COMPLETE SPACES AND TRI-QUOTIENT MAPS

BY

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1. Introduction

The purpose of this note is to study the behavior of complete spaces under various kinds of maps.[1] We first do this for open maps, where we give new proofs for some known results, and then extend these results to tri-quotient maps, an interesting new concept which is introduced and studied in this paper. As an application of our results, we generalize a recent theorem of J. P. R. Christensen [7] on compact-covering images of complete separable metric spaces.[2]

The most familiar completeness properties are complete metrizability and Čech-completeness,[3] and it is these concepts which will mainly concern us in this introduction. In the body of the paper, however, we shall mostly work with sieve-completeness, a particularly well-behaved property which was recently introduced in [5] by J. Chaber, M. M. Čoban, and K. Nagami (see Definition 2.1).[4] Every Čech-complete space is sieve-complete, and it was shown in [5] that the two concepts are equivalent in the presence of paracompactness. The proof of that equivalence in [5] was, however, quite indirect (see Remarks 8.1, 8.2, 8.3), and our first task will be to give a simple, direct proof of this result (see Theorem 3.2). That permits us to prove our basic mapping theorems for sieve-complete spaces, where they are particularly simple, and to indicate in this introduction some consequences of these results for Čech-complete spaces and completely metrizable spaces.

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[1] All maps in this paper are continuous surjections. No separation properties are assumed unless indicated; however, regular spaces are T_1 and paracompact spaces are Hausdorff.

[2] I would like to take this opportunity to thank Eric van Douwen for numerous helpful conversations and suggestions during the summer of 1975. In particular, this paper was originally motivated by his question of whether Christensen's theorem could be generalized to the result stated in Corollary 1.5, and it was he who distilled the concept of a tri-quotient map out of the author's original proof of that result.

[3] A completely regular space is *Čech-complete* if it is a G_δ in one (equivalently, in all) of its Hausdorff compactifications. By a result of E. Čech ([16] or [8, p. 190, Theorem 11]), a metrizable space is completely metrizable (i.e., metrizable by a complete metric) if and only if it is Čech-complete.

[4] Sieve-complete spaces are called *monotonically Čech-complete* in [5]. It follows from [5, Lemma 1.1 and Proposition 2.10] that a regular space is sieve-complete if and only if it is a λ_b-space in the sense of H. H. Wicke [28]. However, sieve-complete spaces seem to be much easier to work with than λ_b-spaces (even when the two are equivalent), and—unlike λ_b-spaces—many results about them are true without assuming any separation properties at all (see, for example, Proposition 4.2 and Theorem 6.3).

We begin by considering open maps. It follows almost immediately from the definitions that such maps preserve sieve-completeness (see Proposition 4.2), and we thus immediately obtain the following theorem of B. Pasynkov [23, Corollary 10(a)].[5][6]

THEOREM 1.1 (B. Pasynkov). *If $f: X \to Y$ is an open map from a Čech-complete space X onto a paracompact space Y, then Y is also Čech-complete.*

COROLLARY 1.2 (F. Hausdorff [12]). *If $f: X \to Y$ is an open map from a completely metrizable space X onto a metrizable space Y, then Y is also completely metrizable.*

Let us now extend the above results to a larger class of maps. For that purpose we introduce *tri-quotient* maps (see Definition 6.1), a new concept which seems to have just the right properties for dealing with completeness. It follows from the definition that all open maps and all perfect maps are tri-quotient (see Theorem 6.5(a), (b)), and that tri-quotient maps are bi-quotient in the sense of [17]. Less obviously, all compact-covering s-maps[7] with first-countable Hausdorff range are tri-quotient (see Theorem 6.5(e)). Finally, just as for open maps (though less trivially), tri-quotient maps preserve sieve-completeness (Theorem 6.3). We thus obtain the following generalizations of Theorem 1.1 and Corollary 1.2.

THEOREM 1.3. *If $f: X \to Y$ is a tri-quotient map from a Čech-complete space X onto a paracompact space Y, then Y is also Čech-complete.*

COROLLARY 1.4. *If $f: X \to Y$ is a tri-quotient map from a completely metrizable space X onto a metrizable space Y, then Y is also completely metrizable.*

An interesting consequence of Corollary 1.4 and Theorem 6.5(e) is the following generalization of a result of J. P. R. Christensen [7].

COROLLARY 1.5. *If $f: X \to Y$ is a compact-covering s-map from a completely metrizable space X onto a metrizable space Y, then Y is also completely metrizable.*[7a]

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[5] This theorem, which is here derived from Theorem 3.2, was used in [5] as a step in the proof of Theorem 3.2. (See Remark 8.3.)

[6] This theorem was also announced, independently, by J. M. Worrell, Jr. [29] and H. H. Wicke [27].

[7] A map $f: X \to Y$ is *compact-covering* if every compact $K \subset Y$ is the image of some compact $C \subset X$; it is an *s-map* if every $f^{-1}(y)$ has a countable base.

[7a] *Added in proof.* Corollary 1.5 was also obtained, independently, by A. V. Ostrovskiĭ, *On compact-covering mappings*, Dokl. Akad. Nauk. SSSR, vol. 227 (1976), pp. 1297–1300 = Soviet Math. Dokl., vol. 17 (1976), pp. 606–610.

For separable X, Corollary 1.5 was proved by Christensen in [7, Corollary 1 to Theorem 1], thereby confirming a conjecture of A. H. Stone and the author in [21, 3.7].

We now introduce another concept. Call a map $f: X \to Y$ *inductively perfect* if there is an $X' \subset X$ such that $f(X') = Y$ and $f \mid X'$ is perfect. If X is Hausdorff, this X' must be closed in X; see, for instance, [18, Corollary 1.5]. Inductively tri-quotient maps, which can be defined analogously, coincide with tri-quotient maps (Lemma 6.4); hence every inductively perfect map is tri-quotient. The following result (which follows from Theorem 6.6) provides a partial converse.

THEOREM 1.6.   *A map $f: X \to Y$ from a Čech-complete space X onto a paracompact space Y is tri-quotient if and only if it is inductively perfect.*[8]

Since perfect maps are compact-covering, so are inductively perfect maps, and we obtain the following corollary.

COROLLARY 1.7.   *Every tri-quotient map $f: X \to Y$ from a Čech-complete space X onto a Hausdorff space Y is compact-covering.*[9]

Since Čech-completeness is inherited by closed subsets [8, p. 144, Theorem 3] and preserved by perfect maps [25], [8, p. 167, Y], Theorem 1.6 provides an alternative approach to Theorem 1.3. Since perfect maps preserve metrizability [22], [24], Theorem 1.6 implies that Corollary 1.4 remains true if the hypothesis on Y is weakened from being metrizable to being paracompact. An analogous refinement of Corollary 1.5 will be given in Theorem 6.7; see also Remark 5.3.

So far, the results about open maps and completeness which we have considered have all generalized to tri-quotient maps. It is not known, however, whether that remains true for the following result, which refines Theorem 1.6 in case X is metrizable and f open.

THEOREM 1.8 [13, Corollary 1.2].   *If $f: X \to Y$ is an open map from a metric space X onto a paracompact space Y, and if every $f^{-1}(y)$ is complete (for the given metric on X), then f is inductively perfect and thus compact-covering.*

Question 1.9.   Does Theorem 1.8 remain true if "open" is weakened to "tri-quotient"? (I don't know the answer even when Y is metrizable.)

The paper is arranged as follows. Sieves and sieve-complete spaces are introduced in Section 2, Section 3 proves the theorem relating sieve-complete spaces to Čech-complete spaces, and Section 4 proves that sieve-completeness is preserved by open maps. In Section 5 we prove a lemma on compact-covering

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[8] For the special case of open maps, the nontrivial ("only if") part of Theorem 1.6 was obtained by Pasynkov in [23, Theorem 8]. That is how Pasynkov proved the result stated in our Theorem 1.1.

[9] For open maps f, Corollary 1.7 was obtained by A. V. Arhangel'skiĭ in [4, Theorem 1.2].

maps. Section 6 introduces and studies tri-quotient maps, especially as they relate to complete spaces, while Section 7 proves that tri-quotient maps are preserved by composition and finite products. Section 8 contains additional remarks and results on sieve-complete spaces. Section 9 is devoted to examples.

2. Sieves and sieve-complete spaces

Recall that a filter base $\mathscr{F}$ *clusters* at x in X if $x \in \bar{F}$ for all $F \in \mathscr{F}$. Two collections of sets $\mathscr{F}$ and $\mathscr{U}$ *mesh* [19, p. 99] if every $F \in \mathscr{F}$ intersects every $U \in \mathscr{U}$.

The following definition is essentially taken from [5, p. 108]. (It should be remarked that the sieves introduced here are not the same as those commonly used in the theory of analytic sets.)

DEFINITION 2.1. A *sieve* on a space X is a sequence of open covers[10] $\{U_\alpha : \alpha \in A_n\}_{n \in N}$ of X (with disjoint A_n), together with functions $\pi_n : A_{n+1} \to A_n$, such that, for all n and $\alpha \in A_n$, $U_\alpha = \bigcup \{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$. A *$\pi$-chain* for such a sieve is a sequence (α_n) such that $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for all n. The sieve is *complete* if, for every π-chain (α_n), every filter base $\mathscr{F}$ on X which meshes with $\{U_{\alpha_n} : n \in N\}$ clusters in X.[11] A space X with a complete sieve is called *sieve-complete*.[12]

We now establish four lemmas about sieves. Lemma 2.2 will be used in the proof of Theorem 3.2, Lemma 2.3 in the proof of Theorem 6.3, and Lemmas 2.1–2.3 in the proof of Theorem 6.7. Lemma 2.4 will only be used in Remark 8.12, and is included here mainly for the sake of completeness.

Let us call a sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ *locally finite* if the indexed family $\{U_\alpha : \alpha \in A_n\}$ is locally finite for all n.

LEMMA 2.2. *If $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is a sieve on a paracompact space X, then there is a locally finite sieve $(\{V_\alpha : \alpha \in A_n\}, \pi_n)$ on X such that $\bar{V}_\alpha \subset U_\alpha$ for all α.*

Proof. Let $\{V_\alpha : \alpha \in A_1\}$ be any locally finite open cover of X such that $\bar{V}_\alpha \subset U_\alpha$ for all $\alpha \in A_1$. Suppose we have $\{V_\alpha : \alpha \in A_k\}$ for $k \leq n$, and let us define $\{V_\beta : \beta \in A_{n+1}\}$. For each $\alpha \in A_n$, $\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ covers the para-compact space $\bar{V}_\alpha$, so there is a locally finite, relatively open cover

$$\{W_\beta : \beta \in \pi_n^{-1}(\alpha)\}$$

of $\bar{V}_\alpha$ such that $\bar{W}_\beta \subset U_\beta$ for all $\beta \in \pi_n^{-1}(\alpha)$; let $V_\beta = W_\beta \cap V_\alpha$ for all $\beta \in \pi_n^{-1}(\alpha)$. Then $\{V_\beta : \beta \in A_{n+1}\}$ has all the required properties, and that completes the proof.

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[10] We allow $U_\alpha = \emptyset$, and also $U_\alpha = U_\beta$ for distinct $\alpha, \beta \in A_n$.

[11] Equivalently, if every ultrafilter on X which contains $\{U_{\alpha_n} : n \in N\}$ converges in X. Or equivalently, if every filter base $\mathscr{F}$ on X, such that each U_{α_n} contains some $F \in \mathscr{F}$, clusters in X.

[12] In [5, Definition 2.1], a complete sieve is called an (mc)-sieve, and a sieve-complete space is called monotonically Čech-complete.

For our next lemma, let us call a sieve $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ *finitely additive* if every collection $\{U_\alpha: \alpha \in A_n\}$, as well as every collection of the form $\{U_\beta: \beta \in \pi_n^{-1}(\alpha)\}$ with $\alpha \in A_n$, is closed under finite unions.

LEMMA 2.3. *Every sieve-complete space X has a finitely additive, complete sieve.*

Proof. Let $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ be a complete sieve on X. For each n, let $\mathscr{B}_n = \{B \subset A_n: B \text{ finite}, B \neq \emptyset\}$, and for every $B \in \mathscr{B}_n$ let $U_B = \bigcup\{U_\alpha: \alpha \in B\}$. Define $p_n: \mathscr{B}_{n+1} \to \mathscr{B}_n$ by $p_n(B) = \{\pi_n(\alpha): \alpha \in B\}$. It is easy to check that $(\{U_B: B \in \mathscr{B}_n\}, p_n)$ is a finitely additive sieve on X. Let us show that it is complete.

Suppose (B_n) is a p-chain, and that $\mathscr{F}$ is a filter base on X which meshes with $\{U_{B_n}: n \in N\}$. Let C_n be the set of all $\alpha \in B_n$ such that U_α intersects every $F \in \mathscr{F}$. Then C_n is finite, $C_n \neq \emptyset$, and $\pi_{n+1}(C_{n+1}) \subset C_n$ for all n. By König's Infinity Lemma (i.e., lim inv $(C_n, \pi_n) \neq \emptyset$ if each C_n is finite, $C_n \neq \emptyset$), there exists a π-chain (α_n) with $\alpha_n \in C_n$ for all n. Clearly $\mathscr{F}$ meshes with $\{U_{\alpha_n}: n \in N\}$, so $\mathscr{F}$ clusters in X. That completes the proof.

According to [5], a sieve $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ on X is a *strong sieve* if $\overline{U}_\beta \subset U_\alpha$ whenever $\alpha \in A_n$ and $\beta \in \pi_n^{-1}(\alpha)$. The following lemma is obtained in [5] as part of a more extensive result [5, Lemma 1.1]; for the sake of completeness, we include a short proof.

LEMMA 2.4. *Every regular, sieve-complete space has a strong complete sieve.*

Proof. Let $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ be a complete sieve on X. It will suffice to define a strong sieve

$$(\{V_{\alpha, \lambda}: (\alpha, \lambda) \in A_n \times \Lambda_n\}, \pi_n \times \phi_n)$$

on X (where $\phi_n: \Lambda_{n+1} \to \Lambda_n$) such that always $V_{\alpha, \lambda} \subset U_\alpha$, since such a sieve will automatically be complete.

Let Λ_1 be any nonempty set, and let $V_{\alpha, \lambda} = U_\alpha$ for all $(\alpha, \lambda) \in A_1 \times \Lambda_1$. Suppose now that everything has been defined up to n. Pick a set M so that card M = exp (card X), let $\Lambda_{n+1} = \Lambda_n \times M$, and define $\phi_n: \Lambda_{n+1} \to \Lambda_n$ by $\phi_n(\lambda, \mu) = \lambda$; then, for each $(\alpha, \lambda) \in A_n \times \Lambda_n$ and each $\beta \in \pi_n^{-1}(\alpha)$, let $\{V_{\beta, (\lambda, \mu)}: \mu \in M\}$ be an indexing of the collection of all open $V \subset U_\beta$ such that $\overline{V} \subset V_{\alpha, \lambda}$. This construction has all the required properties.

Our last lemma follows from [5, Proposition 2.10], and is easily verified.

LEMMA 2.5. *The following properties of a strong sieve $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ on a space X are equivalent.*

(a) *$(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ is a complete sieve.*

(b) *If (α_n) is a π-chain, if $U_\alpha \neq \emptyset$ for all n, and if $C = \bigcap_n U_{\alpha_n}$, then C is nonempty, closed, and compact, and every open $V \supset C$ contains some U_{α_n}.*

3. Čech-complete and sieve-complete spaces

We begin by recalling a result of Z. Frolík [11] and A. V. Arhangel'skiĭ [2] (see [8, p. 143, Theorem 2]).

LEMMA 3.1. *The following properties of a completely regular space X are equivalent.*

(a) X *is Čech-complete.*
(b)   *There is a sequence $(\mathscr{V}_n)$ of open covers of X such that, if $\mathscr{F}$ is a filter base on X and if each $\mathscr{V}_n$ has an element V_n which contains some $F_n \in \mathscr{F}$, then $\mathscr{F}$ clusters in X.*

The following result was obtained, rather indirectly, in [4] (see Remark 8.3). It should be compared to Lemma 3.1.

THEOREM 3.2. *The following properties of a paracompact space X are equivalent.*[13]

(a) X *is Čech-complete.*
(b) X *is sieve-complete.*

Proof. (a) $\Rightarrow$ (b).   Let $(\mathscr{V}_n)$ be as in 3.1(b). We shall construct our sieve by induction on n. For $n = 1$, let $\{U_\alpha : \alpha \in A_1\}$ be the collection of all open subsets of Y which are subsets of some $V \in \mathscr{V}_1$. Suppose we have everything up to and including n. For each $\alpha \in A_n$, let $\{U_\beta : \beta \in A_{n+1}(\alpha)\}$ be the collection of all open subsets of U_α which are also subsets of some $V \in \mathscr{V}_{n+1}$. We may suppose that the index sets $A_{n+1}(\alpha)$, with $\alpha \in A_n$, are disjoint from each other and from all A_k with $k \leq n$. Let $A_{n+1} = \bigcup \{A_{n+1}(\alpha) : \alpha \in A_n\}$, and define $\pi_n : A_{n+1} \to A_n$ by $\pi_n(\beta) = \alpha$ if $\beta \in A_{n+1}(\alpha)$. It is clear that this defines a sieve on X. To see that it is complete, suppose (α_n) is a π-chain and $\mathscr{F}$ a filter base on X which meshes with $\{U_{\alpha_n} : n \in N\}$. Let $\mathscr{F}' = \{F \cap U_{\alpha_n} : F \in \mathscr{F}, n \in N\}$. Then $\mathscr{F}'$ is also a filter base on X, and it satisfies the hypothesis of 3.1(b). Hence $\mathscr{F}'$—and thus also $\mathscr{F}$—clusters in X.

(b) $\Rightarrow$ (a).   We will show that X is a G_δ in any Hausdorff space X^* containing it as a dense subset. By Lemma 2.2, there is a locally finite, complete sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ on X. Let W_n be the set of all $x \in X^*$ at which $\{U_\alpha : \alpha \in A_n\}$ is locally finite. Then each W_n is open in X^*. We will show that $X = \bigcap_n W_n$.

Clearly $X \subset W_n$ for all n. It remains to show that, if $x \in \bigcap_n W_n$, then $x \in X$. Let $B_n = \{\alpha \in A_n : x \in \overline{U}_\alpha\}$. Then B_n is finite, $B_n \neq \emptyset$, and $\pi_n(B_{n+1}) \subset B_n$ for all n. By König's Infinity Lemma (i.e., $\lim \operatorname{inv} (B_n, \pi_n) \neq \emptyset$ if each B_n is finite, $B_n \neq \emptyset$), there is a π-chain (α_n) with $\alpha_n \in B_n$ for all n. Now let

$$\mathscr{F} = \{U \cap X : U \text{ a neighborhood of } x \text{ in } X^*\}.$$

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[13] Paracompactness is not needed for (a) $\Rightarrow$ (b).

Then $\mathscr{F}$ is a filter base on X which meshes with $\{U_{\alpha_n}: n \in N\}$, so $\mathscr{F}$ clusters in X. But clearly x is the only cluster point of $\mathscr{F}$ in X^*, so $x \in X$. That completes the proof.

4. Open images of sieve-complete spaces

Our first result is a lemma which is valid for all (not only open) maps, and which will also be used in the proof of Theorem 6.4.

LEMMA 4.1.   *Suppose that $f: X \to Y$ is a map, and that $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ is a complete sieve on X. Then any sieve $(\{V_\alpha: \alpha \in A_n\}, \pi_n)$ on Y such that $V_\alpha \subset f(U_\alpha)$ for all α is also complete.*

Proof.   Let $\{\alpha_n\}$ be a π-chain and $\mathscr{F}$ a filter base on Y meshing with $\{V_{\alpha_n}: n \in N\}$. Then $f^{-1}(\mathscr{F})$ meshes with $\{U_{\alpha_n}: n \in N\}$, so $f^{-1}(\mathscr{F})$ clusters at some $x \in X$, and hence $\mathscr{F}$ clusters at $f(x)$ in Y. That completes the proof.

PROPOSITION 4.2.   *If $f: X \to Y$ is an open map, and if X is sieve-complete, so is Y.*

Proof.   If $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ is a complete sieve on X, then

$$(\{f(U_\alpha): \alpha \in A_n\}, \pi_n)$$

is a sieve on Y (since f is open), and this sieve is complete by Lemma 4.1.

5. A lemma on compact-covering maps

The following lemma is based on the main idea of Christensen's proof of [7, Theorem 1]. We use ∂ to denote boundary.

LEMMA 5.1.   *Suppose $f: X \to Y$ is compact-covering, with Y Hausdorff and first-countable. Let $y \in Y$, and let (U_n) be an increasing sequence of open subsets of X which cover $\partial f^{-1}(y)$ such that $U_1 \cap f^{-1}(y) \neq \emptyset$. Then there is an open neighborhood V of y in Y and an $n \in N$ such that every compact $K \subset V$ is the image of some compact $C \subset U_n$. (Hence, if $X' = U_n \cap f^{-1}(V)$, the map $f \mid X': X' \to V$ is again compact-covering.)*

Proof.   Case 1. $\partial f^{-1}(y) = \emptyset$. In this case, simply take $V = \{y\}$ and $n = 1$.

Case 2. $\partial f^{-1}(y) \neq \emptyset$. Let $\hat{X} = X - \text{Int } f^{-1}(y)$, and $\hat{f} = f \mid \hat{X}$. Then $\hat{f}: \hat{X} \to Y$ is also compact-covering, and the U_n cover $\hat{f}^{-1}(y)$. It will now suffice to prove our lemma with f replaced by $\hat{f}$; equivalently, we need only show that our lemma is true if the U_n are assumed to cover all of $f^{-1}(y)$.

Suppose it is not. Let (V_n) be a decreasing base for the neighborhoods of y in Y. Then for each n there is a compact $K_n \subset V_n$ which is not the image of any compact $C \subset U_n$. Let $K = \{y\} \cup \bigcup_{n=1}^{\infty} K_n$. Then K is compact, so $K =$

$f(C)$ for some compact $C \subset X$. Let $g = f \mid C$. Since $g^{-1}(y)$ is compact, there is an m such that $g^{-1}(y) \subset U_m$. Now $g: C \to K$ is a closed map, and $U_m \cap C$ is a neighborhood of $g^{-1}(y)$ in C, so there is an $n \geq m$ for which $g^{-1}(V_n) \subset U_m$. Let $C_n = g^{-1}(K_n)$. Then C_n is compact, $f(C_n) = K_n$, and $C_n \subset g^{-1}(V_n) \subset U_m \subset U_n$. This contradicts the assumption that K_n is not the image of any compact subset of U_n, and that completes the proof.

We conclude this section with two remarks on Lemma 5.1.

(5.2)   Lemma 5.1 is analogous to a result of A. H. Stone [24, p. 694, Lemma 1]. In that result, Stone only assumes that f is a quotient map (which—since Y is Hausdorff and first-countable—is weaker than being compact-covering), and concludes only that $f(U_n) \supset V$. He cannot conclude that $f \mid X': X' \to V$ is again a quotient map, so his result, unlike Lemma 5.1, cannot be applied inductively. Stone's lemma was applied in [17, Proposition 3.3(d)] to show that certain quotient maps are bi-quotient, while Lemma 5.1 will be applied in Theorem 6.5(e) to show that certain compact-covering maps are tri-quotient. (See also footnote 17.)

(5.3)   Lemma 5.1 remains true (with essentially the same proof) if, in both the assumption and conclusion, "compact-covering" is weakened to: Every countable, compact subset of the range is the image of some compact (or even merely countably compact) subset of the domain. Consequently, "compact-covering" can also be weakened in this way in Theorem 6.5(e) and Corollary 1.5.

6. Tri-quotient maps

DEFINITION 6.1.   A map $f: X \to Y$ is *tri-quotient* if one can assign to each open U in X an open U^* in Y such that:

(a) $U^* \subset f(U)$.
(b) $X^* = Y$.
(c) $U \subset V$ implies $U^* \subset V^*$.
(d)   If $y \in U^*$ and $\mathscr{W}$ is a cover of $f^{-1}(y) \cap U$ by open subsets of X, then there is a finite $\mathscr{F} \subset \mathscr{W}$ such that $y \in (\bigcup \mathscr{F})^*$.

We call $U \to U^*$ a *tri-quotiency assignment*, or *t-assignment*, for f.

Recall that a map $f: X \to Y$ is *bi-quotient* [17] if, whenever $y \in Y$ and $\mathscr{W}$ is a cover of $f^{-1}(y)$ by open subsets of X, then there is a finite $\mathscr{F} \subset \mathscr{W}$ such that $y \in \mathrm{Int}\, f(\bigcup \mathscr{F})$. Clearly every tri-quotient map is bi-quotient, but Example 9.3 and Theorem 6.3 imply that the converse is false (even for maps between separable metric spaces).

LEMMA 6.2.   *Suppose* $f: X \to Y$ *is a tri-quotient map with t-assignment* $U \to U^*$. *If* $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ *is a finitely additive sieve (see Section 2) on* X, *then* $(\{U_\alpha^*: \alpha \in A_n\}, \pi_n)$ *is a sieve on* Y.

Proof. Let us first show that $\{U_\alpha^*: \alpha \in A_n\}$ covers Y for each n. Let $y \in Y$. Then $y \in X^*$ by 6.1(b). Also $\{U_\alpha: \alpha \in A_n\}$ covers $f^{-1}(y)$ and is preserved by finite unions, so $y \in U_\alpha^*$ for some $\alpha \in A_n$ by 6.1(d). Hence $Y = \bigcup_{\alpha \in A_n} U_\alpha^*$. In exactly the same way one shows that, if $\alpha \in A_n$, then $U_\alpha^* \subset \bigcup \{U_\alpha^*: \beta \in \pi_n^{-1}(\alpha)\}$; the reverse inclusion follows from 6.1(c). That completes the proof.

THEOREM 6.3. *Every tri-quotient image of a sieve-complete space is sieve-complete.*

Proof. This follows from Lemmas 6.2, 2.3, and 4.1.

It should be remarked that Theorem 6.3 is false for *bi*-quotient maps; see Example 9.3.

The following lemma is used in the proofs of Theorems 6.5(c) and 6.6. We call a map $f: X \to Y$ *inductively tri-quotient* if there is an $X' \subset X$ such that $f(X') = Y$ and $f \mid X'$ is tri-quotient.

LEMMA 6.4. *A map $f: X \to Y$ is tri-quotient if and only if it is inductively tri-quotient.*

Proof. To prove the nontrivial part, suppose that, for some $X' \subset X$, $f(X') = Y$, and $f \mid X'$ is tri-quotient. If $U \to U^*$ is a *t*-assignment for $f \mid X'$, then it is easily checked that $U \to (U \cap X')^*$ is a *t*-assignment for f. Hence f is tri-quotient, and the proof is complete.

THEOREM 6.5. *Let $f: X \to Y$ be a map. Then each of the following implies that f is tri-quotient.*

   (a) *f is open.*
   (b) *f is perfect.*
   (c) *f is closed, X is paracompact, and Y is first-countable.*[14]
   (d) *f is compact-covering, and Y is locally compact and Hausdorff.*
   (e) *f is compact-covering,[15] each $\partial f^{-1}(y)$ is Lindelöf, X is regular,[16] and Y is first-countable and Hausdorff.*[17]

Proof. For parts (a), (b), and (d), we shall merely describe *t*-assignments for f, omitting the routine verifications that they satisfy Definition 6.1.
   (a)   Let $U^* = f(U)$.
   (b)   Let $U^* = Y - f(X - U)$.
   (c)   By [15, Proof of Corollary 1.2], f is inductively perfect, so f is tri-quotient by (b) and Lemma 6.4.

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[14] It can be shown (with the aid of [19, Theorem 9.9 and Lemma 9.1]) that "first-countable" can be weakened to "countably bi-k" (see footnote 18).

[15] This can be slightly weakened. See Remark 5.3.

[16] The assumption that X is regular can be omitted if each $f^{-1}(y)$ is *hereditarily* Lindelöf.

[17] This result should be compared to [17, Proposition 3.3(a)], which asserts that every quotient map $f: X \to Y$, with each $f^{-1}(y)$ Lindelöf and Y first-countable Hausdorff, must be bi-quotient.

(d)   Let $U^* = \bigcup \{\text{Int } (f(C)): C \subset U, C \text{ compact}\}$.

(e)   Let $\mathscr{S}$ be the collection of all $S \subset X$ such that $f(S)$ is open in $Y, f \mid S$ (regarded as a map onto $f(S)$) is compact-covering, and $\partial_S (f \mid S)^{-1}(y)$ is Lindelöf for all $y \in Y$. For each open $U \subset X$, let

$$U^* = \bigcup \{f(S): S \in \mathscr{S}, S \subset U\}.$$

The only requirement of Definition 6.1 which is not obviously satisfied is 6.1(d). To check 6.1(d), suppose that $y \in U^*$ and that $\mathscr{W}$ is a cover of $f^{-1}(y) \cap U$ by open subsets of X. Pick $S \in \mathscr{S}$ such that $S \subset U$ and $y \in f(S)$, and let $g = f \mid S$. Then $g^{-1}(y)$ is covered by $\mathscr{W}$. Since X is regular and $\partial_S g^{-1}(y)$ is Lindelöf, $\partial_S g^{-1}(y)$ is covered by countably many open subsets $V_1, V_2, \ldots$ of X such that each $\overline{V}_i$ is a subset of some $W \in \mathscr{W}$ and $V_1 \cap g^{-1}(y) \neq \emptyset$. Let $U_n = \bigcup_{i=1}^{n} V_i$. Now apply Lemma 5.1 to g, obtaining an $X' \subset S$ such that $X' \subset U_n$ for some n, $f(X')$ is open in Y, $y \in f(X')$, and $f \mid X'$ is compact-covering. Let $S' = \overline{X}' \cap f^{-1}(f(X'))$. Then $S' \subset \bigcup \mathscr{F}$ for some finite $\mathscr{F} \subset \mathscr{W}$, $S' \in \mathscr{S}$, and $y \in f(S')$. Hence $y \in (\bigcup \mathscr{F})^*$, and that is all we had to show.

THEOREM 6.6.   *The following are equivalent for any map $f: X \to Y$ from a regular sieve-complete space X onto a paracompact space Y.*

(a)   *f is inductively perfect.*
(b)   *f is tri-quotient.*

Proof. (a) $\Rightarrow$ (b).   This follows (without any conditions on X and Y) from Theorem 6.5(b) and Lemma 6.4.

(b) $\Rightarrow$ (a).   Since X is regular, it has a strong, complete sieve

$$(\{U_\alpha: \alpha \in A_n\}, \pi_n)$$

by Lemma 2.4. Now define the finitely additive, complete sieve

$$(\{U_B: B \in \mathscr{B}_n\}, p_n)$$

on X as in the first paragraph of the proof of Lemma 2.3; it is clear that this is also a strong sieve.

Let $U \to U^*$ be a t-assignment for f. Then $(\{U_B^*: B \in \mathscr{B}_n\}, p_n)$ is a sieve on Y by Lemma 6.2. By Lemma 2.2, there is a locally finite sieve

$$(\{W_B: B \in \mathscr{B}_n\}, p_n)$$

on Y such that $\overline{W}_B \subset U_B^*$ for all $B \in \mathscr{B}_n$ and all n.

Fix $y \in Y$. For each n, let $B_n(y) = \bigcup \{B \in \mathscr{B}_n: y \in \overline{W}_B\}$. Then $B_n(y) \in \mathscr{B}_n$ for all n, and $(B_n(y))$ is a p-chain. Also $f^{-1}(y) \cap U_{B_n(y)} \neq \emptyset$ for all n, for if $B \in \mathscr{B}_n$ is chosen so that $y \in W_B$, then $B \subset B_n(y)$, hence $U_B \subset U_{B_n(y)}$, and therefore

$$y \in W_B \subset U_B^* \subset f(U_B) \subset f(U_{B_n(y)}).$$

Let

$$C_y = \bigcap_{n=1}^{\infty} (f^{-1}(y) \cap U_{B_n(y)}) = \bigcap_{n=1}^{\infty} (f^{-1}(y) \cap \overline{U}_{B_n(y)}).$$

Since $(\{U_B\colon B \in \mathscr{B}_n\}, p_n)$ is complete, it is easy to see (as in Lemma 2.5) that C_y is nonempty and compact.

Define $X' = \bigcup\{C_y\colon y \in Y\}$. Clearly $f(X') = Y$. Let $g = f \mid X'$. Then $g^{-1}(y) = C_y$ is compact for all $y \in Y$. To show that g is closed, suppose $A \subset X'$ is closed and $y \in g(A))^-$, and let us show that $y \in g(A)$.

Let $\mathscr{V}(y)$ be the collection of all neighborhoods of y in Y, and let

$$\mathscr{F} = \{A \cap g^{-1}(V)\colon V \in \mathscr{V}(y)\}.$$

Then $\mathscr{F}$ is a filter base on X. For each n, let

$$V_n = Y - \bigcup\{\overline{W_B}\colon B \in \mathscr{B}_n, y \notin \overline{W_B}\}.$$

Then $V_n \in \mathscr{V}(y)$. If $y' \in V_n$, then $B_n(y') \subset B_n(y)$, so $g^{-1}(y') \subset U_{B_n(y')} \subset U_{B_n(y)}$. Hence $g^{-1}(V_n) \subset U_{B_n(y)}$, so $U_{B_n(y)}$ contains an element of $\mathscr{F}$. By completeness, $\mathscr{F}$ clusters at some $x \in X$. But then $x \in A$; also $g(x) \in \overline{V}$ for every $V \in \mathscr{V}(y)$, so $g(x) = y$. Hence $y \in g(A)$.

That completes the proof.

We conclude this section with a generalization of Corollary 1.5 whose proof uses several of the preceding results.

THEOREM 6.7.   *Suppose $f\colon X \to Y$ is a compact-covering s-map, with X completely metrizable and Y paracompact. Then the following are equivalent.*

(a)   *Y is completely metrizable.*

(b)   *Y is metrizable.*

(c)   *Y is a countably bi-k-space.*[18]

(d)   *f is bi-quotient.*

(e)   *f is tri-quotient.*

(f)   *f is inductively perfect.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c).   Clear.

(c) $\Rightarrow$ (d).   Since f is compact-covering and Y is a Hausdorff k-space, f is a quotient map by [16, Lemma 11.2]. By [19, Theorem 9.8 and Lemma 9.1], a quotient s-map from a metrizable space onto a Hausdorff countably bi-k-space is bi-quotient.

(d) $\Rightarrow$ (e).   It is easily checked that a bi-quotient s-image of a metrizable space (more generally, of a space with point-countable base) is first-countable.[19] Hence f is tri-quotient by Theorem 6.5(e).

(e) $\Rightarrow$ (f).   By Theorem 6.6.

(f) $\Rightarrow$ (a).   This is true because perfect images preserve metrizability [22], [24] and Čech-completeness [25], [8, p. 167, Y].

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[18] Countably bi-k-spaces, which were introduced in [19, Definition 4.E.1], include all first-countable spaces, all locally compact spaces, and (by (8.11) in Section 8) all regular sieve-complete spaces. Every Hausdorff countably bi-k-space is a k-space.

[19] V. V. Filippov [10] has shown that such an image actually has a point-countable base, but we do not need that deeper result here.

7. Operations preserving tri•quotient maps

In this section, it is shown that triquotient maps are preserved by composition and by binary—hence finite—products. I don't know whether they are preserved by infinite products.[20]

THEOREM 7.1. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are maps.*

(a) *If f and g are tri-quotient, so is $g \circ f$.*
(b) *If $g \circ f$ is tri-quotient, so is g.*

Proof. (a) Let $h = g \circ f$. Let $U \to U^f$ and $V \to V^g$ be t-assignments for f and g, respectively. For each open U in X, define $U^h = (U^f)^g$, and let us show that $U \to U^h$ is a t-assignment for h.

It will suffice to verify condition 6.1(d). Suppose, therefore, that $z \in U^h$, and that $\mathscr{W}$ is a cover of $h^{-1}(z) \cap U$ by open subsets of X. Let $E = g^{-1}(z) \cap U^f$. If $y \in E$, then $y \in U^f$ and $\mathscr{W}$ covers $f^{-1}(y) \cap U$, so $y \in (\bigcup \mathscr{F}_y)^f$ for some finite $\mathscr{F}_y \subset \mathscr{W}$. Let $V_y = (\bigcup \mathscr{F}_y)^f$. Then $\{V_y \colon y \in E\}$ is a cover of $E = g^{-1}(z) \cap U^f$ by open subsets of Y, and $z \in (U^f)^g$, so there is a finite $F \subset E$ such that $z \in (\bigcup \{V_y \colon y \in F\})^g$. Let $\mathscr{F} = \bigcup \{\mathscr{F}_y \colon y \in F\}$. Then $\mathscr{F} \subset \mathscr{W}$, $\mathscr{F}$ is finite and (by 6.1(c) for f)

$$z \in (\bigcup \{(\bigcup \mathscr{F}_y)^f \colon y \in F\})^g \subset ((\bigcup \mathscr{F})^f)^g = (\bigcup \mathscr{F})^h.$$

That establishes 6.1(d).

(b) Let $h = g \circ f$. Suppose $U \to U^h$ is a t-assignment for h, and let us show that $V \to (f^{-1}(V))^h$ must be a t-assignment for g.

We need only check 6.1(d). So suppose that $z \in V^g$ and that $\mathscr{W}$ is a cover of $g^{-1}(z) \cap V$ by open subsets of Y. Then $z \in (f^{-1}(V))^h$ and $f^{-1}(\mathscr{W})$ is a cover of $h^{-1}(z)$ by open subsets of X, so there is a finite $\mathscr{F} \subset \mathscr{W}$ such that $z \in (\bigcup f^{-1}(\mathscr{F}))^h$. But then

$$z \in (f^{-1}(\bigcup \mathscr{F}))^h = (\bigcup \mathscr{F})^g,$$

and that completes the proof.

It should be remarked that, in Theorem 7.1(b), nothing can be concluded about f. (For example, f could be arbitrary and g a constant map.)

THEOREM 7.2. *If $f\colon X \to Y$ and $g\colon X' \to Y'$ are tri-quotient, so is*

$$f \times g\colon X \times X' \to Y \times Y'.$$

Proof. Let $h = f \times g$. Let $U \to U^f$ and $U' \to (U')^g$ be t-assignments for f and g, respectively, and let us define a t-assignment $W \to W^h$ for h.

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[20] *Bi*-quotient maps are preserved by arbitrary products, as are open maps, perfect maps, and thus also inductively perfect maps. This last result and Theorem 6.6 imply that any product of tri-quotient maps $f_\alpha\colon X_\alpha \to Y_\alpha$, with each X_α regular sieve-complete and each Y_α paracompact, must be tri-quotient.

Let Φ be the family of all finite collections $\mathscr{F}$ of open rectangles in $X \times X'$. For each $\mathscr{F} = \{U_j \times U_j'\}_{j=1}^n$ in Φ, define the open set $G(\mathscr{F})$ in $Y \times Y'$ by

$$G(\mathscr{F}) = \left(\bigcup_{j=1}^n U_j \right)^f \times \bigcap_{j=1}^n (U_j')^g.$$

For all open W in $X \times X'$, let

(*) $W^h = \bigcup \{G(\mathscr{F}): \mathscr{F} \in \Phi, \bigcup \mathscr{F} \subset W\}.$

It will now suffice to verify condition 6.1(d).

Suppose $(y, y') \in W^h$, and $\mathcal{O} = \{S_\alpha \times S_\alpha': \alpha \in A\}$ is a cover of $h^{-1}(y, y') \cap W$ by basic open subsets of $X \times X'$. Choose $\mathscr{F} = \{U_j \times U_j'\}_{j=1}^n$ in Φ such that $\bigcup \mathscr{F} \subset W$ and $(y, y') \in G(\mathscr{F})$. Then $\mathcal{O}$ also covers $h^{-1}(y, y') \cap (\bigcup \mathscr{F})$. Note that $h^{-1}(y, y') = f^{-1}(y) \times g^{-1}(y')$.

Let $U = \bigcup_{j=1}^n U_j$. Fix $x \in f^{-1}(y) \cap U$. Pick $j(x) \le n$ so that $x \in U_{j(x)}$, and let $A(x) = \{\alpha \in A: x \in S_\alpha\}$. Then $\{S_\alpha': \alpha \in A(x)\}$ is a cover of $g^{-1}(y) \cap U_{j(x)}'$ by open subsets of X', and $y' \in (U_{j(x)}')^g$, so there is a finite $F(x) \subset A(x)$ such that $y' \in (V'(x))^g$ where $V'(x) = \bigcup \{S_\alpha': \alpha \in F(x)\}$. Let $V(x) = \bigcap \{S_\alpha: \alpha \in F(x)\}$; then $V(x)$ is an open neighborhood of x in X.

Now $\{V(x): x \in f^{-1}(y) \cap U\}$ is a cover of $f^{-1}(y) \cap U$ by open subsets of X, and $y \in U^f$, so there is a finite subset $\{x_1, \ldots, x_m\}$ of $f^{-1}(y) \cap U$ such that $y \in (\bigcup_{i=1}^m V(x_i))^f$. Let $F = \bigcup_{i=1}^m F(x_i)$, let $\mathscr{P} = \{S_\alpha \times S_\alpha': \alpha \in F\}$, and let $P = \bigcup \mathscr{P}$. Then $\mathscr{P}$ is a finite subcollection of $\mathcal{O}$, and we will complete the proof by showing that $(y, y') \in P^h$.

Let $\mathscr{E} = \{V(x_i) \times V'(x_i)\}_{i=1}^m$. Then $\mathscr{E} \in \Phi$ and $(y, y') \in G(\mathscr{E})$. Now

$$V(x_i) \times V'(x_i) \subset \bigcup \{S_\alpha \times S_\alpha': \alpha \in F(x_i)\}$$

for all $i \le m$, so $\bigcup \mathscr{E} \subset P$. Hence $(y, y') \in P^h$ by (*), and that completes the proof.

8. Miscellaneous results and remarks on sieve-complete spaces

(8.1) [5, Theorem 3.7]. A regular space is sieve-complete if and only if it is an open image of a paracompact Čech-complete space.

(8.2). As observed in [5], the "only if" part of (8.1) follows from the analogous result for λ_b-spaces obtained by H. H. Wicke in [27, Theorem 4.5], together with the equivalence, in regular spaces, of sieve-completeness and the λ_b condition (see footnote 4). (It can also be proved directly.) The "if" part of (8.1) follows from Proposition 4.2.

(8.3). As indicated in [5, Corollary 3.8], Theorem 3.2 follows from (8.1) and the theorem of Pasynkov stated in our Theorem 1.1.

(8.4). Analogously to (8.1), one can prove: A topological space is sieve-complete if and only if it is an open image of a space satisfying condition 3.1(a).

(8.5). A countable product of sieve-complete spaces is sieve-complete. (Outline of proof: Let $X = \prod_{i=1}^\infty X_i$ be such a product. For each i, let

$(\{U_\alpha: \alpha \in A_{n,\,i}\}, \pi_{n,\,i})_{n \in N}$ be a complete sieve on X_i. Let $A_n = \prod_{i=1}^{n} A_{n,\,i}$. Define $\pi_n: A_{n+1} \to A_n$ by

$$\pi_n(\alpha_1, \dots, \alpha_{n+1}) = (\pi_{n,\,1}(\alpha_1), \dots, \pi_{n,\,n}(\alpha_n)).$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in A_n$, define $U_\alpha \subset X$ by

$$U_\alpha = \left(\prod_{i=1}^{n} U_{\alpha_i}\right) \times \left(\prod_{i=n+1}^{\infty} X_i\right).$$

One can now show (using the first part of footnote 11) that $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ is a complete sieve on X.)

(8.6) (E. van Douwen). Remark (8.5) can be extended as follows: A non-empty product $X = \prod_{\lambda \in \Lambda} X_\lambda$ is sieve-complete if and only if every X_λ is sieve-complete and X_λ is compact for all but countably many λ. (Proof of "only if": Suppose X is sieve-complete, with complete sieve $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$. Then each X_λ is sieve-complete by Proposition 4.2. Now let (α_n) be any π-chain such that $U_{\alpha_n} \neq \emptyset$ for all n. Then there is a countable $\Lambda' \subset \Lambda$ such that $p_\lambda(U_{\alpha_n}) = X_\lambda$ for all $\lambda \notin \Lambda'$ and all n (where $p_\lambda: X \to X_\lambda$ is the projection). But then X_λ is compact if $\lambda \notin \Lambda'$, for if $\mathscr{F}$ is a filter base in X_λ for such a λ, then $p_\lambda^{-1}(\mathscr{F})$ meshes with $\{U_{\alpha_n}: n \in N\}$, hence $p_\lambda^{-1}(\mathscr{F})$ clusters in X, so $\mathscr{F}$ clusters in X_λ.)

(8.7) [5, Theorem 3.4 and Proposition 3.6]. If $f: X \to Y$ is a perfect map, then X is sieve-complete if and only if Y is sieve-complete. (Proof: For "only if," this follows from Theorems 6.3 and 6.5(b) For "if," it is easy to check that, if $(\{U_\alpha: \alpha \in A_n\}, \pi_n)$ is a complete sieve on Y, then $(f^{-1}(U_\alpha): \alpha \in A_n\}, \pi_n)$ is a complete sieve on X.)

(8.8). Sieve-completeness is inherited by closed subsets (trivially), and, in regular spaces, by G_δ-subsets (less trivially; see [5, Proposition 2.13]). This latter result, once established, immediately implies Theorem 3.2(a) $\to$ (b).

(8.9) [5, Proposition 2.2]. A space X is sieve-complete if and only if every point has a sieve-complete open neighborhood. (Use 4.2.) If X is regular, the neighborhoods need not be open. (Use 8.8.)

(8.10). Every regular, sieve-complete space is a Baire space (i.e., the intersection of countably many dense open sets is dense).

(8.11) [5, Proposition 4.4]. Every regular, sieve-complete space X is of countable type (i.e., every compact $C \subset X$ is contained in a compact $K \subset X$ of countable character in X; see [3, Definition 3.7]).

(8.12). Analogously to a complete sieve, one can define a *countably complete sieve* by restricting the filter base $\mathscr{F}$ in Definition 2.1 to be countable (equivalently, by requiring that, if (α_n) is a π-chain and $x_n \in U_{\alpha_n}$ for all n, then the sequence (x_n) clusters in X). A space with a countably complete sieve is called *countably sieve-complete.*[21] Results 2.3–2.4, 4.1–4.2, 6.3, and 8.7–8.10 in this

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[21] In regular spaces, this is equivalent to being a λ_c-space in the sense of H. H. Wicke [28]. Compare footnote 4.

paper remain valid, with the same proofs, for countably complete sieves and countably sieve-complete spaces. Lemma 2.5 also remains valid, provided "compact" is changed to "countably compact." This last result implies that a paracompact space is countably sieve-complete if and only if it is sieve-complete, and hence that Theorem 3.2 remains valid with "sieve-complete" weakened to "countably sieve-complete." Every countably compact space is countably sieve-complete, but not necessarily sieve-complete.

9. Examples

Our first example explains the significance of the paracompactness assumption in several of our theorems. (That Theorem 1.1 is false without this assumption also follows from earlier examples of M. E. Estill (= M. E. Rudin) [9] (see also [1, 3.2.3]), and of H. H. Wicke [26] (where Y is collectionwise normal).)

Example 9.1. An open, compact-covering map $f: X \to Y$, with each $f^{-1}(y)$ finite, X completely metrizable, and Y a metacompact, completely regular space which is not Čech-complete.

Proof. In [5], Example 2.9 constructs a metacompact, completely regular space Y which is locally completely metrizable but not Čech-complete. Let $\mathscr{V}$ be a point-finite open cover of Y by completely metrizable subsets. Let X be the topological sum $\sum \{V: V \in \mathscr{V}\}$, and let $f: X \to Y$ be the natural map. Clearly X is completely metrizable, and each $f^{-1}(y)$ is finite. Moreover, any map obtained in this way from an open cover of a Hausdorff space Y is easily seen to be compact-covering. Hence $f: X \to Y$ has all the required properties.

Our next example explains the significance of the first-countability assumption in Theorem 6.5(c) and (e) (and thus also in Lemma 5.1), and of the assumption in Theorem 6.7(c) that Y be countably bi-k. I am grateful to E. van Douwen for calling this example to my attention.

Example 9.2. A closed, compact-covering map $f: X \to Y$, with X complete separable metric and Y paracompact but not Čech-complete.

Proof. Such an example is given in [1, Example 2.4.2.]

The following example shows that Theorem 6.3 is false for bi-quotient maps, and hence that not all bi-quotient maps are tri-quotient.

Example 9.3. The rationals (which are not completely metrizable) are a bi-quotient image of the irrationals (which are completely metrizable).

Proof. This was proved in [21, Corollary 1.2 and observation on p. 631].

Example 9.4. A nonmetrizable, first-countable, paracompact space Y which is a bi-quotient s-image of a completely metrizable space X.

Proof. Let Y be the reals with the irrationals made discrete (see [14, p. 375]). As observed in [14], this Y is (hereditarily) paracompact. Clearly Y is first-countable, but Y is not metrizable since the rationals are a closed, non-G_δ subset. That Y is a bi-quotient s-image of a completely metrizable space was proved in [20, Remarks 2 and 3]. That completes the proof.

The previous two examples help to explain the significance of the compact-covering assumption in Theorem 6.7: Without this assumption, Example 9.4 shows that (c) and (d) need not imply (b), while Example 9.3 shows that (b) need not imply (a). It should be remarked that the space X in Example 9.4 cannot be chosen separable, since every bi-quotient image of a second-countable space is second-countable [17, Proposition 3.4].

Our next example explains the need for the requirement in Corollary 1.5 and Theorem 6.7 that f be an s-map, and for the requirement in Theorem 6.5 that Y be locally compact.

Example 9.5. (a) If Y is a metrizable space, then Y is a compact-covering image of a completely metrizable space X.

(b) If Y is the rationals, then Y is a compact-covering, bi-quotient image of a completely metrizable space X.

Proof. (a) Let X be the topological sum of the compact subsets of Y, with the obvious map $f: X \to Y$.

(b) By (a) and Example 9.3, there are maps $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$, with X_1 and X_2 completely metrizable, such that f_1 is compact-covering and f_2 is bi-quotient. Then the sum $f_1 \oplus f_2: X_1 \oplus X_2 \to Y$ is compact-covering and bi-quotient. That completes the proof.

Our last two examples show why the assumption that X is sieve-complete cannot be omitted from Theorem 6.6(b) $\to$ (a), even when f is open (and hence tri-quotient by Theorem 6.5(a)).

Example 9.6. An open map $f: X \to Y$, with X and Y separable metric, which is not compact-covering (and hence not inductively perfect).

Proof. Such a map was constructed in [13, Example 4.1].

Example 9.7 (E. van Douwen). An open map $f: X \to Y$, with X and Y separable metric, which is compact-covering but not inductively perfect.

Proof. Let Y be a subset of $I = [0, 1]$ such that card $Y = c$ but every compact subset of Y is countable. Let $P = Y \times I$, and let $\pi: P \to Y$ be the projection onto the first coordinate. Let $\{K_\alpha: \alpha < c\}$ enumerate all compact subsets of Y, and let $\{F_\alpha: \alpha < c\}$ enumerate all those closed subsets of P which π maps onto Y.

Since each K_α is countable, we can inductively pick $y_\alpha \in Y$ for $\alpha < c$ such that $y_\alpha \notin \bigcup_{\beta < \alpha} (K_\beta \cup \{y_\beta\})$, and then pick $t_\alpha \in I$ such that $(y_\alpha, t_\alpha) \in F_\alpha$. Let $X = P - \{(y_\alpha, t_\alpha): \alpha < c\}$, and let $f = \pi \mid X$.

Our construction implies that the y_α are distinct, so X is obtained from P by removing at most one point from each $\pi^{-1}(y)$. Hence f is open.

To see that f is compact-covering, let $K \subset Y$ be compact. Then $K = K_\alpha$ for some $\alpha < c$. Pick $t \in I - \{t_\beta: \beta \leq \alpha\}$. Then $K \times \{t\}$ is a compact subset of X and $f(K \times \{t\}) = K$.

It remains to show that f is not inductively perfect: Suppose there were an $X' \subset X$ such that $f(X') = Y$ and $f \mid X'$ is perfect. Since $\pi: P \to Y$ is a continuous extension of $f \mid X'$, the set X' must be closed in P (see, for instance, [18, Corollary 1.5]), so $X' = F_\alpha$ for some α. But that is impossible, since, by our construction, $F_\alpha \not\subset X$ for all α. The proof is now complete.

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