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STABLE OPERATIONS ON COMPLEX *K*-THEORY

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1. Introduction

Let **K** be the spectrum representing classical (periodic) complex K-theory. A stable operation (of degree zero) on complex K-theory should then correspond to an element of the K-cohomology group $\mathbf{K}^0(\mathbf{K})$; equivalently, it should correspond to a map of spectra $\mathbf{f}: \mathbf{K} \to \mathbf{K}$. (It will be convenient if the word "map" means a homotopy class, and is restricted to maps of degree zero.) Two maps from **K** to **K** are well known: Ψ^1, the identity map, and Ψ^{-1}, the map induced by complex conjugation. One may then form integral linear combinations $\lambda\Psi^1 + \mu\Psi^{-1}$, where $\lambda, \mu \in \mathbf{Z}$. It has been conjectured, and some have tried to prove, that in this way one obtains all the maps from **K** to **K**. Although some of our colleagues have found it hard to believe, we will show that this conjecture is false; there are uncountably many maps from **K** to **K**. We deduce this from a result which has other applications in K-theory.

2. Study of *K*-homology

Let $\mathbf{K}_*(\mathbf{K})$ be the K-homology of the spectrum **K**. It has been sufficiently described by Adams, Harris, and Switzer [3]; but these authors omit the following fundamental result.

THEOREM 2.1. $\mathbf{K}_*(\mathbf{K})$, *considered as a left module over* $\pi_*(\mathbf{K})$, *is free on a countably infinite set of generators* (*of degree zero*).

Because of the structure of $\pi_*(\mathbf{K})$, any graded module M_* over $\pi_*(\mathbf{K})$ which is zero in odd degrees satisfies $M_* \cong \pi_*(\mathbf{K}) \otimes_{\mathbf{Z}} M_0$. So Theorem 2.1 will follow from the following result.

THEOREM 2.2. $\mathbf{K}_0(\mathbf{K})$ *is a free abelian group on a countably infinite set of generators.*

In order to prove this, recall that according to [3] we have an embedding $\mathbf{K}_0(\mathbf{K}) \subset \mathbf{K}_0(\mathbf{K}) \otimes \mathbf{Q} = \mathbf{Q}[w, w^{-1}]$ where $w = u^{-1}v$ (u and v being as in [3]). Let $F(n, m)$ be the intersection of $\mathbf{K}_0(\mathbf{K})$ with the **Q**-module generated by w^n, $w^{n+1}, \ldots, w^m$.

LEMMA 2.3. $F(n, m)/F(n, m-1)$ *and* $F(n, m)/F(n+1, m)$ *are free abelian groups of rank* 1.

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Proof. We give the proof for $F(n, m)/F(n, m-1)$; the proof for $F(n, m)/F(n+1, m)$ is parallel.

An element of $F(n, m)$ may be written in the form $\sum_{n \le r \le m} c_r w^r$, where the coefficients c_r lie in $\mathbf{Q}$. We can define an embedding

$$F(n, m)/F(n, m-1) \to \mathbf{Q}$$

by sending $\sum_{n \le r \le m} c_r w^r$ to the coefficient c_m of w^m. We wish to determine the image I of this embedding. It is a subgroup of $\mathbf{Q}$, and clearly contains $\mathbf{Z}$, since w^m belongs to $F(n, m)$. The result will follow if we show that there is an integer M such that the image I is contained in $(1/M)\mathbf{Z}$. We prove this by localization; it will be sufficient to prove the following.

(i) For each prime p there is a power p^e such that

$$I \subset (1/p^e)\mathbf{Z}_{(p)}$$

(where $\mathbf{Z}_{(p)}$ means the localization of $\mathbf{Z}$ at p, as usual.)

(ii) For all but a finite number of primes p we can take $p^e = 1$.

So let p be a prime. Then in $\mathbf{K}^0(\mathbf{K}; \mathbf{Z}_{(p)})$ we have an element Ψ^k for each integer k prime to p; and we have $\langle \Psi^k, w^r \rangle = k^r$. Let r run over the range $n \le r \le m$, and let k run over an equal number of distinct integers k_n, $k_{n+1}, \ldots, k_m$ prime to p; then the matrix with entries k^n is nonsingular, for we will show that its determinant Δ is nonzero. In fact, by removing from Δ a factor $(k_n k_{n+1} \cdots k_m)^n$, we obtain a Vandermonde determinant, which is nonzero because $k_n, k_{n+1}, \ldots, k_m$ are distinct. We can therefore choose coefficients λ_k in $\mathbf{Z}_{(p)}$ such that

$$\left\langle \sum_k \lambda_k \Psi^k, w^r \right\rangle = \begin{cases} 0 & \text{if } n \le r < m \\ \Delta & \text{if } r = m. \end{cases}$$

In particular, for any element $x = \sum_{n \le r \le m} c_r w^r$ in $F(n, m)$ we have

$$\left\langle \sum_k \lambda_k \Psi^k, x \right\rangle = \Delta c_m.$$

But certainly we have $\langle \sum_k \lambda_k \Psi^k, x \rangle \in \mathbf{Z}_{(p)}$; therefore $c_m \in (1/\Delta)\mathbf{Z}_{(p)}$. Moreover, for $p - 1 \ge m - n + 1$ we can arrange for Δ to be nonzero mod p, for we can arrange for $k_n, k_{n+1}, \ldots, k_m$ to be distinct mod p. This completes the proof.

Proof of Theorem 2.2. This follows immediately from Lemma 2.3. Suppose, as an inductive hypothesis, that we have found a base for $F(n, m)$; we may also suppose that the base contains $m - n + 1$ elements. Then Lemma 2.3 allows one to extend the base to a base for $F(n, m+1)$ or $F(n-1, m)$; we may also assert that this base contains $m - n + 2$ elements. The induction does start, because the case $n = m$ of Lemma 2.3 is to be interpreted as saying that $F(n, n)$ is a free abelian group of rank 1. (The proof even shows that $F(n, n)$ has a base consisting of the element w^n.) It is natural to arrange the induction so that alternate steps increase m and decrease n, but the induction may be conducted in any way provided that $m \to +\infty$ and $n \to -\infty$. The induction constructs a base for $\bigcup F(n, m) = \mathbf{K}_0(\mathbf{K})$. This proves Theorem 2.2, and Theorem 2.1 follows.

3. Maps from K to K

These are described by the following result.

THEOREM 3.1. *The Kronecker product gives an isomorphism*

$$\mathbf{K}^*(\mathbf{K}) \to \mathrm{Hom}_{\pi_*(\mathbf{K})}(\mathbf{K}_*(\mathbf{K}), \pi_*(\mathbf{K})).$$

Proof. This follows immediately from Theorem 2.1, by using the universal coefficient theorem in K-theory. The basic ideas for the proof of such a theorem were given by Atiyah [4], but in the context of the Künneth theorem for spaces. A discussion in the context of the universal coefficient theorem for spectra is given in [1]; it lacks a treatment of the convergence of the spectral sequence, but this may be supplied from the indications given in [2].

COROLLARY 3.2. $\mathbf{K}^1(\mathbf{K}) = 0$; $\mathbf{K}^0(\mathbf{K})$ *is uncountable.*

This follows immediately from Theorems 2.1 and 3.1.

COROLLARY 3.3. $\mathbf{K}^0(\mathbf{K})$ *contains maps not of the form* $\lambda\Psi^1 + \mu\Psi^{-1}$, *where* λ, $\mu \in \mathbf{Z}$.

This follows immediately from Corollary 3.2.

We will now show how to construct a map which is not of the form $\lambda\Psi^1 + \mu\Psi^{-1}$. For a map of the form $\phi = \lambda\Psi^1 + \mu\Psi^{-1}$ we have

$$\langle \phi, 1 \rangle = \lambda + \mu, \qquad \langle \phi, w \rangle = \lambda - \mu;$$

so $\langle \phi, 1 \rangle = 0$ and $\langle \phi, w \rangle = 0$ imply $\phi = 0$, and in particular $\langle \phi, w^2 \rangle = 0$. Let h be the composite

$$F(0, 2) \longrightarrow F(0, 2)/F(0, 1) \xrightarrow{\cong} \mathbf{Z},$$

where the isomorphism comes from Lemma 2.3; then we have $h(1) = 0$, $h(w) = 0$ but $h(w^2) \neq 0$. (In fact calculation shows that $h(w^2) = \pm 24$, but this is irrelevant.) We will now extend h to an element of

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{K}_0(\mathbf{K}), \mathbf{Z}) = \mathrm{Hom}^0_{\pi_*(\mathbf{K})}(\mathbf{K}_*(\mathbf{K}), \pi_*(\mathbf{K})).$$

In fact, according to the proof of Theorem 2.2, a base of $F(0, 2)$ may be extended to a base of $\mathbf{K}_0(\mathbf{K})$, and so h may be extended over $\mathbf{K}_0(\mathbf{K})$ by giving it arbitrary values on the remaining basis elements. Applying Theorem 3.1, we obtain a map $\phi \in \mathbf{K}^0(\mathbf{K})$ such that $\langle \phi, 1 \rangle = 0$, $\langle \phi, w \rangle = 0$ but $\langle \phi, w^2 \rangle \neq 0$; this map ϕ is not of the form $\lambda\Psi^1 + \mu\Psi^{-1}$.

References

1. J. F. Adams, *Lectures on generalized cohomology*, Lecture Notes in Mathematics, no. 99, Springer 1969, especially pp. 1–45.
2. ———, *Algebraic topology in the last decade*, Proceedings of Symposia in Pure Mathematics, vol. 22, Amer. Math. Soc., 1971, especially p. 11.
3. J. F. Adams, A. S. Harris, and R. M. Switzer, *Hopf algebras of cooperations for real and complex K-theory*, Proc. London Math. Soc., vol. 23 (1971), pp. 385–408.
4. M. F. Atiyah, *Vector bundles and the Künneth formula*, Topology, vol. 1 (1962), pp. 245–248.

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