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ON A LEMMA OF MARCINKIEWICZ

BY

CALIXTO P. CALDERON

Introduction

Given any closed set F in R (real line), we shall call the distance from any point x to F, the distance function; it will be denoted by $\delta(x; F)$, or simply by $\delta(x)$. Throughout this paper, we shall be concerned with operators

$$T(f) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{x-y} G\left[\frac{\delta(x)-\delta(y)}{x-y}\right] f(y)\, dy. \tag{0.1}$$

Here, $\delta(x)$ denotes the distance function; $G(s)$ is a function satisfying

$$|G(s) - G(0)| < K|s|, \quad |s| \leq 1. \tag{0.2}$$

$f(x)$ stands for a function belonging to the Lebesgue class $L^p(R)$, $1 \leq p \leq \infty$. If $x \in F$ and $G(s) = s$, T reduces to the classical Marcinkiewicz integral (see [3]). If we allow x to take values all over R, $T(f)$ becomes a particular case of the operator studied in [1].

Another interesting case arises when $G(s) = s^\lambda$ where, $\lambda > 1$. When $x \in F$ this is the case of the Marcinkiewicz integral $J_\lambda(x)$ (see [3, p. 252]).

We may consider also the situations

$$G(s) = s/(1+s^2), \qquad G(s) = 1/(1+s^2). \tag{0.3}$$

These situations arise in the case of a double layer potential, more precisely, when considering the L^p behavior of the Cauchy-type integral

$$U(z) = \text{p.v.} \frac{1}{2\pi i} \int_\Gamma \frac{1}{s-z} f(s)\, ds \tag{0.4}$$

where Γ is the curve $z = x + i\,\delta(x; F)$. The proof shows that the boundary could be given by the more general expression

$$z = x + i\phi(x)\,\delta(x, F) \quad \text{where } \phi(x) \in C^{1+\varepsilon}(R),\ \varepsilon > 0.$$

Throughout the proof we are going to keep the notation introduced in [3] for the various Marcinkiewicz integrals. The letter A will always denote the complement of F.

The main theorem

The results are summarized as follows.

THEOREM. *Suppose that G satisfies condition* (0.2), $G(0) = 0$ *and* $|A| < \infty$.

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Then we have the following.

(i) $\left(\int_{-\infty}^{\infty} |T(f)|^r \, dx\right)^{1/r} < C_{r;p} |A|^{(p-r)/pr} \| f \|_p$

where $1 < p \leq \infty$, $1/p \leq 1/r < 1 + 1/p$ *and* $C_{r;p}$ *depends on p and r only.*

If μ *is a finite Borel measure defined on R we have the following instead.*

(ii) $|E(|T(\mu)| > \lambda)| < (C_r/\lambda^r) \| \mu \|^r$

for $1/2 \leq r \leq 1$. *Here* $\| \mu \|$ *stands for the total variation of* μ; C_r *depends on r and the measure of A.*

Suppose now that either A has infinite measure or $G(0) \neq 0$. *In that case we have these results.*

(iii) $\| T(f) \|_r < C_r \| f \|_r$, $1 < r < \infty$. *Here* C_r *depends on r only.*

(iv) $|E(|T(\mu)| > \lambda)| < (C_1/\lambda) \| \mu \|$.

In all the cases, $T(f)$ *is defined point wise* $a \cdot e$ *as a principal value; furthermore the operator* $T^*(f) = \sup_{\varepsilon>0} |T_\varepsilon(f)|$ *where*

$$T_\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} \frac{1}{(x-y)} G\left[\frac{\delta(x)-\delta(y)}{(x-y)}\right] f(y) \, dy$$

satisfies the same inequalities as $T(f)$.

Proof. Consider $x \in F$. In that case we have

$$T_\varepsilon(f)(x) = G(0) H_\varepsilon(f)(x) + \tilde{T}_\varepsilon(f)(x) \tag{1.1}$$

where $H_\varepsilon(f)(x)$ stands for the truncated Hilbert transform

$$H_\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} \frac{1}{(x-y)} f(y) \, dy \tag{1.2}$$

and $\tilde{T}_\varepsilon(f)$ denotes the operator associated with $\tilde{G}(s) = G(s) - G(0)$. It follows from (0.2) that

$$|\tilde{T}_\varepsilon(f)| < K \int_A \frac{\delta(y)}{(x-y)^2} |f(y)| \, dy = K J_1(x; |f|, F). \tag{1.3}$$

Here J_1 is the Marcinkiewicz integral introduced in [3, p. 252]. Letting $H^*(f)$ denote $\sup_{\varepsilon>0} |H_\varepsilon(f)|$, we have

$$T^*(f) \leq |G(0)| H^*(f) + K J_1(x; |f|, F), \quad x \in F. \tag{1.4}$$

Suppose now that $G(0) = 0$ and $|A| < \infty$. In this case we have, for $x \in F$,

$$\begin{aligned} |T_\varepsilon(f)| &\leq K \int_{|x-y|>\varepsilon} \frac{\delta(y)}{(x-y)^2} |f(y)| \, dy \\ &\leq \text{p.v.} \left| K \int_{-\infty}^{\infty} \frac{\delta(y)}{(x-y)^2} |f(y)| \, dy \right| \\ &= K |C(\delta, |f|)(x)|. \end{aligned} \tag{1.5}$$

Consequently

$$T^*(f)(x) \le K\,|C(\delta,\ |f|)(x)|. \tag{1.6}$$

Here we use the notation $C(\delta,\ |f|)(x)$ for the classical commutator singular integral studied in [1].

Our next step is to describe the behavior of $T^*(f)$ on A, that is when $x \in A$. Let us express A as $\bigcup_1^\infty (a_k,\ b_k)$, where the $(a_k,\ b_k)$ are pairwise disjoint. We shall denote by c_k the middle point of $(a_k,\ b_k)$. Without loss of generality we may assume that $x \in (a_k,\ c_k)$ since the case $x \in (c_k,\ b_k)$ could be handled in a similar manner. Consider as before, $T_\varepsilon(f) = G(0)H_\varepsilon + \tilde{T}_\varepsilon(f)$ where $\tilde{T}_\varepsilon(f)$ could be dominated in the following way:

$$\begin{aligned} |\tilde{T}_\varepsilon(f)(x)| \le\ & |\tilde{G}(1)|\ |H_\varepsilon(f_{k;0})(x)| \\ & + K\,|H_\varepsilon(|f_{k;1}|)(x)| \\ & + \sum_{j \ne k} \int_{|x-y|>\varepsilon} \frac{\delta(x)+\delta(y)}{(x-y)^2}\,|f_j(y)|\,dy. \end{aligned} \tag{1.7}$$

Here $f_{j;0} = f$ if $x \in (a_j,\ c_j)$ and zero otherwise, $f_{j;1} = f$ if $x \in (c_j,\ b_j)$ and zero otherwise and $f_j = f_{j;0} + f_{j;1}$. If $x \in (a_k,\ c_k)$ then $\delta(x) = x - a_k$. On the other hand, if $y \in (a_j b_j)\ j \ne k$, we have $|x-y| \ge |x - a_k|$. Consequently

$$\delta(x) \sum_{j \ne k} K \int_{|x-y|>\varepsilon} \frac{|f_j(y)|}{(x-y)^2}\,dy \le 2KM(|f|)(x). \tag{1.8}$$

Here $M(|f|)(x)$ denotes the Hardy-Littlewood maximal function. If $y \in (a_j,\ b_j)$, $j \ne k$, then $\delta(y) = \min(|y-a_j|,\ |y-b_j|) \le |x-y|$. Consequently

$$\begin{aligned} K \sum_{j \ne k} \int_{|x-y|>\varepsilon} & \frac{\delta(y)}{(x-y)^2}\,|f_j(y)|\,dy \\ & \le 2K \int \frac{\delta(y)}{(x-y)^2+\delta(y)^2}\,|f(y)|\,dy = 2KH_1'(x,\ |f|,\ F) \end{aligned} \tag{1.9}$$

where $H_1'(x,\ |f|,\ F)$ is the Marcinkiewicz integral defined in [3, 2.4, p. 253]. The estimates (1.7), (1.8), and (1.9) give

$$\begin{aligned} T^*(f)(x) \le\ & |G(0)|\,H^*(f)(x) + KH^*(f_{k;0})(x) \\ & + KH^*(|f_{k;1}|)(x) + 2KM(|f|)(x) \\ & + 2KH_1'(x,\ |f|,\ F). \end{aligned} \tag{1.10}$$

Similar estimates are valid for $x \in (c_k, b_k)$. Therefore

$$(1.11)\qquad \int_{a_k}^{b_k} (T^*(f))^p\,dx \le (4K)^p \int_{a_k}^{b_k} (H'_1(|f|))^p\,dx + (4K)^p \int_{a_k}^{b_k} M(|f|)^p\,dx + |G(0)| \int_{a_k}^{b_k} (H^*(f))^p\,dx + C_p^p\,|4K|^p \int_{a_k}^{b_k} |f|^p\,dx, \quad 1 < p < \infty.$$

Here C_p^p stands for the type constant of the maximal Hilbert transform.

By combining (1.11) and (1.4) we get (iii). Suppose now that $|A| < \infty$. Let r be such that $1/p < 1/r < 1 + 1/p$. Let $1/q = 1/r - 1/p$, $1 < p \le \infty$. If $G(0) = 0$, (1.11) yields

$$(1.12)\qquad \left(\int_A (T^*(f))^l\,dx\right)^{1/l} \le C_l\left(\int_A |f|^l\,dx\right)^{1/l}, \quad 1 < l < \infty.$$

In turn, if $f \in L^\infty(R)$, we have, from (1.12),

$$(1.13)\qquad \left(\int_A (T^*(f))^l\,dx\right)^{1/l} \le C_l|A|^{1/l}\|f\|_\infty.$$

Holder's inequality yields

$$(1.14)\qquad \left(\int_A T^*(f)^r\,dx\right) \le \left(\int_A T^*(f)^p\,dx\right)^{r/p}(|A|)^{r/q} \le |A|^{r/q}C_p^{r/p}\left(\int_A |f|^p\,dx\right)^{r/p}.$$

An application of Theorems A and B in [2] to $|C(\delta, f)|$ yields similar results for $T^*(f)$ in F (see 1.6). Collecting results we get (i). In order to prove (iv) we have to consider (1.10) specialized for the case of a measure μ, namely

$$(1.15)\qquad T^*(\mu)(x) \le |G(0)|H^*(\mu)(x) + KH^*(\mu_{k;0})(x) + KH^*(V_{k;1})(x) + 2KM(V)(x) + 2KH'_1(x; V, F)(x), \quad x \in (a_k, c_k),$$

where

$$(1.16)\qquad \mu_{k;0}(I) = \mu[I \cap (a_k, c_k)], \quad \mu_{k;1}(I) = \mu[I \cap (c_k, b_k)]$$

for all intervals I.

Similar definitions hold for $V_{k;i}$, $i = 0, 1$, where V denotes the variation of μ.

We also have $\mu_k = \mu_{k;0} + \mu_{k;1}$ and $V_k = V_{k;0} + V_{k;1}$. Letting

$$L(\mu)(x) = |G(0)|H^*(\mu)(x) + 2KM(V)(x) + 2KH'_1(x, V, \dot{F})$$

and taking (1.15) into account we have

$$(1.17) \quad |E(T^*(\mu) > \lambda) \cap (a_k, b_k)| \leq |E(L(\mu) > \lambda/2) \cap (a_k, b_k)| + \frac{C}{\lambda}\int_{a_k}^{b_k} dV.$$

The above inequality and (1.4) specialized to the case of a measure give (iv). In order to show (ii) consider first $x \in A$ and $1/r = 1/q + 1$, $1 \leq q < \infty$. We assume in this case that $|A| < \infty$ and $G(0) = 0$. Consider $\lambda > 0$ and suppose that $\lambda/\|\mu\| < 1$. Then

$$(1.18) \qquad |A| < (\|\mu\|/\lambda)^r |A|.$$

If $\lambda/\|\mu\| \geq 1$ then

$$(1.19) \qquad T^*(\mu) \leq \left(\frac{\lambda}{\|\mu\|}\right)^{r/q} T^*(\mu).$$

Consequently

$$(1.20) \quad |E(T^*(\mu) > \lambda)| \leq |E(T^*(\mu) > \lambda^{1-(r/q)}\|\mu\|^{r/q}| \leq (C/\lambda^r)\|\mu\|^r.$$

The last inequality follows from the case $r = 1$. By combining (1.18) and (1.20) we have

$$(1.21) \qquad |A \cap E(T^*(\mu) > \lambda)| < (C/\lambda^r)\|\mu\|^r.$$

In A, $T(\mu) \leq K|C(\delta, v)|$ and the corresponding inequality follows from Theorem B in [2]. This concludes the proof of (ii).

References

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University of Illinois at Chicago Circle
Chicago, Illinois