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RESIDUAL MEASURES[1]

BY

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Abstract

We define residual measures on topological spaces as a simultaneous generalization of the normal Radon measures of Dixmier and the category measures of Oxtoby. We examine the regularity, τ-smoothness, tightness, and support properties of residual measures. We show that residual measures without support exist iff real-valued measurable cardinals exist. In the compact setting we associate with any compact Hausdorff space X a larger Stonian compact Hausdorff space, the Gleason space of X, such that there is a bijective correspondence between the residual measures on these spaces and the residual Radon measures on these spaces. Hence, we lift the question of existence of certain types of residual measures to the Stonian setting of Dixmier.

1. Residual measures on completely regular Baire spaces

Let X be a topological space. Let $\Sigma(X) = \Sigma$ denote the σ-algebra of subsets of X with the property of Baire. Let $\eta(X) = \eta$ denote the σ-ring of meager subsets of X. Since $A \in \eta$ implies $B \in \eta$ for all $B \subset A$, η is a complete σ-ideal in Σ. When ordered by inclusion η is Dedekind complete. The Boolean algebra Σ/η is a complete Boolean algebra. We let $\mathscr{A} = \mathscr{A}(X)$ denote the Boolean algebra of all regular open sets in X and we let $\bar{\mathscr{A}} = \bar{\mathscr{A}}(X)$ denote the Boolean algebra of all regular closed sets in X. When X is a Baire space, $\mathscr{A}$, $\bar{\mathscr{A}}$ and Σ/η are isomorphic Boolean algebras. If $\triangle$ denotes symmetric difference the isomorphism between $\mathscr{A}$ and Σ/η assigns to $\theta \in \mathscr{A}$ its equivalence class $\theta \triangle \eta = \{\theta \triangle N : N \in \eta\} \in \Sigma/\eta$ and the isomorphism between $\mathscr{A}$ and $\bar{\mathscr{A}}$ assigns to $\theta \in \mathscr{A}$ its closure, $\bar{\theta}$, which lies in $\bar{\mathscr{A}}$. These results may be found in [7], [11], or [20].

The Borel sets $\mathscr{B} = \mathscr{B}(X)$ in X have as a σ-ideal the σ-ring $\mathscr{B} \cap \eta$ of meager Borel sets. The completion, measure theoretically, of $\mathscr{B} \cap \eta$ is η. The σ-algebra Σ is obtained as the completion of $\mathscr{B}$ with respect to $\mathscr{B} \cap \eta$ since $\Sigma = \{\theta \triangle N : N \in \eta, \theta \in \mathscr{A}\}$.

DEFINITION. (1) $\mu \in M^+(X, \Sigma)$ is a residual measure iff $\eta \subset \mu^{-1}\{0\}$.

(2) A residual measure is a *category* measure iff $\eta = \mu^{-1}\{0\}$ iff $\mu(\theta) > 0$ when θ is a nonempty open set.

Remark. (1) Residual measures are just those elements of $M^+(X, \Sigma)$ arising from countably additive functions from Σ/η into $[0, \infty)$.

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(2) $\mu \in M^+(X, \Sigma)$ is residual iff $\mu(\partial C) = 0$ when C is a closed set.

(3) If μ is residual and ν is absolutely continuous with respect to μ then ν is residual. Consequently, if $f \in L^{1+}(X, \Sigma, \mu)$, then $f \cdot \mu$ is residual. In particular if $A \in \Sigma$ then $\chi_A \cdot \mu$ is residual. The restriction of μ to $A \in \Sigma$ is a residual measure on A. If ν is a measure on $A \in \Sigma$ it may be extended to a residual measure on X by setting $\nu(X \backslash \dot{A}) = 0$ iff, when $A = \theta \bigtriangleup N$ with θ open and $N \in \eta$ then ν is residual on $\theta \backslash N$ and $\nu(A \cap N) = 0$.

(4) Residual measures μ can also be characterized by the fact that if f is a bounded upper semicontinuous function with lower semicontinuous regularization $\hat{f}$ then $\int f \, d\mu = \int \hat{f} \, d\mu$. See Proposition 2 of [2].

(5) Residual measures are uniquely determined by their restriction to the Borel algebra, hence may be regarded as Borel measures.

(6) Any X which possesses a category measure is a Baire space.

We recall from [22, p. XII] that a measure μ is regular iff it is regular with respect to the paving $\mathscr{F} = \mathscr{F}(X)$ of closed sets in X. It is easily verified, from the identity $\mu(F) = \mu(F^0) = \mu(\overline{F^0})$ for residual measures, that a residual measure is regular iff it is regular with respect to the paving $\mathscr{A}$. It is shown in Proposition 22.3 of [11] that any category measure on a regular Baire space is regular. We may ask whether any residual measure on a regular Baire space is regular.

In [22, p. XII] a Borel measure μ is said to be τ-smooth iff whenever $\{F_\alpha\} \subset \mathscr{F}$ is a decreasing family (i.e., filtering to the left) with intersection F one has $\mu(F) = \lim_\alpha \mu(F_\alpha)$. It is known, [22], that any τ-smooth Borel measure μ on a completely regular space has a closed support and is regular. For residual measures we have the following proposition.

PROPOSITION 1. *Let μ be a residual measure on the topological space X. Consider the following:*

(a) *μ is τ-smooth;*

(b) supp (μ) *exists;*

(c) *μ is regular.*

When X is a completely regular space, (a)$\Leftrightarrow$(b). *When X is a regular Baire space,* (b)$\Rightarrow$(c). *All three are equivalent if X is compact.*

Proof. Let X be a regular Baire space. If supp (μ) exists it must belong to $\mathscr{A}$. On supp (μ), μ is a category measure. Hence, by Proposition 22.3 of [11], μ is regular on supp (μ). The regularity of μ on X follows immediately. Thus (b) implies (c).

Let X be a completely regular space. That (a) implies (b) is known.

Let supp (μ) exist. To show that μ is τ-smooth let $\{F_\alpha\} \subset \mathscr{F}$ be a decreasing family with intersection F. Let $\lim_\alpha \mu(F_\alpha) = \rho \geq \lambda = \mu(F) = \mu(F^0)$. We must show that $\rho = \lambda$. By considering $F'_\alpha = F_\alpha \backslash F^0$ we may assume that $F^0 = \emptyset$ hence that $\lambda = 0$. If $\{F_n: n \in Z^+\} \subset \{F_\alpha\}$ is a decreasing sequence with intersection F_∞ and with $\mu(F_\infty) = \lim_{n\to\infty} \mu(F_n) = \rho$ we find that $\mu(F_\infty \backslash F_\alpha) = 0$ for all α. If we let $F''_\alpha = F_\alpha \cap F_\infty$ then $\{F''_\alpha\} \subset \mathscr{F}$ decreases to F and $\mu(F_\infty \backslash F''_\alpha) = 0$ for all α. Consequently, $F^0_\infty \backslash F''_\alpha = F^0_\infty \backslash F_\alpha$ is an open subset of $X \backslash \text{supp}\,(\mu)$ for all α. Since

$$\bigcup_\alpha (F^0_\infty \backslash F_\alpha) = F^0_\infty \backslash \left(\bigcap_\alpha F_\alpha \right) = F^0_\infty \backslash F \subset X \backslash \text{supp}\,(\mu)$$

we have $\rho = \mu(F_\infty) = \mu(F^0_\infty) = \mu(F^0_\infty \backslash F) + \mu(F) = 0 = \lambda$. This establishes that (b) $\Rightarrow$ (a).

If X is compact, any regular μ is Radon hence is τ-smooth. Consequently, in this case, (c) $\Rightarrow$ (a). ▮

COROLLARY 1.1. *Any category measure on a completely regular space is τ-smooth.*

Remark. The equivalence of τ-smoothness and regularity for Borel measures on a completely regular space X is the definition of Borel measure compactness of X in [4]. Consequently, if X is Borel measure compact (a), (b), and (c) are equivalent for residual measures.

A cardinal number n is said to be real-valued measurable, or an **R**-cardinal, iff there is a discrete space T of cardinality n and a positive countably additive μ on 2^T such that $\mu(\{t\}) = 0$ for all $t \in T$. A cardinal which isn't an **R**-cardinal is called a non-**R**-cardinal. These cardinals were first investigated by Ulam. Solovay [21] has introduced a more restrictive definition of real-valued measurable cardinals where the measures satisfy a stronger additivity condition. Since no new phenomena seem to arise if we consider a higher degree of additivity, we will not be concerned with this notion of real-valued measurability.

It is known that the existence of **R**-cardinals is independent of the Zermelo-Fraenkel axioms of set theory even with the axiom of choice. If **R**-cardinals exist the least such cardinal must be very large (weakly inaccessible) but maybe as small as the cardinality of the continuum. Note that if n is an **R**-cardinal all larger cardinals are **R**-cardinals. Non-**R**-cardinals are closed under finite products and under sums with at most a non-**R**-cardinal number of summands. $\aleph_0$ is a non-**R**-cardinal. We refer the reader to [21] for more specific details.

We call a topological space **R**-Lindelof iff any collection $\{\theta_\beta\}$ of open sets has a subcollection, $\{\theta'_\alpha: \alpha \in \Gamma\}$, with the same union with card (Γ) a non-**R**-cardinal. We say that the space is weakly **R**-Lindelof iff no disjoint collection of opens has an **R**-cardinal as its cardinal number. Any topological space with a base whose cardinal is a non-**R**-cardinal is **R**-Lindelof.

PROPOSITION 2. (a) *If X is a completely regular space which is weakly* **R**-*Lindelof then any residual measure is τ-smooth.*

(b) *If X is metrizable and has a dense set whose cardinal number is a non-***R**-*cardinal all residual measures on X are τ-smooth.*

Proof. (a) By Proposition 1, if μ is residual we need only show that supp (μ) exists. If it exists it is the complement of the union, U, of all open sets which are

μ-negligible. Supp (μ) exists iff $\mu(U) = 0$. Let us assume that $\mu(U) > 0$. Let $\mathscr{G}$ be a maximal collection $\{\omega_\alpha: \alpha \in \Gamma\}$ of disjoint open sets which have the property that $\mu(\omega_\alpha) = 0$ for all $\alpha \in \Gamma$. Let $V = \bigcup \mathscr{G}$. We assert that $\mu(V) = \mu(U) > 0$. Note that $V \subset U$ and that if U didn't lie in $\bar{V}$ there would exist an open $\omega \subset \bar{\omega} \subset U \backslash \bar{V}$ with $\emptyset \neq \omega$ and $\mu(\omega) = 0$ contradicting the maximality of $\mathscr{G}$. Since $U \backslash V \subset \bar{V} \backslash V \in \eta$, $\mu(U \backslash V) = 0$ so $\mu(V) = \mu(U) > 0$.

On the discrete space Γ define $\tilde{\mu}(\Gamma') = \mu(\bigcup \{\omega_\alpha: \alpha \in \Gamma'\})$, $\Gamma' \subset \Gamma$. Observe that $\tilde{\mu}$ is positive countably additive, that if $\alpha \in \Gamma$ then $\tilde{\mu}(\{\alpha\}) = 0$, and that $\tilde{\mu}(\Gamma) > 0$. The existence of $\tilde{\mu}$ implies that card (Γ) is an **R**-cardinal. This contradicts the assumption on X. We conclude that $\mu(U) = 0$ and that supp $(\mu) = U^c$.

(b) If X is a metric space, $\{x_\alpha: \alpha \in \Gamma\}$ is a dense subset and N_α is a countable base of neighborhoods for x_α for $\alpha \in \Gamma$ then $N = \bigcup_\alpha N_\alpha$ is a base for X whose cardinality is at most $\aleph_0 \cdot$ card (Γ). If card (Γ) is a non-**R**-cardinal then X is weakly **R**-Lindelof. ▮

Remarks. (1) (b) may be generalized to include the case where X has a base for its uniform structure and a dense subset both of which have a non-**R**-cardinal number.

(2) Because of the uncertainty on the size of **R**-cardinals and because the continuum hypothesis could be assumed to hold the only consequence of (b) that we may be sure of is that in a separable metric space all residual measures are τ-smooth but this is already a consequence of the fact that all Borel measures on a separable metric space have a support.

PROPOSITION 3. *The following are equivalent.*

(a) *There exists an* **R**-*cardinal.*

(b) *There exists a residual non-τ-smooth measure on some compact Hausdorff space.*

(c) *There exists a residual, regular, non-τ-smooth measure on some locally compact Hausdorff space.*

Proof. It is clear that both (b) and (c) imply (a). Conversely given (a) we will construct examples of (c) and of (b).

Let T be a discrete space whose cardinal is real-valued measurable. Let $\mu \neq 0$ be a countably additive real function on 2^T with $\mu(\{t\}) = 0$ for all $t \in T$. It is easily verified that μ is a regular, residual measure on the locally compact T which has no support. This establishes that (a) $\Rightarrow$ (c). Let T^* be the one point compactification, $T \cup \{\infty\}$, of T. We verify that $\Sigma(T^*) = 2^{T^*}$ and that $\eta(T^*) = \{\emptyset, \{\infty\}\}$. Extend μ to $\Sigma(T^*)$ by setting $\mu(A) = \mu(A \cap T)$ for all $A \subset T^*$. The measure μ on T^* is residual but is easily seen to have no support. Consequently (a) $\Rightarrow$ (c). ▮

We recall from [22] that a Borel measure μ on X is said to be tight or Radon iff, for any Borel set A, $\mu(A) = \sup \{\mu(K): K \subset A, K \in \mathscr{K}(X)\}$ where $\mathscr{K}(X)$ is the paving on X consisting of compacts. We will reserve the term Radon for measures on locally compact spaces. If X is a complete metric space or is a locally compact space all τ-smooth measures are tight, [22]. Conversely any tight measure is τ-smooth. Therefore an equivalent to part (c) of the preceding proposition is the existence of a regular residual measure on a locally compact space which isn't a Radon measure.

Corollary 1.1 shows that when X is a complete metric space, a locally compact space or a residual Borel subset of such a space all category measures on X are tight. Hence, all residual measures with support are tight on these spaces.

The following proposition shows that on metric spaces nonzero residual measures usually have to be purely atomic.

PROPOSITION 4. *Let X be metrizable.*

(1) *There is a nonzero τ-smooth residual measure on X iff X has an isolated point.*

(2) *If* **R**-*cardinals don't exist, nonzero residual measures exist on X iff X has an isolated point.*

(3) *If X is separable, or more generally, is weakly* **R**-*Lindelof there is a nonzero residual measure on X iff X has an isolated point.*

Proof. If X has an isolated point x then δ_x is a nonzero residual measure which is τ-smooth. Conversely, if μ is a residual measure and $x \in X$ with $\mu(\{x\}) > 0$ then x is isolated. For the rest of the proof we assume that X has no isolated points.

(1) If μ is a τ-smooth residual measure, by Proposition 1, it has a support Y and μ is a category measure on Y. Theorem 18 of [12], shows that if $Y = \emptyset$ then Y is the closure of its countable set of isolated points and that each isolated point y of Y has $\mu(\{y\}) > 0$. Since this is impossible, $\mu = 0$.

(2) If there are no **R**-cardinals, Proposition 2(b) shows that all residual measures on X are τ-smooth.

(3) Proposition 2(a) shows that all residual measures on X are τ-smooth. ∎

We now construct an example of a compact Hausdorff space X which isn't **R**-Lindelof such that all residual measures are τ-smooth. In fact, all residual measures are zero. This example is built on the space $[0, 1]$ which, by Proposition 4, admits no nonzero residual measures. We let T be a discrete space with an **R**-cardinal, let T^* denote the one point compactification of T and let $X = [0, 1] \times T^*$. That X is not weakly **R**-Lindelof is immediate. If μ is a residual measure on X we define the measure m on $[0, 1]$ by setting $m(A) = \mu(A \times T^*)$. It is easily checked that m is residual on $[0, 1]$ hence is 0. Consequently $\mu(X) = 0$.

The preceding example brings forth the question of existence of residual measures on product spaces. The following proposition answers most of this question.

If μ is a measure on a product space $X \times Y$, its marginal distribution ν on X is that measure such that $\nu(A) = \mu(A \times Y)$ when $A \times Y$ is μ-measurable.

In [1] or [13], the notion of a pseudobase of a topological space is defined as a generalization of a base of a topological space. A collection $\mathscr{B}$ of open sets is a pseudobase of a topological space iff every open set contains a member of $\mathscr{B}$. Any space possessing a countable pseudobase is separable.

PROPOSITION 5. (a) *If X is a topological space which has no nonzero residual measures and Y is a topological space then $X \times Y$ has no nonzero residual measures.*

(b) *Let X be a topological space with a countable pseudobase and let Y be a topological space. If μ and ν are residual measures on X and Y respectively, then $\mu \times \nu$ is a residual measure on $X \times Y$.*

(c) *If $\{X_\alpha: \alpha \in \Gamma\}$ is an infinite collection of topological spaces possessing nonempty nondense open subsets there is no nonzero residual measure on $X = \prod \{X_\alpha: \alpha \in \Gamma\}$.*

Proof. (a) This is immediate from the preceding example.

(b) We must show that when $N \in \eta(X \times Y)$ then $(\mu \times \nu)(N) = 0$. We may assume that N is a Borel set; hence is $\mu \times \nu$ measurable. For any $y \in Y$ let $N(y) = \{x: (x, y) \in N\} \subset X$. The category version, (1.1) of [13], of Fubini's Theorem shows that, except for a meager set of y in Y, $N(y)$ is meager in X. Thus, except for a ν-negligible set of y in Y, $N(y)$ is μ-negligible. The classical Fubini Theorem shows that $(\mu \times \nu)(N) = 0$.

(c) Let Γ_0 be a countable subset of Γ and let

$$X_0 = \prod \{X_\alpha: \alpha \in \Gamma_0\}.$$

Part (a) of this proposition shows that to establish (c) it is enough to show that if μ is a residual measure on X_0 then $\mu = 0$. Let $\{\Gamma_j: j \in N\}$ be a partition of Γ_0 with card $(\Gamma_j) = j + 1$ for $j \in N$. Since there is a nondense nonempty open set in X_α there are at least two disjoint open sets in X_α for all $\alpha \in \Gamma$. There is a disjoint family G_j of nonempty open subsets of $X_j = \prod \{X_\alpha: \alpha \in \Gamma_j\}$ with card $(G_j) = 2^{j+1}$ for any $j \in N$. Let μ_j be the marginal distribution of μ on X_j. For any j there is a $\theta_j \in G_j$ with $\mu_j(\theta_j) \leq 2^{-j-1}$. If we set $F_j = X_j \backslash \theta_j$ and $F = \prod_{j=1}^{\infty} F_j \subset X_0$ then

$$\mu_j(F_j) \geq (1 - 2^{-j-1})\mu_j(X_j) = (1 - 2^{-j-1})\mu(X_0)$$

hence $\mu(F) \geq [1 - \sum_{j=1}^{\infty} 2^{-j-1}] \cdot \mu(X_0) = \frac{1}{2}\mu(X_0)$. Since F is nowhere dense in X_0 and μ is residual, $0 = \mu(F) = \frac{1}{2}\mu(X_0)$. This establishes (c). ∎

Remarks. (1) (b) holds for finite products as long as one factor has a countable pseudobase. The limitation of the validity of (b) to products with one factor having a countable pseudo base appears from the category version of Fubini's theorem in [13].

(2) If X is a topological space such that any open set is dense either X is

meager or every nonempty open set is residual. In the first case there are no residual measures on X. In the second case the nonempty open sets form a filter base whose associated filter $\mathscr{F}$ has the property that countable intersections of its elements are nonempty; in fact, they are dense. There is, in this case, a uniquely determined category measure on X with $\mu(Y) = 1$ if $Y \in \mathscr{F}$.

Let X_α be a space such that every nonempty open set is dense for $\alpha \in \Gamma$. If X is $\prod \{X_\alpha: \alpha \in \Gamma\}$ then every nonempty open set in X is dense. It may be shown that X is not meager iff each X_α is not meager. Consequently, if each X_α has a nonzero residual measure for all $\alpha \in \Gamma$ so does X.

Let T be an infinite set with the cofinite topology. Any nonempty open set is dense in T and T is meager iff it is countable. Consequently, if T is uncountable T^N possesses a nonzero category measure. This fact shows that (c) of Proposition 5 doesn't have any significant extensions.

(3) The usual cases of interest for applying (c) are products of metric spaces, of compact Hausdorff spaces, or locally compact Hausdorff spaces. Products of discrete spaces have been examined before. In particular (c) applies to an infinite power of a discrete space such as N^N or $\{0, 1\}^m$ for some infinite cardinal number m. Sudderth and Purves have established results in [17] having as a corollary the validity of (c) for X^N where X is an infinite discrete space.

(4) If X has no nonzero residual measure, every Borel measure on X assigns full measure to some meager $\mathscr{F}_\sigma$.

2. Residual measures on compact Hausdorff spaces

If X is a compact Hausdorff space, by Proposition 1, the residual Radon measures, the τ-smooth residual measures, and the residual measures with support are the same.

Dixmier [2] appears to be the first to consider residual Radon measures on any compact Hausdorff space. The compact Hausdorff spaces he works with are the Stonian or extremally disconnected spaces in which the closure of every open set is open. When X is a Stonian compact Dixmier defines a *normal measure* on X to be a residual Radon measure. $\eta^+(X)$ denotes the cone of all normal measures on X and $\eta(X) = \eta^+(X) - \eta^+(X)$ is the vector space of all *signed normal* measures on X.

Stonian compact Hausdorff spaces are *totally disconnected* in that the Boolean algebra of clopen sets forms a basis for the topology. Between the totally disconnected compact Hausdorff spaces and the extremally disconnected compact Hausdorff spaces are the *basically disconnected* compact Hausdorff spaces in which every Baire open set has open closure and every closed Baire set has closed interior, [5].

It is known that a compact Hausdorff space X is totally disconnected iff the unit ball of $\mathscr{C}(X)$ is the uniformly closed convex hull of its extreme points. X is basically disconnected iff $\mathscr{C}(X)$ is Dedekind σ-complete as a vector lattice. X is extremally disconnected iff $\mathscr{C}(X)$ is a Dedekind complete vector lattice [5], [19].

If $\mathscr{B}$ is any Boolean algebra its Stone space $X_{\mathscr{B}}$ is constructed either as the set of all maximal ideals in $\mathscr{B}$ or of all ultrafilters on $\mathscr{B}$. We use the latter definition. To any $A \in \mathscr{B}$ one associates the subset $[A]$ of $X_{\mathscr{B}}$ consisting of all ultrafilters on $\mathscr{B}$ containing A. The Stone correspondence Φ: $A \to [A]$ is an isomorphism of $\mathscr{B}$ onto a Boolean algebra of subsets of $X_{\mathscr{B}}$. Taking $\Phi(\mathscr{B})$ as a base for a topology on $X_{\mathscr{B}}$ we obtain the Stone topology on $X_{\mathscr{B}}$ which is known to be compact, Hausdorff and totally disconnected with $\Phi(\mathscr{B})$ the algebra of clopen sets in $X_{\mathscr{B}}$. Any totally disconnected compact Hausdorff space is (homeomorphic to) the Stone space of its algebra of clopen sets. $X_{\mathscr{B}}$ is basically disconnected iff $\mathscr{B}$ is σ-complete and $X_{\mathscr{B}}$ is Stonian iff $\mathscr{B}$ is complete.

It is useful to note that any clopen set is a regular open set and a regular closed set. A compact Hausdorff space is Stonian iff every regular open set is clopen iff every regular closed set is clopen. Consequently, if X is a Stonian compact Hausdorff space, X is the Stone space of $\mathscr{A}(X)$ or of $\Sigma(X)/\eta(X)$.

If μ is an additive real function of bounded variation on the Boolean algebra $\mathscr{B}$ (i.e., $\mu \in BA(\mathscr{B})$) there is a corresponding Radon measure $\tilde{\mu}$ on $X_{\mathscr{B}}$ defined by the requirement that $\tilde{\mu}([A]) = \mu(A)$ for all $A \in \mathscr{B}$. The correspondence $\mu \to \tilde{\mu}$ is known to be a Banach lattice isomorphism from $BA(\mathscr{B})$ (with the variation norm) onto $\mathscr{M}(X_{\mathscr{B}})$. Under this isomorphism the cone $FA^+(\mathscr{B})$ of positive elements of $BA(\mathscr{B})$ is carried onto $\mathscr{M}^+(X_{\mathscr{B}})$. The Banach lattice $CA(\mathscr{B})$ of countably additive elements of $BA(\mathscr{B})$ is isomorphic under this isomorphism to the closed sublattice of $\mathscr{M}(X_{\mathscr{B}})$ consisting of those $\tilde{\mu}$ with $|\tilde{\mu}|(N) = 0$ for any nowhere dense Baire compact set N. If we extend Luxemburg's definition [9] of a normal measure on $\mathscr{B}$ from the case with $\mathscr{B}$ complete to the case where $\mathscr{B}$ is an arbitrary Boolean algebra we find that they are precisely those $\mu \in FA^+(\mathscr{B})$ such that $\tilde{\mu}$ is a residual measure in $\mathscr{M}^+(X_{\mathscr{B}})$. With $\mathscr{B}$ complete, μ is normal on $\mathscr{B}$ for Luxemburg's definition iff $\tilde{\mu}$ is normal on the Stonian compact Hausdorff space $X_{\mathscr{B}}$.

The importance of normal measures on Stonian spaces is demonstrated in this theorem due essentially to Dixmier in [2].

THEOREM. *Let X be a compact Hausdorff space. In order that $\mathscr{C}(X)$ be a dual Banach space it is necessary and sufficient that X be Stonian and that $\eta^+(X)$ separate functions in $\mathscr{C}(X)$. If this is the case, $\eta(X)$ is the unique subspace of $\mathscr{M}(X)$ predual to $\mathscr{C}(X)$.*

The Stonian compact Hausdorff spaces X with $\eta^+(X)$ separating $\mathscr{C}(X)$ are called *hyperstonian* by Dixmier. $\eta^+(X)$ separates $\mathscr{C}(X)$ iff for every open set $\theta \neq \emptyset$ there is a nonzero normal measure μ with supp $(\mu) \subset \theta$. If this is the case it is possible to construct a dense open $Y \subset X$ and a $\nu \in \mathscr{M}^+(Y)$ such that on Y all elements of $\eta^+(X)$ are absolutely continuous with respect to ν and such that $\nu(A) = 0$ in Y iff $A \in \eta(Y)$. The mapping $\mu \to d\mu|_Y/d\nu$ is a Banach lattice isomorphism of $\eta(X)$ onto $L^1(Y, \nu)$. This was established by Dixmier in [2]. Note that supp $(\nu) = Y$ and that in general $\nu(Y) = \infty$. In fact, ν can be constructed so that $\nu(Y) < \infty$ iff X has a category measure. We have $\mathscr{C}(X)$ isomorphic to $L^\infty(Y, \nu)$ as a Banach lattice since $\eta(X)$ is isomorphic to $L^1(Y, \nu)$.

If $(S, \mathscr{B}, \mu)$ is a positive measure space with $L^\infty(S, \mathscr{B}, \mu)$ the dual of $L^1(S, \mathscr{B}, \mu)$ then $L^\infty(S, \mathscr{B}, \mu)$ is isomorphic with $\mathscr{C}(X)$ for some compact hyperstonian X, $L^1(S, \mathscr{B}, \mu)$ is isomorphic to $\eta(X)$ and $L^{\infty\prime}(S, \mathscr{B}, \mu)$ is isomorphic to $\mathscr{M}(X)$ as Banach lattices. X is the Stone space of the complete Boolean measure algebra of μ which is $\mathscr{B}$ modulo the μ-negligible sets. In this case we may define the dense open set $Y \subset X$ as $\bigcup \{[A]: \mu(A) < \infty\}$ where $[A]$ is the clopen set in X corresponding to the equivalence class of $A \in \mathscr{B}$ in the measure algebra of μ. Defining $\nu([A]) = \mu(A)$ for $A \in \mathscr{B}$ we obtain an element ν of $\mathscr{M}^+(Y)$ with $\nu(N) = 0$ if $N \in \eta(Y)$. We find that $L^1(Y, \nu)$, $\eta(X)$, and $L^1(S, \mathscr{B}, \mu)$ are isomorphic as Banach lattices as are $L^\infty(Y, \nu)$, $\mathscr{C}_b(Y)$, $\mathscr{C}(X)$, and $L^\infty(S, \mathscr{B}, \mu)$ and as are $L^{\infty\prime}(Y, \nu)$, $\mathscr{M}(X)$ and $L^{\infty\prime}(S, \mathscr{B}, \mu)$.

Of particular importance is the case where the Stonian compact Hausdorff space X has a category measure. This is true iff X is the Stone space of the measure algebra of a finite measure iff the union of all the supports of normal measures on X is a dense σ-compact open set. Such a space X is called a *measurable* Stonian compact Hausdorff space in accordance with the terminology of [12].

One might suppose that if X were a compact Hausdorff space such that the residual Radon measures separated $\mathscr{C}(X)$ then X would automatically be hyperstonian. That this is not the case may be seen by considering compact Hausdorff spaces X which have a dense set of isolated points. All scattered compact Hausdorff spaces, as in [19], are of this form. The discrete measures concentrated on the isolated points are the only residual Radon measures on such a space. Consequently, if X is a compact Hausdorff space with a dense set T of isolated points then $l^{1+}(T)$ is (isomorphic to) the set of residual Radon measures on X. If X were to be hyperstonian then $\mathscr{C}(X)$, when restricted to T, would be $l^\infty(T) = C_b(T)$. This is the case, when T is infinite, iff X is the Stone-Cech compactification, βT, of T. When T is infinite there are uncountably many compactifications X of T other than βT. The Alexandroff compactification $X = T \cup \{\infty\}$ is a totally disconnected compact Hausdorff space which isn't extremally disconnected yet the residual Radon measures separate $\mathscr{C}(X)$. If we let $T \subset \mathbf{R}^2$ be the set

$$\bigcup_{n=1}^{\infty} \{(k/n, 1/n): 0 \le k \le n\}$$

then $X = \bar{T}$ is the compact set $[[0, 1] \times \{0\}] \cup T$. No point $(\lambda, 0)$ with $0 \le \lambda \le 1$ has a base of clopen neighborhoods in X. Thus, X is a compact metric space with a category measure which isn't totally disconnected. Note that Theorem 18 of [12], or Proposition 4, show that a compact metric space X has a category measure iff it is the closure of its isolated points iff the residual Radon measures on X separate $\mathscr{C}(X)$. Finally, we note that the compact Hausdorff spaces X such that all residual Radon measures are discrete and separate $\mathscr{C}(X)$ are precisely those X with dense isolated points.

If $\mathscr{B}$ is the Boolean algebra of clopen subsets of a totally disconnected compact Hausdorff space X then X has a dense set of isolated points iff every element of $\mathscr{B}\backslash\{\emptyset\}$ contains an atom of $\mathscr{B}$. We recall [16] that a dense subset S of a Boolean algebra $\mathscr{B}$ is one such that every $A \in \mathscr{B}\backslash\{\emptyset\}$ contains an element of S other than $\emptyset$. Thus, X has a dense set of isolated points iff the atoms of $\mathscr{B}$ are dense in $\mathscr{B}$. Examples of this exist in profusion. If $\mathscr{B}$ is any Boolean algebra of subsets of some set S such that all singletons belong to $\mathscr{B}$ they form the dense set of atoms of $\mathscr{B}$. Particular examples are Blackwell spaces of [10], Borel algebras of T_1 spaces, Baire algebras of second countable T_1 spaces and all larger algebras.

Examples of compact Hausdorff spaces X such that all residual Radon measures are purely nonatomic and separate $\mathscr{C}(X)$ are hard to come by except for hyperstonian X. These are precisely the Stone spaces of measure algebras for purely nonatomic positive measure spaces.

We will now show that on the Gleason space X_Σ of a compact Hausdorff space X the existence of residual measures (either Radon or not) is equivalent with their existence on X. The Gleason space X_Σ is the Stone space of the complete Boolean algebra $\mathscr{A}(X)$ (or $\bar{\mathscr{A}}(X)$ or $\Sigma(X)/\eta(X)$). If we let $\mathscr{A} = \mathscr{A}(X)$, $\bar{\mathscr{A}} = \bar{\mathscr{A}}(X)$, $\Sigma = \Sigma(X)$ and $\eta = \eta(X)$, the corresponding sets for X_Σ are denoted by $\mathscr{A}'$, $\bar{\mathscr{A}}'$, Σ', η'. Since X_Σ is Stonian, $\mathscr{A}' = \bar{\mathscr{A}}'$ is the Boolean algebra of clopen sets in X_Σ. Since X_Σ is the Stone space of $\mathscr{A}$ all of the algebras $\mathscr{A}$, Σ/η, Σ'/η' and $\mathscr{A}'$ are isomorphic. There is a unique continuous projection $\pi_\Sigma\colon X_\Sigma \to X$ which assigns to any $x \in X_\Sigma$ (which is an ultrafilter in $\mathscr{A}(X)$) its unique limit point $\pi_\Sigma(x)$ in X when x is considered as a (convergent) filter base in X. X_Σ is the minimal Stonian space admitting a continuous surjection onto X. If Y is Stonian and $\pi_Y\colon Y \to X$ is a continuous surjection there is a continuous surjection $\pi_{\Sigma,Y}\colon Y \to X_\Sigma$ such that $\pi_Y = \pi_\Sigma \circ \pi_{\Sigma,Y}$. These facts about X_Σ are to be found in [6]. The mapping π_Σ is *irreducible* in that there is no proper closed subset Z of X_Σ with $\pi_\Sigma(Z) = X$ [19, p. 448]. Equivalently, $A \subset X_\Sigma$ is dense iff it has dense image in X under π_Σ. It follows [1, p. 58] that X and X_Σ have the same density character and that if $\mathscr{U}$ is a pseudobase of X the inverse image is a pseudobase of X_Σ. If X is the Stone space of a Boolean algebra $\mathscr{B}$, it is shown in [1] that X_Σ is the Stone space of the completion of $\mathscr{B}$ as a Boolean Algebra.

The following lemma shows that the isomorphism between Σ/η and Σ'/η' is effected by π_Σ.

LEMMA 6. *Let X be a compact Hausdorff space and let X_Σ be its Gleason space.*

(i) *$\pi_\Sigma^{-1}(\eta)$ is cofinal in η'.*
(ii) *$\pi_\Sigma^{-1}(\Sigma)$ has completion with respect to $\pi_\Sigma^{-1}(\eta)$ equal to Σ'.*
(iii) *If $\theta \in \mathscr{A}$ has $[\theta]$ as its corresponding clopen set in X_Σ then $[\theta] = \overline{\pi_\Sigma^{-1}(\theta)}$.*

Proof. (i) Let N be closed and nowhere dense in X. Since π_Σ is irreducible, $\pi_\Sigma^{-1}(X\backslash N)$ is a dense open set in X_Σ so $\pi_\Sigma^{-1}(N) \in \eta'$. Consequently, $\pi_\Sigma^{-1}(\eta) \subset \eta'$.

Let N be closed and nowhere dense in X_Σ and let $\theta = X\backslash\pi_\Sigma^{-1}(\pi_\Sigma(N))$. Since π_Σ is irreducible and $\pi_\Sigma(\theta \cup N) = X$, $\theta \cup N$ is dense in X_Σ; hence θ is dense in X_Σ and $\pi_\Sigma(\theta) = X\backslash\pi_\Sigma(N)$ is a dense open set in X. Since $\pi_\Sigma(N) \in \eta$ we have $N \subset \pi_\Sigma^{-1}(\pi_\Sigma(N)) \in \pi_\Sigma^{-1}(\eta)$. This is sufficient to establish the cofinality of $\pi_\Sigma^{-1}(\eta)$ in η'.

(iii) It is easy to show that $\pi_\Sigma^{-1}(\theta) \subset [\theta]$ if $\theta \in \mathscr{A}$ and that

$$\pi_\Sigma^{-1}(X\backslash\bar{\theta}) \subset [X\backslash\bar{\theta}] = X_\Sigma\backslash[\theta].$$

Since $\pi_\Sigma^{-1}(\partial\theta) \in \eta'$ we must have $\pi_\Sigma^{-1}(\theta)$ dense in $[\theta]$; hence $\overline{\pi_\Sigma^{-1}(\theta)} = [\theta]$.

(ii) If $A \in \Sigma$ there is a $\theta \in \mathscr{A}$ and $N \in \eta$ with $A = \theta \bigtriangleup N$. Since $\pi_\Sigma^{-1}(\theta)$ is open and $\pi_\Sigma^{-1}(N) \in \eta'$ we have

$$\pi_\Sigma^{-1}(A) = \pi_\Sigma^{-1}(\theta) \bigtriangleup \pi_\Sigma^{-1}(N) \in \Sigma'.$$

Thus, $\pi_\Sigma^{-1}(\Sigma) \subset \Sigma'$.

If $A \in \Sigma'$ there is an $N \in \eta'$ and a $\theta \in \mathscr{A}$ with

$$A = [\theta] \bigtriangleup N = \pi_\Sigma^{-1}(\theta) \bigtriangleup [N \cup \partial\theta].$$

Thus, A differs at most by an element of η' from an element of $\pi_\Sigma^{-1}(\Sigma)$. This is sufficient to show that Σ' is a completion of $\pi_\Sigma^{-1}(\Sigma)$ with respect to its ideal $\pi_\Sigma^{-1}(\eta)$. ∎

Since $\pi_\Sigma^{-1}(\Sigma) \subset \Sigma'$ we may define a measure μ on Σ from a measure μ' on Σ' by setting $\mu(A) = \mu'(\pi_\Sigma^{-1}(A))$ for $A \in \Sigma$. This is analogous to the process of defining a Radon measure μ on X from a Radon measure μ' on X_Σ. The two processes are the same for μ' on Σ' which are regular. If μ' is a residual measure then μ is also a residual measure since $\pi_\Sigma^{-1}(\eta) \subset \eta'$. If μ' is normal on the Stonian space X_Σ then μ is a residual Radon measure on X. The following shows that for residual measures the transformation $\mu' \to \mu$ is invertible.

PROPOSITION 7. *Let X be a compact Hausdorff space with Gleason space X_Σ. If μ is a residual measure on X there is a unique residual measure μ' on X_Σ such that $\mu'([\theta]) = \mu(\theta)$ for all $\theta \in \mathscr{A}$. The measure μ' is normal iff μ is Radon.*

Proof. The uniqueness of the residual measure μ' is immediate for if $A \in \Sigma'$ there is a unique $\theta \in \mathscr{A}$ and $N \in \eta'$ such that $A = [\theta] \bigtriangleup N$. We must have $\mu'(A) = \mu'([\theta]) = \mu(\theta)$.

Since μ is a countably additive positive finite measure on Σ/η so is μ' on Σ'/η' hence μ' is well defined as a residual measure on X_Σ.

By Proposition 1, μ' is a normal measure iff supp (μ') exists and μ is Radon iff supp (μ) exists. To establish the proposition we need to show that supp $(\mu') \neq \emptyset$ iff supp $(\mu) \neq \emptyset$.

If supp $(\mu') \neq \emptyset$, it is a clopen set $[\theta]$ with $\theta \in \mathscr{A}$. If $\emptyset \neq \theta_0 \subset \theta$ and $\theta_0 \in \mathscr{A}$ we have $\mu(\theta_0) = \mu'([\theta_0]) \neq 0$. Thus $\emptyset \neq \theta \subset$ supp (μ). Conversely, if supp $(\mu) \neq \emptyset$ it is $\bar{\theta}$ for some $\bar{\theta} \in \bar{\mathscr{A}}$, and if $\emptyset \neq \theta_0 \subset \theta$ is in $\mathscr{A}$ then $\mu'([\theta_0]) = \mu(\theta_0) \neq 0$. Thus $\emptyset \neq [\theta] \subset \text{supp}(\mu')$. ∎

COROLLARY 7.1. (1) *X has a nonzero residual measure iff X_Σ does.*

(2) *X has a nonzero residual Radon measure iff X_Σ does.*

(3) *X_Σ is hyperstonian iff the residual Radon measures on X separate $\mathscr{C}(X)$.*

(4) *X has a category measure iff X_Σ is a measurable Stonian compact Hausdorff space.*

Proof. We will only establish (3). X_Σ is hyperstonian iff given a clopen subset $[\theta]$ with $\theta \in \mathscr{A}$ there is a normal measure $\mu' \neq 0$ with $\text{supp}(\mu') \subset [\theta]$. This is equivalent to the existence of a nonzero residual Radon measure μ on X with $\text{supp}(\mu) \subset \bar{\theta}$. This suffices to establish (3). ∎

COROLLARY 7.2. (1) *If X is an infinite product of nontrivial compact Hausdorff spaces then X_Σ isn't hyperstonian.*

(2) *If X is a compact metric space in which the isolated points aren't dense then X_Σ isn't hyperstonian.*

Proof. (1) By Proposition 5, X has no nonzero residual measures; hence X_Σ has no nonzero normal measures.

(2) By Proposition 4 there are no nonzero residual Radon measures on the complement of the isolated points of X. Since this set has nonempty interior, the residual Radon measures don't separate $\mathscr{C}(X)$. ∎

Remarks. (1) The residual Radon measures μ on a compact Hausdorff space X form the positive cone of a Banach lattice which is isomorphic as a Banach lattice to $\eta(X_\Sigma)$. The residual measures on X form the positive cone of another Banach lattice isomorphic to the corresponding Banach lattice of measures on X_Σ.

(2) Propositions 2 and 6 imply that there is a Stonian compact Hausdorff space X with a non-Radon residual measure iff **R**-cardinals exist. The example generated by the proof of Proposition 2 is the following. Let T be discrete with cardinal number an **R**-cardinal and let T^* be its Alexandroff compactification. The Gleason space $(T^*)_\Sigma$ is the desired example. This is known to be βT. This example is the Stone space of a complete atomic Boolean algebra for which the atoms are dense.

(3) As a consequence of (1) of Corollary 6.2, when $X = \{0, 1\}^n$ for an infinite cardinal number n then X_Σ isn't hyperstonian. This was established in [8] for $n > \text{card}([0, 1])$.

One might expect that (b) of Proposition 5 would have the following analogy. If X and Y are compact Hausdorff spaces such that Y has a countable pseudobase and X_Σ and Y_Σ are hyperstonian then so is $(X \times Y)_\Sigma$. This is true but reduces to the trivial special case where Y is the closure of a countable set of isolated points as shown below.

COROLLARY 7.3. *Let Y be a compact Hausdorff space with a countable pseudobase and with Y_Σ hyperstonian. Y is the closure of countably many isolated points.*

Proof. It is immediate that the number of isolated points of Y is countable. Since the maximum number of disjoint nonzero residual Radon measures on Y must be countable, Y must have a category measure. The Stonian space Y_Σ has a countable pseudobase $\mathscr{U}$. The set $\bar{\mathscr{U}} = \{\bar{\theta}: \theta \in \mathscr{U}\}$ is also a countable pseudobase. The algebra $\mathscr{B}$ of clopen sets generated by $\bar{\mathscr{U}}$ is countable hence has a metrizable Stone space X. There is a continuous projection $p: Y_\Sigma \to X$; hence if $\pi_Y: Y_\Sigma \to Y$ and $\pi_X: X_\Sigma \to X$ are the canonical maps there is a continuous projection $q: Y_\Sigma \to X_\Sigma$ with $\pi_X \circ q = p$. If $\tilde{\mathscr{B}}$ is the inverse image of $\mathscr{A}(X_\Sigma)$ in $\mathscr{A}(Y_\Sigma)$ it is complete. If $\mathscr{F} \subset \tilde{\mathscr{B}}$ has supremum Ψ in $\tilde{\mathscr{B}}$ then $\overline{\bigcup \mathscr{F}} \subset \Psi$. If $\Psi \backslash \overline{\bigcup \mathscr{F}} \neq \emptyset$ there is a $\theta \in \bar{\mathscr{U}} \backslash \{\emptyset\}$ with $\theta \subset \Psi \backslash \overline{\bigcup \mathscr{F}}$. This contradicts the definition of Ψ so $\Psi = \overline{\bigcup \mathscr{F}}$. If $V \in \mathscr{A}(Y_\Sigma)$ then the union of $\mathscr{F} = \{\theta \in \bar{\mathscr{U}}: \theta \subset V\}$ is easily seen to be dense in V hence V is the supremum of $\mathscr{F}$ in $\tilde{\mathscr{B}}$. Thus $\tilde{\mathscr{B}} = \mathscr{A}(Y_\Sigma)$ hence $X_\Sigma = Y_\Sigma$ with q the identity map. Since Y_Σ has a category measure so does X. Consequently, by Theorem 18 of [12], X is the closure of its isolated points and all residual Radon measures on X are atomic. It easily follows that all residual Radon measures on Y_Σ and on Y are atomic. This implies that Y is the closure of its isolated points. ∎

Remark. (1) The question whether X_Σ and Y_Σ hyperstonian for compact Hausdorff spaces X and Y implies that $(X \times Y)_\Sigma$ is hyperstonian remains unanswered. An equivalent formulation of this question has both X and Y Stonian. Another equivalent question is whether $X \times Y$ possesses a category measure when both X and Y do.

(2) We should remark that the equation $(X \times Y)_\Sigma = (X_\Sigma \times Y_\Sigma)_\Sigma$ is true whereas the equation $(X \times Y)_\Sigma = X_\Sigma \times Y_\Sigma$ is false when X and Y are infinite compact Hausdorff spaces. Indeed, no product of infinite compact Hausdorff spaces is Stonian or even basically disconnected: To see this one can find a disjoint sequence $\{\theta_n: n \in N\}$ of nonempty Baire open sets in X and a disjoint sequence $\{\Psi_n: n \in N\}$ of nonempty Baire open sets in Y. The Baire open set $\bigcup_{n=1}^{\infty} (\theta_n \times \Psi_n)$ in $X \times Y$ doesn't have open closure.

We have associated to any residual measure μ on a compact Hausdorff space X a unique residual measure μ' on X_Σ satisfying $\mu'([\theta]) = \mu(\theta)$ if $\theta \in \mathscr{A}(X)$. Since X_Σ is the Stone space of $\mathscr{A}(X)$ the Stone correspondence assigns to μ a unique Radon measure $\tilde{\mu}$ on X_Σ satisfying $\tilde{\mu}([\theta]) = \mu(\theta)$ if $\theta \in \mathscr{A}(X)$. Consequently μ' and $\tilde{\mu}$ agree on clopen subsets of X_Σ and are identical iff $\tilde{\mu}$ is a normal measure iff μ' is a Radon measure iff μ is a Radon measure. In general we can only say that μ' and $\tilde{\mu}$ agree on the Baire algebra of X_Σ but not on the Borel algebra. Since μ' is residual, $\tilde{\mu}$ annihilates all compact nowhere dense Baire subsets of X_Σ but $\tilde{\mu}$ doesn't annihilate all nowhere dense compact sets.

PROPOSITION 8. *Let X be a Stonian compact Hausdorff space. There is a non-Radon residual measure on X iff there is a positive Radon measure $\bar{\mu} \neq 0$ on X* *such that* supp $(\bar{\mu}) \subset \eta(X)$ *and such that* $\bar{\mu}$ *annihilates all compact nowhere dense Baire sets in* X.

Proof. If a non-Radon residual measure μ exists on X $(= X_\Sigma)$ there is a Radon measure $\tilde{\mu}$ on X annihilating all compact nowhere dense Baire sets but with $\tilde{\mu}(N) > 0$ for some compact nowhere dense N. Let $\bar{\mu} = \chi_N \cdot \mu$.

Conversely, if $\bar{\mu}$ exists there is a residual measure μ on X corresponding to $\bar{\mu}$ under the Stone correspondence. Since $\bar{\mu}$ isn't residual, μ isn't Radon. ∎

COROLLARY 8.1. *Let* X *be a Stonian compact Hausdorff space with no nonzero normal measures. There are enough positive Radon measures on* X *which annihilate compact nowhere dense Baire sets yet which have nowhere dense support to separate* $\mathscr{C}(X)$ *iff every open set in* X *is assigned strictly positive measure by some residual measure.*

Proof. Immediate. ∎

COROLLARY 8.2. *The following are equivalent.*

(a) **R**-*cardinals don't exist.*

(b) *If* $\mu \in \mathscr{M}^+(X)$ *for a Stonian compact Hausdorff space* X *and* $\mu(K) = 0$ *for any compact nowhere dense Baire set* K *then* μ *is residual.*

(c) *If* $\mathscr{B}$ *is a complete Boolean algebra, any* $\mu \in CA^+(\mathscr{B})$ *is normal in the sense of Luxemburg.*

Proof. (c) is just a restatement of (b).

Let X be Stonian and let $\mu \in \mathscr{M}^+(X)$ annihilate all Baire compact nowhere dense sets. Let ν be the corresponding countably additive function on $\mathscr{A}(X)$ and let $\tilde{\nu}$ be the corresponding residual measure on X. If (a) is true then $\tilde{\nu} = \mu$ by Proposition 3 hence (b) holds. If (a) is false, by (b) of Proposition 3, there is a residual non-Radon measure on some compact Hausdorff space X, hence, by Proposition 7, there is a residual non-Radon measure on X_Σ which suffices to establish (b) using Proposition 8. ∎

Remark. (1) In [2] it is shown that if X is Stonian and every clopen set θ has $\mu(\theta) > 0$ for some $\mu \in \mathscr{M}^+(X)$ with nowhere dense support which annihilates all nowhere dense compact Baire sets then X is purely nonhyperstonian so that there are no normal measures on X. It is shown that **R**-cardinals exist iff such a Stonian space X exists.

(2) Dixmier canonically decomposes a Stonian space into three clopen sets X_1, X_2, and X_3. X_1 is hyperstonian, $X_2 \cup X_3$ is purely nonhyperstonian, X_2 is the largest purely nonhyperstonian open set in which every meager set is nowhere dense and X_3 is the largest open set in which a meager set is dense. We may decompose $X_2 \cup X_3$ into two clopen sets Y_1 and Y_2. Y_1 is the largest open set such that every open subset is assigned nonzero measure by some residual measure. On Y_2 there is a dense open set θ which receives full measure from each residual measure yet every point in θ has a neighborhood which is assigned measure 0 by each residual measure. It is not known which, if any, of $Y_1 \cap X_2$, $Y_1 \cap X_3$, $Y_2 \cap X_2$ and $Y_2 \cap X_3$ are automatically empty.

(3) Does there exist a Boolean algebra $\mathscr{B}$ and a $\mu \in CA^+(\mathscr{B})$ such that $\tilde{\mu} \in \mathscr{M}(X_{\mathscr{B}})$ isn't residual? More generally, does there exist a compact Hausdorff space X and a $\mu \in \mathscr{M}^+(X)$ which isn't residual yet annihilates all compact nowhere dense Baire sets? If **R**-cardinals exist the answer is yes. The converse to this is conjectured to be true.

Note added in Proof. Lacey and Cohen, in the paper *On injective envelopes of Banach spaces* (J. Functional Analysis, vol. 4 (1969), pp. 11–30), have established Lemma 6 and (3) of Corollary 7.1. They also remark that no perfect separable compact Hausdorff space has a nonzero residual Radon measure hence its Gleason space isn't hyperstonian.

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