both are defined. So, in effect, there is one correspondence, and it is defined whenever  $A$  acts on  $G$  with  $(|A|, |G|) = 1$ . We give an algorithm for this map, and state some properties thereof.

**5.1 THEOREM.** *Assume Hypotheses 3.2,  $A$  solvable, and  $|G|$  odd. Then  $\pi_1(G, A) = \pi_2(G, A)$ .*

*Proof.* We induct on  $|\Gamma|$ . By Theorem 3.1, by Lemma 4.9, and by the solvability of  $A$ , we may assume  $|A| = p$ , a prime. We may also assume  $[G, A]C < G$ , as otherwise both maps are the identity on  $\text{IRR}(G)$ . Choose  $[G, A]C \leq H < G$  maximal so that  $H$  is  $A$ -invariant. Let  $K = [G, A]$  and  $L = K \cap H$ . Note  $[G, A] \leq L$ , so that  $L \triangleleft KHA = \Gamma$ . Also  $H/L = C_{G/L}(A)$ . Furthermore,  $K/L$  is an elementary abelian chief factor of  $\Gamma$ .

Let  $\chi \in \text{IRR}_A(G)$ . We show  $\chi\pi_1(G, A) = \chi\pi_2(G, A)$ . Let  $\theta \in \text{IRR}_A(K)$  with  $[\chi_K, \theta] \neq 0$ . By Lemmas 3.5 and 4.4 we may assume via induction on  $|G|$  that  $I_G(\theta) = G$ . By Lemma 2.4(b), we may let  $\phi$  be the unique  $A$ -invariant irreducible constituent of  $\theta_L$ . Now Lemma 4.4 and the maximality of  $H$  imply  $I_G(\phi)$  is  $H$  or  $G$ .

Choose the unique  $\eta \in \text{IRR}_A(H)$  such that  $p \nmid [\chi_H, \eta]$ , and let  $\beta = \chi\sigma(G, H, A)$ . By Corollary 3.3 and Theorem 4.6, it suffices to show  $\eta = \beta$ . As  $\phi$  is the unique  $A$ -invariant constituent of  $\chi_L$ ,  $[\eta_L, \phi] \neq 0$  and  $[\beta_L, \phi] \neq 0$ . In particular, if  $I_G(\phi) = H$ ,  $\eta = \beta$ . So we assume  $I_G(\phi) = G$ .

As  $K/L$  is an abelian chief factor of  $\Gamma$  and as  $I_\Gamma(\theta) = \Gamma = I_\Gamma(\phi)$ , then either  $\theta_L = \phi$  or  $\phi$  is fully ramified with respect to  $K/L$ . If  $\theta_L = \phi$ , Lemma 2.1 implies  $\eta = \chi_H = \beta$ . So, we assume  $\phi$  is fully ramified with respect to  $K/L$ .

If  $K = G$  then  $C \leq H \triangleleft G$ ,  $\chi_H$  has a unique  $A$  invariant constituent, and we're done. Choose  $K \leq N \triangleleft G$  with  $|G:N|$  prime. By Lemmas 3.5 and 4.4; we may assume that  $\chi_N = \mu \in \text{IRR}(N)$ . Let  $M = N \cap H$  and note that  $|H:M| = |G:N|$ . Let  $\gamma = \mu\sigma(N, M, A)$ . By induction, by Corollary 3.3, and by Lemmas 3.4 and 4.8, both  $\eta$  and  $\beta$  extend  $\gamma$ . In particular,  $\eta = \lambda\beta$  for some  $\lambda \in \text{IRR}(H/M)$ .

Let  $F$  be a conjugacy class of subgroups as in Theorem 4.1 and note  $A$  acts on  $F$ . By Lemma 2.3, we can choose an  $A$ -invariant  $V \in F$ . Now  $[V, A] \leq [G, A] \cap V = L$ . So  $V \leq C_{G/L}(A)$  and hence  $V = H$ . Let  $\psi \in \text{Ch}(G/K)$  as in Theorem 4.1. Note that  $\chi_H = \psi_H \beta$  and  $\psi^2$  is rational valued.

Now  $\chi_L$  is a multiple of  $\phi$ , and so, by Lemma 2.5, every constituent of  $\chi_H$  is  $A$ -invariant. Observe that  $\beta + 2\Xi = \chi_H = a\eta + p\Lambda$  with  $p \nmid a$  and  $\Xi, \Lambda \in \text{Ch}(H)$ . Now

$$(\psi\chi)_H = a\psi_H\eta + p\psi_H\Lambda = a\lambda\psi_H\beta + p\psi_H\Lambda = a\lambda\chi_H + p\psi_H\Lambda.$$

Hence  $\lambda\eta = \lambda^2\beta$  is the unique  $\alpha \in \text{IRR}(H)$  such that  $p \nmid [(\psi\chi)_H, \alpha]$ . Observe that if  $\beta_1$  is the complex conjugate of  $\beta$  then

$$[(\psi\chi)_H, \lambda^2\beta] = [\psi_H^2\beta, \lambda^2\beta] = [(\psi^2)_H\beta\beta_1, \lambda^2] = [\psi^2\beta\beta_1, \lambda^{-2}] = [(\psi\chi)_H, \lambda^{-2}\beta].$$

As  $\beta$  extends  $\gamma$ ,  $\lambda^2 = \lambda^{-2}$  and  $\lambda^4 = 1_H$ . As  $|G|$  is odd,  $\lambda = 1_H$  and  $\eta = \beta$ . This proves the theorem.

The above theorem says we really just have one correspondence. Of course, the Odd-Order Theorem implies that  $\pi_1(G, A)$  or  $\pi_2(G, A)$  exists under Hypotheses 3.2. We will now write  $\pi(G, A)$  for  $\pi_1(G, A)$  or  $\pi_2(G, A)$ . Whenever  $C < G$  and  $A$  is nonabelian simple,  $[G, A]C < G$ . Hence, the following contains an algorithm for finding  $\pi(G, A)$ .

5.2 COROLLARY. Assume Hypotheses 3.2. Let  $T \trianglelefteq A$  and  $B = C_G(T)$ . Then the following hold.

- (a)  $\pi(G, A): \text{IRR}_A(G) \rightarrow \text{IRR}(C)$  is one-to-one and onto.
- (b)  $\chi\pi(G, T) \in \text{IRR}_A(B)$  for  $\chi \in \text{IRR}_A(G)$ , and

$$\pi(G, A) = \pi(G, T)\pi(B, A/T).$$

- (c) Assume  $A$  is simple. If  $|A|$  is prime, set  $p = |A|$  and let  $C \leq H \leq G$  be

$A$ -invariant. Otherwise set  $p = 2$ , and choose  $[G, A]C \leq H \leq G$  where  $H$  is  $A$ -invariant. If  $\chi \in \text{IRR}_A(G)$ , there exists a unique  $\beta \in \text{IRR}_A(H)$  such that  $p \nmid [\chi_H, \beta]$ . Also  $\chi\pi(G, A) = \beta\pi(H, A)$ .

*Proof.* See Lemma 4.9 and Theorems 3.1, 4.6, and 5.1.

Note that  $\pi(G, A)$  is uniquely determined by the action of  $A$  on  $G$ . We mention a few properties of  $\pi(G, A)$ .

5.3 LEMMA. Assume Hypotheses 3.2. Let  $N \trianglelefteq \Gamma$  with  $N \leq G$ . Let  $\chi \in \text{IRR}_A(G)$ ,  $\theta \in \text{IRR}_A(N)$ ,  $\varepsilon = \chi\pi(N, A)$ , and  $\delta = \theta\pi(G, A)$ . The following hold.

- (a)  $[\chi_N, \theta] \neq 0$  if and only if  $[\varepsilon, \delta] \neq 0$ .
- (b) If  $NC = G$  and  $T = I_G(\theta)$ , then  $T \cap C = I_C(\delta)$ , and

$$(\psi\pi(T, A))^C = \psi^G\pi(G, A) \text{ for } \psi \in \text{IRR}(T|\theta).$$

*Proof.* This is immediate from Lemmas 3.4, 3.5, 4.4, and 4.8 and from Theorem 5.1.

5.4 LEMMA. *Assume Hypotheses 3.2. Let  $\chi \in \text{IRR}_A(G)$  and let  $\beta = \chi\pi(G, A)$ . Then:*

- (a)  $\beta(1) \mid |G:C|\chi(1)$ .
- (b) *If  $G$  is solvable,  $\chi(1)/\beta(1)$  is an integer dividing  $|G:C|$ .*

*Proof.* For solvable  $A$ , Glauberman [5] proves (a), and (b) is an exercise in Chapter 13 of [7] (solvable  $A$ ). We will just prove (b).

Induct on  $|\Gamma|$ . We may assume by induction and Theorem 5.1(b) that  $A$  is simple. Let  $G/N$  be a chief factor of  $\Gamma$ . Then, as  $G/N$  is abelian, either  $C \leq N$  or  $NC = G$ .

If  $C \leq N$  then, by Corollary 2.4(b),  $\chi_N$  has a unique  $A$ -invariant irreducible constituent  $\psi$ . As  $[G, A]C \leq N$ ,  $\psi\pi(N, A) = \beta$ . As  $\chi(1)/\psi(1) \in Z$  and  $\chi(1)/\psi(1) \mid |G:N|$ , we are done if  $C \leq N$ .

So, assume  $NC = G$ . Note  $|G:N|$  is prime. Let  $\theta \in \text{IRR}_A(N)$  be a constituent of  $\chi_N$ . Lemma 5.3 allows us to assume via induction that  $\chi_N = \theta$ . Furthermore, Lemma 5.3 implies  $\beta_{N \cap C} = \theta\pi(N, A)$ . The result follows by induction.

The next result is due to Isaacs. He proves it for solvable  $A$  in his discussion of Glauberman correspondence in Chapter 13 of [7]. The existence of Isaacs correspondence is sufficient for the general case, as the same proof works. Under Hypotheses 3.2, we note that application of Lemma 2.3 shows that  $K \rightarrow K \cap C$  is a one-to-one map from the  $A$ -fixed conjugacy classes onto the conjugacy classes of  $C$  (for a proof see Corollary 13.10 of [7]).

5.5 THEOREM. *Assume Hypotheses 3.2. Then the actions of  $A$  on  $\text{Irr}(G)$  and on the conjugacy classes of  $G$  are permutation isomorphic.*

*Proof.* See Lemma 13.23, Corollary 13.10, and Theorem 13.24 of [7]; note that the solvability of  $A$  can be removed from the hypothesis of 13.24.

We mention one final property of  $\pi(G, A)$ . We omit the proof, which is quite straightforward. For  $\alpha \in \text{IRR}(H)$ , let  $Q(\alpha)$  be the (Galois) extension of  $Q$  obtained by adjoining the values  $\alpha(h)$ ,  $h \in H$ , to  $Q$ . Under Hypotheses 3.2, let  $\chi \in \text{IRR}_A(G)$  and  $\beta = \chi\pi(G, A)$ . Then  $Q(\chi) = Q(\beta)$  and  $\chi^\tau\pi(G, A) = \beta^\tau$  for  $\tau$  in the Galois group of  $Q(\chi)$  over  $Q$ .

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UNIVERSITY OF WISCONSIN  
MADISON, WISCONSIN