ORDERING ON A SEMISIMPLE LIE GROUP AND CHARACTER VALUES

BY

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1. Introduction

We first recall some results of Kostant in [1]. Let G be a connected, semisimple, real Lie group, $\mathfrak{g}$ its Lie algebra. Let $\hat{G}$ be the set of all equivalence classes of irreducible finite-dimensional representations of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$, $G = KAN$ the associated decomposition of G. An element h of G is called *hyperbolic* if $h = \exp(x)$, with $x \in \mathfrak{g}$ such that ad (x) is **R**-diagonalizable. In this case, one shows that h is conjugate in G to an element of A; or, equivalently, x is conjugate under the action of Ad (G) to an element of $\mathfrak{a}$.

The ordering between hyperbolic elements is defined in the following manner: Let $W = W(\mathfrak{g}, \mathfrak{a})$ be the Weyl group. For $x \in \mathfrak{a}$, let $\mathfrak{a}(x)$ be the convex hull of the orbit of x under W. Then $x \leq y$ $(x, y \in \mathfrak{a})$ iff $x \in \mathfrak{a}(y)$. This ordering is transported, by using the exponential isomorphism, to A and then to the set of all hyperbolic elements: let h, k be hyperbolic, respectively conjugate to the elements a and b of A; then we set $h \leq k$ iff $a \leq b$. (For the consistency of this definition see [1].)

For $\lambda \in \hat{G}$, let π_λ be a representation of the class λ: we define the (non-normalized) character χ_λ on G by $\chi_\lambda(x) = \operatorname{Tr}(\pi_\lambda(x))$. Then the following theorem holds (Kostant [1, part 6, Theorem 6.1].

THEOREM. *Let f, g be two hyperbolic elements of G, $f \leq g$. Then for all $\lambda \in \hat{G}$, $\chi_\lambda(f) \leq \chi_\lambda(g)$.*

We shall prove the following converse, in the case where $\mathfrak{g}$ is a normal real form of its complexification $\mathfrak{g}_{\mathbf{C}}$, i.e., when $\mathfrak{a}_{\mathbf{C}} = \mathfrak{a} + i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ (see [2, p. 6]).

THEOREM. *Let G be a real semisimple connected Lie group; assume that $\mathfrak{g}$ is a normal real form of $\mathfrak{g}_{\mathbf{C}}$. Let $f, g \in G$ be two hyperbolic elements. If $\chi_\lambda(f) \leq \chi_\lambda(g)$ for all $\lambda \in \hat{G}$, then $f \leq g$.*

We shall use a sequence of lemmas. Let $\mathfrak{a}_+$ be the adherence of the fundamental Weyl chamber in $\mathfrak{a}$. Using the properties of conjugacy, we see that it is enough to consider the case $f = \exp(x)$, $g = \exp(y)$, for x, y in $\mathfrak{a}_+$. In fact we first contend that x and y are in the Weyl chamber $(\mathfrak{a}_+)^0$, interior of $\mathfrak{a}_+$.

We define a preorder relation on $\mathfrak{a}$, denoted $\ll$, by $x \ll y$ $(x, y \in \mathfrak{a})$ iff for all $\lambda \in \hat{G}$, $\chi_\lambda(\exp(x)) \leq \chi_\lambda(\exp(y))$.

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Using the Killing duality, denoted $\langle \ , \ \rangle$, we identify $\mathfrak{a}$ with its (real) dual; we can identify an element λ of $\hat{G}$ with the associated representation of $\mathfrak{g}$, and this one to its highest weight, considered as being in $\mathfrak{a}$—thus in fact $\lambda \in \mathfrak{a}_+$.

LEMMA 1. *Let* $x \in (\mathfrak{a}_+)^0$, $\lambda \in \hat{G}$. *Then* $\chi_\lambda(\exp(tx)) \sim_{t\to+\infty} \exp(t\langle\lambda, x\rangle)$ (*the equivalence is between two functions of* t).

Proof. Let $\{\alpha_i, i = 1, \ldots, r\}$ be the simple roots in $\mathfrak{a}$; let Π_λ be the set of weights of the representation λ. Then

$$\chi_\lambda(\exp(tx)) = \exp(t\langle\lambda, x\rangle) + \sum_{\rho\in\Pi\lambda,\, \rho\neq\lambda} m_{\lambda,\rho}\, exp\,(t\langle\rho, x\rangle)$$

for certain integers $m_{\lambda,\rho}$. If $\rho \in \Pi_\lambda$, $\rho \neq \lambda$, we know that $\rho = \lambda - \sum_{i=1}^r n_i\alpha_i$, with the integers n_i not all zero. But by definition of the Weyl chamber, $\langle\alpha_i, x\rangle > 0$ for all $x \in (\mathfrak{a}_+)^0$, $i = 1, \ldots, r$. Thus it is clear that $\langle\rho, x\rangle < \langle\lambda, x\rangle$ for all $\rho \in \Pi_\lambda$, $\rho \neq \lambda$; the comparison of the exponentials then proves Lemma 1.

Now we use Weyl's formula giving the formula for χ_λ:

$$\chi_\lambda(\exp(x)) = \frac{\sum_{\sigma\in W} \det(\sigma) \exp\langle\sigma(\lambda + \rho), x\rangle}{\sum_{\sigma\in W} \det(\sigma) \exp\langle\sigma(\rho), x\rangle}$$

with $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_{i=1}^r \lambda_i$, λ_i the fundamental dominant weights.

We know that the denominator $D(x)$ is given by

$$D(x) = \sum_{\sigma\in W} \det(\sigma) \exp\langle\sigma(\rho), x\rangle = \prod_{\alpha>0} (e^{\langle\alpha/2, x\rangle} - e^{-\langle\alpha/2, x\rangle})$$

from which we deduce that for all $x \in (\mathfrak{a}_+)^0$, $D(x) > 0$, and the quotient in Weyl's formula makes sense. (For these results see [3, pp. 138–139].)

LEMMA 2. *Let* $x, y \in (\mathfrak{a}_+)^0$, *such that* $x \ll y$, *Then* $nx \ll ny$, *for all* $n \in \mathbf{N}$ *such that* $(n - 1)\rho \in \hat{G}$.

Proof. By Weyl's formula and the hypothesis on x and y,

$$(*) \qquad \frac{\sum \det(\sigma) \exp\langle\sigma(\lambda + \rho), x\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), x\rangle} \leq \frac{\sum \det(\sigma) \exp\langle\sigma(\lambda + \rho), y\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), y\rangle}$$

for all $\lambda \in \hat{G}$.

Let $n \in \mathbf{N}$ be such that $(n - 1)\rho \in \hat{G}$, whence $n\lambda + (n - 1)\rho \in \hat{G}$. Replacing λ by $(n\lambda + (n - 1)\rho)$ in $(*)$, we get

$$(**) \qquad \frac{\sum \det(\sigma) \exp\langle\sigma(\lambda + \rho), nx\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), x\rangle} \leq \frac{\sum \det(\sigma) \exp\langle\sigma(\lambda + \rho), ny\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), y\rangle}$$

for all $\lambda \in \hat{G}$,

whence finally, rearranging,

$$\chi_\lambda(\exp(nx)) \le C(n, x, y)\chi_\lambda(\exp(ny)) \quad \text{for all } \lambda \in \hat{G},$$

with

$$C(n, x, y) = \frac{\sum \det(\sigma) \exp\langle\sigma(\rho), x\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), nx\rangle} \cdot \frac{\sum \det(\sigma) \exp\langle\sigma(\rho), ny\rangle}{\sum \det(\sigma) \exp\langle\sigma(\rho), ny\rangle}$$

(thus *C does not depend on* λ).

We want to prove that in fact $\chi_\lambda(\exp(nx)) \le \chi_\lambda(\exp(ny))$. This is clear because of the following lemma (it is here that we use the non-normalization of the characters).

LEMMA 3. *Let $f, g \in G$. If there exists a constant $C > 0$ such that $\chi_\lambda(f) \le C\chi_\lambda(g)$ for all $\lambda \in \hat{G}$ then $\chi_\lambda(f) \le \chi_\lambda(g)$ for all $\lambda \in \hat{G}$. (C can be taken equal to* 1.)

Proof. Evident because of the complete reducibility and of the formula $\chi_\lambda \cdot \chi_\mu = \chi_{\lambda\otimes\mu}$ (consider $\chi_{\otimes^m\lambda}$, $m \to \infty$).

The Lemma 2 is so completely proved.

COROLLARY OF LEMMA 2. *Let $x, y \in (\mathfrak{a}_+)^0$. Then* $\exp(x) \ll \exp(y)$ *implies* $\exp(x) \le \exp(y)$.

Proof. Let $\mathfrak{a}_\rho$ be the cone generated by the simple roots in $\mathfrak{a}$. Then it is easy to see that $x \in \mathfrak{a}$ is in $\mathfrak{a}_\rho$ if and only if $\langle\lambda_i, x\rangle \ge 0$ for all $i = 1, \ldots, r$. Moreover [1, Lemma 3.3, p. 429], an element x of $\mathfrak{a}_+$ is in $\mathfrak{a}(y)$ if and only if $y - x \in \mathfrak{a}_\rho$. Let $\{\lambda_i, i = 1, \ldots, r\}$ be the fundamental weights. We know that there exists $K \in \mathbf{N}$ such that $K\lambda_i \in \hat{G}$ for all i. Let $\mu_i = K\lambda_i$, and $J \subset \mathbf{N}$ be the set of integers defined by Lemma 2. (J is infinite.)

For all $n \in J$, $nx \ll ny$ by Lemma 2, whence $\chi_{\mu_i}(\exp(nx)) \le \chi_{\mu_i}(\exp(ny))$. But, if we let $n \to \infty$ in J, we have by Lemma 1,

$$\chi_{\mu_i}(\exp(nx)) \sim \exp(n\langle\mu_i, x\rangle), \qquad \chi_{\mu_i}(\exp(ny)) \sim \exp(n\langle\mu_i, y\rangle)$$

as functions of n, whence $\langle\mu_i, x\rangle \le \langle\mu_i, y\rangle$; therefore $\langle\lambda_i, y - x\rangle \ge 0$ for all i, i.e., $y - x \in \mathfrak{a}_\rho$. So $x \in \mathfrak{a}(y)$, whence $\exp(x) \le \exp(y)$, which proves the corollary.

By conjugation, we see that the theorem is proved for all the regular elements of $\mathfrak{a}$; we conclude by using the evident continuity of the order $\le$.

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