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QUASI-REGULAR IDEALS OF SOME ENDOMORPHISM RINGS

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1. Introduction

If α is an endomorphism of the abelian p-group G such that $x\alpha = x$ for all x in G of order p then α is one-to-one and onto [5; 13.1, p. 279]. It follows that the set Ann $G[p]$ of all endomorphisms of G annihilating $G[p]$ is a quasi-regular (two-sided) ideal of the endomorphism ring End G of G. In general, not every element of Ann $G[p]$ is nilpotent which shows that the Jacobson radical $J(\text{End } G)$ of End G need not be nil. It is an easy exercise in ring theory to verify that an ideal J of a ring R with identity is quasi-regular if there exists a quasi-regular ideal L of R such that $(J + L)/L$ is nil. Thus, for endomorphism rings of abelian p-groups, the famous problem whether the Jacobson radical needs to be nil reduces to the question whether $J(\text{End } G)$ is a nil extension of the quasi-regular ideal $L = \text{Ann } G[p]$.

In this article we show that the answer to this question is affirmative if G is totally projective. In general, this is not the case: if G is unbounded and torsion-complete, then $J(\text{End } G)$ contains elements no power of which annihilate $G[p]$ [5; 14.6, p. 287].

Throughout the following, G denotes a totally projective abelian p-group, where p is some fixed prime. A complete description of $J(\text{End } G)$ is given in 3.8: if λ denotes the length of G then $J(\text{End } G)$ consists of all ε in End G for which there exists a finite sequence of ordinals

$$0 = \beta_0 < \beta_1 < \cdots < \beta_n < \beta_{n+1} = \lambda$$

such that $p^{\beta_i}G[p]\varepsilon \leq p^{\beta_{i+1}}G$ for $i = 0, 1, \ldots, n$. It follows that an ideal of End G is quasi-regular if and only if its restriction to $G[p]$ is a nil ring.

The proof largely depends on a strong decomposition theorem for totally projective p-groups (2.3) which may be of independent interest.

2. Tools

Notation and terminology will follow [2], [3], [4] unless explained otherwise. The word "ideal" will always mean two-sided ideal. A ring is called nil if all of its elements are nilpotent. Given fully invariant subgroups $A \leq B$ of G, the set

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of all ε in End G such that $B\varepsilon \leq A$ is denoted by Ann (B/A). Clearly, Ann (B/A) is an ideal of End G. We shall make frequent use of the following result.

2.1. If J is an ideal of End G such that $J \mid G[p]$ is nil then J is quasi-regular.

Let σ be an ordinal. Since G is totally projective, every endomorphism of $p^\sigma G$ can be extended to an endomorphism of G [7; 3.9, p. 252]. Thus, the restriction of $J(\text{End } G)$ to $p^\sigma G$ is a quasi-regular ideal of End $p^\sigma G$, and [5; 14.2 and 14.4, pp. 284, 286] implies the following fact.

2.2. If $\varepsilon \in J(\text{End } G)$ then $p^\sigma G[p]\varepsilon \leq p^{\sigma+1}G$ for every ordinal σ.

In order to construct certain endomorphisms, the following decomposition theorem will be needed. If Σ is a set of ordinals, sup Σ denotes the smallest ordinal that is greater than or equal to every σ in Σ. As customary, $\tau = \{\sigma: \sigma < \tau\}$.

2.3. THEOREM. *Let G be a totally projective p-group of length λ, let $\tau \leq \lambda$ be a limit ordinal, and let T be a set of ordinals such that $T \subseteq \tau$ and $\tau = \sup T$. Then there exist $\Sigma \subseteq T$ and subgroups A and B of G satisfying the following:* (i) $\tau = \sup \Sigma$; (ii) *A has length τ and B has length λ*; (iii) $G = A \oplus B$; (iv) *For all* $\sigma \in \Sigma$, $p^\sigma G[p] = p^\sigma A[p] \oplus p^{\sigma+1}B[p]$.

Proof. If $\lambda = \omega = \tau$ then G is a direct sum of cyclic groups [7; 3.5, p. 251] and 2.3 holds with Σ any infinite subset of T such that $T \backslash \Sigma$ is infinite. Suppose that $\lambda > \omega$ and let f be the Ulm-Kaplansky function of G, i.e.,

$$f(\mu) = rk(p^\mu G[p]/p^{\mu+1}G[p])$$

for every μ. It suffices to construct $\Sigma \subseteq T$ such that $\tau = \sup \Sigma$ and functions g and h from the ordinals to the cardinals satisfying the following conditions

(2.4) $f(\mu) = g(\mu) + h(\mu)$ for every μ.
(2.5) $\tau = \sup \{\mu + 1: g(\mu) \neq 0\}$.
(2.6) $\lambda = \sup \{\mu + 1: h(\mu) \neq 0\}$.
(2.7) For each limit ordinal $\rho < \tau$ such that $\rho + \omega < \tau$ and for each $t < \omega$, $\sum_{\rho+\omega \leq \mu < \tau} g(\mu) \leq \sum_{t \leq n < \omega} g(\rho + n)$.
(2.8) For each limit ordinal $\rho < \lambda$ such that $\rho + \omega < \lambda$ and for each $t < \omega$, $\sum_{\rho+\omega \leq \mu < \lambda} h(\mu) \leq \sum_{t \leq n < \omega} h(\rho + n)$.
(2.9) $h(\sigma) = 0$ for all $\sigma \in \Sigma$.

In fact, by [3; 83.6, p. 100], there exist totally projective groups A and B whose Ulm-Kaplansky functions are g and h respectively. By (2.4), the Ulm-Kaplansky function of $A \oplus B$ is f, so that $G \simeq A \oplus B$ by [3; 83.3, p. 98]. Property (iv) is a direct consequence of (2.9). Since f is the Ulm-Kaplansky function of G,

(2.10) $\sum_{\rho+\omega \leq \mu < \lambda} f(\mu) \leq \sum_{t \leq n < \omega} f(\rho + n)$, for each limit ordinal ρ such that $\rho + \omega < \lambda$ and all $t < \omega$;

furthermore,

(2.11) $\sup\{\mu + 1: f(\mu) \neq 0, \mu < \sigma\} = \sigma$ if $\sigma = \lambda$ or $\sigma < \lambda$ is a limit ordinal [3; 83.6, p. 100].

For convenience, let $I_\rho = \{\mu: \rho \leq \mu < \rho + \omega\}$ and put $T_\rho = T \cap I_\rho$. For the construction of Σ, we distinguish two cases.

Case 1. $\tau = \nu + \omega$ for some $\nu < \tau$. We may assume, without loss of generality, that either $\nu = 0$ or ν is a limit ordinal [6; pp. 295f, 271f]. Let $t < \omega$ and consider the cardinals

$$k_t = \sum_{\nu + t \leq \mu \in T_\nu} f(\mu) \quad \text{and} \quad l_t = \sum_{\nu + t \leq \mu \in I_\nu \setminus T_\nu} f(\mu).$$

Let $m = \min\{k_t + l_t: t < \omega\}$. Since $\nu < \nu + \omega = \tau \leq \lambda$, (2.11) implies

$$k_t + l_t = \sum_{\nu + t \leq \mu \in I_\nu} f(\mu) \geq \aleph_0,$$

for each $t < \omega$. Hence, $m \geq \aleph_0$. If $k_t < m$ for some $t < \omega$, put $\Sigma = T_\nu$. Suppose that $k_t \geq m$ for all $t < \omega$. Then T_ν contains an infinite subset T' such that, for all $t < \omega$, $\sum_{\nu + t \leq \mu \in T'} f(\mu) \geq m$. In this case, pick any subset Σ of T' such that both Σ and $T' \setminus \Sigma$ are infinite. In either case, $\sup \Sigma = \tau$ and

$$\sum_{\nu + t \leq \mu \in I_\nu \setminus \Sigma} f(\mu) = \sum_{\nu + t \leq \mu \in I_\nu} f(\mu) \geq \aleph_0 \quad \text{for all } t < \omega. \tag{2.12}$$

Case 2. $\rho < \tau$ implies $\rho + \omega < \tau$. Let $\Delta = \{\rho < \tau: T_\rho \neq \emptyset, \rho \text{ limit}\}$. For each $\rho \in \Delta$, pick $\sigma_\rho \in T_\rho$ and let $\Sigma = \{\sigma_\rho: \rho \in \Delta\}$. Then $\sup \Sigma = \tau$ [6; pp. 295, 296] as desired. In either case, the set Σ has been constructed. In order to define the functions g and h, consider an ordinal ρ such that either $\rho = 0$ or ρ is a limit ordinal for which $\rho + \omega \leq \tau$. Let

$$M_\rho = \{\mu \in I_\rho: 0 \neq f(\mu) < \aleph_0, \mu \in \Sigma\}.$$

If M_ρ is finite, put $P_\rho = \emptyset$; otherwise, pick $P_\rho \subseteq M_\rho$ such that both P_ρ and $M_\rho \setminus P_\rho$ are infinite. Let

$$M = \bigcup \{M_\rho: \rho + \omega \leq \tau, \rho = 0 \text{ or } \rho \text{ limit}\},$$

$$P = \bigcup \{P_\rho: \rho + \omega \leq \tau, \rho = 0 \text{ or } \rho \text{ limit}\},$$

and define g and h by

$$g(\mu) = \begin{cases} 0 & \text{if } \tau \leq \mu, \\ 0 & \text{if } \mu \in M \setminus P, \\ f(\mu) & \text{if } \mu \in P \cup (\tau \setminus M), \end{cases}$$

$$h(\mu) = \begin{cases} f(\mu) & \text{if } \tau \leq \mu, \\ f(\mu) & \text{if } \mu \in \tau \setminus (P \cup \Sigma) \\ 0 & \text{if } \mu \in P \cup \Sigma. \end{cases}$$

Then (2.9) is satisfied. The fact that $f(\mu)=f(\mu)+f(\mu)$ whenever $\mu \in \tau\backslash(M \cup \Sigma)$ implies (2.4); (2.5) and (2.6) follow from (2.11) and (2.12), recalling that either both M_ρ and P_ρ are infinite or both are finite. The same argument implies

$$\aleph_0 + \sum_{\rho+t\leq\mu\in M_\rho} f(\mu) \leq \sum_{\rho+t\leq\mu\in I_\rho\backslash M_\rho} f(\rho)$$

for all $t < \omega$, whenever ρ is a limit ordinal such that $\rho + \omega < \tau$ or $\rho = 0$. Thus, (2.7) follows from (2.10). It remains to verify (2.8). Let ρ be a limit ordinal such that $\rho + \omega < \lambda$. Because of (2.11), we may assume $\rho < \tau$. If $\rho + \omega < \tau$, (2.8) is a consequence of (2.10), (2.11), and the properties of M_ρ and N_ρ; if $\rho = \nu$ where $\nu + \omega = \tau$, observe (2.12).

The following easy set theoretical result will be needed.

2.13. LEMMA. *Let $\tau = \{\sigma\colon \sigma < \tau\}$ be a limit ordinal and let $f\colon \tau \to \tau$ be a function such that $f(\sigma) > \sigma$ for all $\sigma \in \tau$. Then there exists a subset $T \subseteq f(\tau)$ such that* $\sup T = \tau$ *and, for every $\Sigma \subseteq T$,* $\sup \Sigma = \tau$ *implies* $\sup [f^{-1}(\Sigma)] = \tau$.

Proof. Enlarge the domain of f by setting $f(\tau) = \tau$, ignoring the abuse of notation. Define ordinals η_σ inductively by $\eta_0 = 0$ and $\eta_\mu = f(\sup\{\eta_\sigma\colon \sigma < \mu\})$. Then there exists $\nu \leq \tau$ such that $\eta_\nu = \tau$. Let ν be minimal with respect to this property. One verifies that the set $T = \{\eta_\sigma\colon \sigma < \nu\}$ meets the requirements.

2.14. LEMMA. *Let G be totally projective of length λ and let $\tau \leq \lambda$ be a limit ordinal. Let $\varepsilon \in \operatorname{End} G$ such that, for all $\sigma < \tau$, $p^\sigma G[p]\varepsilon \nsubseteq p^\tau G$ and $p^\sigma G[p]\varepsilon \leq p^{\sigma+1}G$. Then, for all $k < \omega$, there are $A_k \leq G$, $w_k \in A_k$ and ordinals τ_k satisfying the following.*

(i) *$G = \bigoplus_{k<\omega} A_k \oplus C$ for some $C \leq G$.*
(ii) *For each $k < \omega$, $p^{\tau_k}A_k = \langle w_k\rangle = \mathbf{Z}(p)$ and $\tau_k < \tau_{k+1} < \tau$.*
(iii) *There exist ϕ, $\psi \in \operatorname{End} G$ such that, for all $k < \omega$, $w_k\phi\varepsilon\psi = w_{k+1}$.*

Proof. By hypothesis, for each $\sigma < \tau$, there exists $y_\sigma \in p^\sigma G[p]$ such that $y_\sigma\varepsilon \notin p^\tau G$. Define $f\colon \tau \to \tau$ by $f(\sigma) = h(y_\sigma\varepsilon)$. Then f satisfies the hypothesis of 2.13 and there exists $T \subseteq \tau$ as described in 2.13. In particular, $\sup T = \tau$ and 2.3 is applicable. Hence, there are A, $B \leq G$ of length τ and λ, respectively, and $\Sigma \subseteq T$ such that $G = A \oplus B$, $\sup \Sigma = \tau$ and, for all $\mu \in \Sigma$,

$$p^\mu G[p] = p^\mu A[p] \oplus p^{\mu+1}B[p].$$

Let $\Delta = \{\sigma < \tau\colon h(y_\sigma\varepsilon) \in \Sigma\}$ and let $\pi\colon G \to A$ be the natural projection annihilating B. Then $\Delta = f^{-1}(\Sigma)$, hence, by 2.13,

$$\tau = \sup \Delta, \tag{2.15}$$

and

$$0 \neq y_\sigma\varepsilon\pi \in A \quad \text{for all } \sigma \in \Delta.$$

Since A is totally projective [3; (A), p. 89] of length τ and τ is a limit ordinal, there exist $H_\sigma \leq A$ such that $A = \oplus_{\sigma<\tau} H_\sigma$, $p^{\sigma+1}H_\sigma = 0$ for all $\sigma < \tau$ [3; (e), p. 97]. Clearly, every $a \in A$ has finite support. Thus, for each $\sigma \in \Delta$, there exist ordinals η_σ, ρ_σ such that $\sigma < \eta_\sigma \leq \rho_\sigma < \tau$ and $0 \neq y_\sigma \varepsilon\pi \in \oplus_{\eta_\sigma \leq \mu \leq \rho_\sigma} H_\mu$. By (2.15), we may select countably many $\sigma_k \in \Delta$, $k < \omega$, such that $\sigma_{k+1} \geq \rho_{\sigma_k}$ for all $k < \omega$. Simplifying our notation without going through a formal renaming process, we write y_k instead of y_{σ_k} and ρ_k instead of ρ_{σ_k}. Let $v_k = h(y_k)$ and $\mu_k = h(y_k \varepsilon\pi)$. Then $\sigma_k \leq v_k < \mu_k \leq \rho_k \leq \sigma_{k+1}$, and

$$y_k \varepsilon\pi \in L_k \quad \text{where} \quad L_k = \bigoplus_{\mu_k \leq \mu \leq \rho_k} H_\mu. \tag{2.16}$$

Clearly,

$$G = \bigoplus_{k<\omega} L_k \oplus H \tag{2.17}$$

for some $H \leq G$. By [1; 3.3, p. 15], every totally projective p-group L is a direct sum of subgroups each of which has a p-basis with exactly one minimal element; and the lengths of those summands cannot exceed the length of L. Using the fact that L_k has length $\rho_k + 1$ it follows that, for each $k < \omega$, L_k has a decomposition of the form $L_k = A_k \oplus B_k$, where $p^{\tau_k} A_k = \mathbf{Z}(p)$ for some ordinal τ_k such that $\mu_k \leq \tau_k \leq \rho_k$. Since $\rho_k \leq \sigma_{k+1} < \mu_{k+1}$, we have $\tau_k < \tau_{k+1}$ for all $k < \omega$. Let $w_k \in A_k$ such that $p^{\tau_k} A_k = \langle w_k \rangle$. Then $\tau_k = h(w_k) \leq \rho_k \leq \sigma_{k+1} \leq v_{k+1} = h(y_{k+1})$. Thus, for each $k < \omega$, there is a homomorphism from A_k to G mapping w_k to y_{k+1} [7; 3.9, p. 252]. Since, for suitable $C \leq G$, $G = \oplus_{k<\omega} A_k \oplus C$, there exists $\phi \in \text{End } G$ such that $w_k \phi = y_{k+1}$ for all $k < \omega$ [2; 8.1, p. 40]. Likewise, $h(y_k \varepsilon\pi) = \mu_k \leq \tau_k = h(w_k)$, and, recalling (2.16) and (2.17), the same argument implies the existence of $\psi' \in \text{End } G$ such that $y_k \varepsilon\pi\psi' = w_k$ and thus, $w_k \phi\varepsilon\pi\psi' = y_{k+1} \varepsilon\pi\psi' = w_{k+1}$ for all $k < \omega$. Setting $\pi\psi' = \psi$, the conclusion follows.

3. Main results

In the following proposition, G need not be totally projective.

3.1. PROPOSITION. *Let* $\varepsilon \in \text{End } G$ *and assume the validity of* (i), (ii), (iii) *of* 2.14. *Then* $\varepsilon \notin J(\text{End } G)$.

Proof. Assume, by way of contradiction, that $\varepsilon \in J(\text{End } G)$. Let $\pi \in \text{End } G$ be the natural projection of G onto $\oplus_{k<\omega} A_k$ corresponding to the decomposition (i). Put $\beta = \phi\varepsilon\psi\pi$. Then

$$w_k \beta = w_{k+1} \quad \text{for all } k < \omega, \tag{3.2}$$

$C\beta = 0$, and $\beta \in J(\text{End } G)$. Hence $1 - \beta$ is an automorphism and there exists $\gamma \in \text{End } G$ such that $(1 - \beta)^{-1} = 1 - \gamma$. A straightforward computation shows that

$$\beta = \beta\gamma - \gamma, \ \beta\gamma = \gamma\beta, \tag{3.3}$$

and (3.2) implies

$$(3.4) \qquad w_{k+1} = w_{k+1}\gamma - w_k\gamma \quad \text{for all } k < \omega.$$

Let $y = w_0(1-\gamma)$ and let $z_n = w_0 + w_1 + \cdots + w_n$. Then, by (3.4),

$$\begin{aligned} z_n &= w_0 + (w_1\gamma - w_0\gamma) + (w_2\gamma - w_1\gamma) + \cdots + (w_n\gamma - w_{n-1}\gamma), \\ &= w_0(1-\gamma) + w_n\gamma. \end{aligned}$$

Thus, $y - z_n = (-w_n)\gamma$ has height at least τ_n which implies that, for all $k < \omega$, the component of y in the kth summand of the decomposition 2.14 (i) is w_k. This is plainly impossible and the proof is completed.

3.5. THEOREM. *Let G be totally projective of length λ and let $\varepsilon \in J(\text{End } G)$. Then, for each $0 < \tau \leq \lambda$, there exists $\sigma < \tau$ such that $p^\sigma G[p]\varepsilon \leq p^\tau G$.*

Proof. Assume, by way of contradiction, that, for all $\sigma < \tau$, $p^\sigma G[p]\varepsilon \not\leq p^\tau G$. Then, by 2.2, ε satisfies the hypothesis of 2.14 and (i), (ii), (iii) hold. Apply 3.1.

3.6. THEOREM. *Let G be totally projective of length λ and let $\varepsilon \in J(\text{End } G)$. Then there exist finitely many ordinals*

$$0 = \beta_0 < \beta_1 < \cdots < \beta_n < \beta_{n+1} = \lambda$$

such that, for $i = 0, \ldots, n$, $p^{\beta_i}G[p]\varepsilon \leq p^{\beta_{i+1}}G$.

Proof. Use 3.5 together with the fact that every properly decreasing sequence of ordinals terminates after finitely many steps [6; p. 270].

If ε has the properties stated in 3.6 then $\varepsilon \in \bigcap_{i=0}^{n} \text{Ann } (p^{\beta_i}G[p]/p^{\beta_{i+1}}G[p])$, and $\varepsilon \mid G[p]$ is nilpotent. Recalling 2.1, we have the following result.

3.7. COROLLARY. *Let G be a totally projective p-group and let J be an ideal of* End G. *Then J is quasi-regular if and only if J induces in $G[p]$ a nil ring of endomorphisms.*

The description of the Jacobson radical of End G is now complete.

3.8. THEOREM. *If G is a totally projective p-group of length λ then*

$$J(\text{End } G) = \bigcup_{n<\omega} \left(\bigcup_{0=\beta_0<\beta_1<\cdots<\beta_{n+1}=\lambda} \left[\bigcap_{i=0}^{n} \text{Ann } (p^{\beta_i}G[p]/p^{\beta_{i+1}}G[p]) \right] \right).$$

Proof. Let J denote the right hand side of this equation. Then $\varepsilon \mid G[p]$ is nilpotent for every $\varepsilon \in J$. Thus, using 3.6 and 3.7, it remains to show that J is an ideal. This follows from the fact that, if

$$0 = \beta_0 < \beta_1 < \cdots < \beta_{n+1} = \lambda \quad \text{and} \quad 0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m+1} = \lambda$$

are ordinals such that $\{\beta_i\}_{i\leq n+1} \subseteq \{\gamma_i\}_{i\leq m+1}$ and $p^{\beta_i}G[p]\varepsilon \leq p^{\beta_{i+1}}G$ for $0 \leq i \leq n$, then $p^{\gamma_i}G[p]\varepsilon \leq p^{\gamma_{i+1}}G$ for $0 \leq i \leq m$.

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