ON FINITE GROUPS IN WHICH THE GENERALIZED FITTING GROUP OF THE CENTRALIZER OF SOME INVOLUTION IS EXTRASPECIAL

BY

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1. Introduction

A finite group G is said to be of *characteristic* 2 *type* if $F^*(C_G(t))$ is a 2-group for each involution t in G. It seems probable that in the near future the problem of determining the finite simple groups will be reduced to determining the simple groups of characteristic 2 type. The principal model for investigation of the characteristic 2 type groups is Thompson's work on N-groups. There Thompson argues on abelian normal subgroups of 2-locals. As an extreme case, he must consider the situation where, for some maximal 2-local M, abelian normal subgroups of M have order at most 2. Hence $Z(M) = \langle z \rangle$ is of order 2, $M = C_G(z)$, and $F^*(M)$ is an extraspecial 2-group. Since many of the sporadic simple groups possess such centralizers, it seems likely that this will be a troublesome case in most suitably general classification problems.

Thompson's analysis of this situation may be divided into two sections. In Lemma 13.63 he proves that z is weakly closed in $F^*(M)$. The remainder of Section 13 is then devoted to eliminating this case The following theorem supplies this latter analysis in general.

THEOREM. *Let G be a finite group and z an involution in G such that $F^*(C_G(z)) = Q$ is an extraspecial 2-group of width at least 2. Then one of the following hold:*

(1) $z \in Z(G)$.

(2) $\langle z^G \rangle = F^*(G)$ *is isomorphic to $U_m(2)$, the m-dimensional unitary group over $GF(2)$, or the second Conway group Co_2.*

(3) $z \notin O_2(C_G(t))$ *for some involution $t \in Q$. In particular $F^*(C_G(t))$ is not a 2-group, so G is not of characteristic 2 type.*

(4) *z is fused in G to some noncentral involution of Q.*

The proof depends upon work of B. Fischer and F. Timmesfeld on groups generated by $\{3, 4\}$-transpositions. See [6] or [1] for notation and terminology. Certain results in Sections 4 and 5, in particular Lemma 5.12, may be of independent interest. Co_2 is identified using a result of F. Smith [14].

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2. A preliminary lemma

(2.1) *Let $L \cong L_2(2^n)$ or $Sz(2^n)$, $n \geq 2$, and let V be an irreducible, L, $GF(2)$-module. Let $T \in Syl_2(L)$ and $S = \Omega_1(T)$. Assume $m([V, S]) \leq n + 1$. Then $L \cong L_2(2^n)$ and either*

(1) *V is the natural module for L, or*
(2) *$n = 2$ and V is the natural module for $O_4^-(2)$.*

Proof. Let $t \in S^\#$ and $m = m([V, t])$. If $L \cong Sz(2^n)$ then t inverts an element x of prime order p where p divides $2^{4n} - 1$ but not $2^i - 1$ for $i < 4n$. Thus $m([V, x]) \geq 4n$, so

$$m \geq m([V, x, t]) = m([V, x])/2 \geq 2n.$$

Thus $n + 1 \geq m([V, S]) \geq m \geq 2n$, a contradiction.

So $L \cong L_2(2^n)$. Let $F = GF(2^n)$, $A = \mathrm{Aut}\,(F)$, and M the natural L, F-module. Then

$$V \otimes F = \bigoplus_{a \in A} N^a$$

for some irreducible L, F-module N. Moreover $N = \otimes_{b \in B} M^b$ for some $B \subseteq A$. This allows us to determine m. In particular as $m \leq n + 1$ we conclude either V is the natural module for L or n is even and V is the natural module for $O_4^-(2^{n/2})$, or $n = 8$ and V is induced by the permutation module for L on 9 letters. In the last case $m(V) = 8$ and $[V, S]$ is a hyperplane of V, contrary to hypothesis. In the second case $[V, S]$ is of codimension $n/2$, so $n = 2$.

(2.2) *Let Q be a 2-subgroup of $G = \langle Q^G \rangle$ with $Q = U \otimes P$ where $U \leq Z(G)$, P is extraspecial, $\Phi(Q) = \Phi(P) = \langle z \rangle$, and $Q \trianglelefteq M = C_G(z)$. Assume M contains a Sylow 2-subgroup of G, $z \in O_2(G)$, and M is the unique maximal subgroup of G containing Q. Set $V = \langle z^G \rangle$ and let K be the largest normal subgroup of G contained in M. Then either*

(1) *$|V : C_V(G)| = 4$ and $G/K \cong S_3$, or*
(2) *$G/K \cong A_5$ and $V/C_V(G)$ is the natural module for $O_4^-(2)$.*

Proof. Set $\bar{G} = G/K$. As $z \in O_2(G)$ and M contains a Sylow 2-subgroup of G, we conclude V is abelian, $K = C_G(V)$, and $O_2(\bar{G}) = 1$. As $\langle z \rangle = \Phi(Q) \leq K$, $\bar{Q}$ is elementary abelian. Hence as M is the unique maximal subgroup of G containing Q, and as $Q \trianglelefteq M$, $\bar{M} = N_{\bar{G}}(\bar{Q}) \geq C_{\bar{G}}(\bar{x})$ for each $\bar{x} \in \bar{Q}^\#$. Also if $1 \neq \bar{Q} \cap \bar{Q}^g$ then $\bar{Q}^g \leq C(\bar{Q} \cap \bar{Q}^g) \leq \bar{M}$, so by uniqueness of M, $M = M^g$ and $g \in M$. Hence $\bar{Q}$ is a TI-set in $\bar{G}$. As $\bar{M}$ is the unique maximal subgroup containing $\bar{Q}$ and $O_2(\bar{G}) = 1$, $\bar{Q}$ is strongly closed in $\bar{G}$ (e.g., 2.14 in [16]). Hence by 3.3 in [1], $\bar{M}$ is strongly embedded in $\bar{G}$. Therefore by Bender's Theorem and uniqueness of M, $\bar{G} \cong L_2(2^n)$, $Sz(2^n)$, or D_{2p}, for some odd prime p.

Let $R = Q \cap K$ and $Z = Z(R)$. Then $R = U \otimes (P \cap R)$ and $U(Q \cap V) \leq Z$. Let $n = m(\bar{G})$, $k = m(U)$, and $m + k = m(Z)$. As P is extraspecial, $m \leq m(P/R \cap P) + 1 = n + 1$. Set $\tilde{V} = V/C_V(G)$. $[V, Q] \leq V \cap Q \leq Z$, so $m([\tilde{V}, Q]) \leq m \leq n + 1$. Moreover $m(Z(Q)) = k + 1$ so

$$m(C(Q) \cap [V, Q]U) \leq k + 1.$$

Suppose $\bar{M} \cong D_{2p}$. Let $x \in Q - K$ invert y with $\bar{y}$ of order p. Then

$$UV = C_{UV}(y) \otimes [V, y] \quad \text{and} \quad m([V, Q]) = m([V, y, x]) = m([V, y])/2 = i/2.$$

Also $[V, Q] \leq Z(Q)$. Hence $k + 1 \geq m(C(Q) \cap [V, Q]U) = k + i/2$, so that $i = 2$, $p = 3$, and $|V/C_V(G)| = 4$.

So assume $\bar{G} \cong L_2(2^n)$ or $Sz(2^n)$, $n \geq 2$. Let $\tilde{W}$ be an irreducible submodule of $\tilde{V}$. Then $m([\tilde{W}, Q]) \leq m([\tilde{V}, Q]) \leq n + 1$, so by 2.1, $\bar{G} \cong L_2(2^n)$ and either $\tilde{W}$ is the natural module for $\tilde{\bar{G}}$ or $n = 2$ and $\tilde{W}$ is the natural module for $O_4^-(2)$. Let $x \in Q - K$. Then x inverts an element y of odd order with $W = C_V(G) \otimes [W, y]$, so $C_W(x) = C_V(G) \otimes [W, y, x]$ and hence $C_{\tilde{W}}(x) = C_W(x)/C_V(G)$. In particular Q centralizes an element w in $[W, Q] - U$, so $Z(Q) = U\langle z \rangle = U\langle w \rangle \leq UW$ and hence $V = \langle z^G \rangle \leq W$.

Suppose $\tilde{V}$ is the natural module for $L_2(2^n)$. Then

$$U[V, Q] \leq C(Q) \quad \text{with} \quad m(U[V, Q]) \geq k + n > k + 1,$$

a contradiction. So $n = 2$ and $\tilde{V}$ is the natural module for $O_4^-(2)$. This completes the proof of 2.2.

3. $\{3, 4\}^+$-transpositions

Let G be a finite group. A *set of* $\{3, 4\}^+$*-transpositions* is a G-invariant collection D of involutions such that D generates G and for each a, b in D, ab has order at most 4, and $[a, b] \in D$ when ab has order exactly 4. Groups generated by $\{3, 4\}^+$-transpositions have been classified by Timmesfeld. We record some of his results.

(3.1) *Assume $O_2(G) = 1$. Then D is partitioned into subsets D_i, $1 \leq i \leq r$, such that:*

(1) $[D_i, D_j] = 1$ *for* $i \neq j$.

(2) $\langle D_i \rangle$ *is transitive on* D_i.

(3) *If $a, b \in D$ with $ab \in D$ then a is conjugate to b in G, and hence lie in the same block D_i.*

Proof. See 4.1.5 and 4.1.6 in [13].

(3.2) *Assume G is transitive on D, $O_2(G) = 1$, and all numbers from 1 through 4 occur as an order of a product of two elements of D. Then $G/Z(G)$ is isomorphic to one of the following groups:*

(1) $L_n(2)$, $n \geq 3$

(2) $Sp_n(2)$, $n \geq 6$
(3) $\Omega_n^\varepsilon(2)$, $n \geq 8$
(4) $U_3(3)$
(5) ${}^3D_4(2)$
(6) $F_4(2)$
(7) ${}^2E_6(2)$
(8) $E_6(2)$
(9) $E_7(2)$
(10) $E_8(2)$

Moreover D is a class of root involutions of G.

Proof. This is the main theorem of [12].

A set D of $\{3, 4\}^+$-transpositions of G is a *set of 3-transpositions* of G if ab has order at most 3 for each a, b in D.

(3.3) *Assume D is a conjugacy class of 3-transpositions of G and $L = F^*(G) = E(G)$. Then $L/Z(L)$ is one of the following:*

(1) *A_n, and D is the set of transpositions.*
(2) *$U_n(2)$, $Sp_n(2)$, or $\Omega_n(2)$, and D is the set of transvections.*
(3) *$\Omega_n(3)$, and D is a set of reflections.*
(4) *F_n, and D is uniquely determined.*

Proof. This is the main theorem of [5].

(3.4) *Assume $G \trianglelefteq H$, $F^*(H) = L = E(H)$, G is transitive on D, and $O_2(H) = 1$. Let $a \in D$ and assume $P \trianglelefteq C_H(a)$ with P extraspecial. Then one of the following hold:*

(1) *$P \cap D \nsubseteq \{a\}$ and either $P = O_2(C_L(a))$ and $L/Z(L) \cong L_n(2)$, $\Omega_n^\varepsilon(2)$, ${}^3D_4(2)$, ${}^2E_6(2)$, or $E_n(2)$, or $H \cong G_2(2)$ and $P = O_2(C_H(a))$.*
(2) *$G/Z(G) \cong U_n(2)$ and $F^*(C_H(a)) = P$ is of width $n - 2$.*
(3) *$L \cong U_3(3)$ and $P \cong Q_8$.*
(4) *$L \cong L_3(2)$ and $P \cong D_8$.*
(5) *$H \cong Sp_6(2)$ and $P \cong Q_8 * Q_8$.*

Proof. Without loss we take L simple and $H \leq \mathrm{Aut}\,(L) = A$. Assume first $a = [b, c]$ for some $b, c \in D$. Then by 1.2.2 in [12], $\langle b, c\rangle \leq O_2(C_L(a)) = Q$. Moreover L is described in 3.2 and D is a class of root involutions.

Now $C_A(a)$ is described in [3]. By inspection we find either (i) Q is the unique extraspecial normal subgroup of $C_H(a)$, or (ii) $L \cong F_4(2)$ or $Sp_n(2)$ and $C_H(a)$ has no normal extraspecial subgroup, or (iii) (3) or (4) holds, or (iv) $H = \mathrm{Aut}\,(U_3(3)) = G_2(2)$ and $P = O_2(C_H(a))$, or (5) holds. Therefore the lemma holds in this case.

So assume D is a set of 3-transpositions of G. Then the pair L, D is described in 3.3. $C_A(a)$ is described in [3] and [5]. By inspection $O_2(C_H(a))$ is abelian unless $G = L \cong U_n(2)$, in which case Q is the unique nonabelian normal 2-subgroup of $C_H(a)$ and is extraspecial of width $n - 2$. The proof is complete.

Let V be an n-dimensional orthogonal space over $GF(2)$ with bilinear form $(\ , \ )$ and quadratic form f of sign ε. Let $H = \mathrm{Aut}\,(V) \cong O_n^\varepsilon(2)$. Recall $v \in V^\#$ is *singular* if $f(v) = 0$, and *nonsingular* otherwise. $U \leq V$ is *totally singular* if each point of U is singular.

(3.5) *Let t be an involution in H, let $m = m([V, t])$, and let*

$$V(t) = \{v \in V: (v, v^t) = 0\}.$$

Then:

(1) $m \leq n/2$.

(2) $m(V/V(t)) \leq 1$ *with equality if m is odd.*

(3) *t is fused to an involution s in H if and only if $m = m([V, s])$ and $m(V/V(t)) = m(V/V(s))$.*

(4) $[V, t]^\perp = C_V(t)$.

Proof. See Sections 7 and 8 in [3].

If t is an involution in H we say t is of *type* a_m if $m([V, t]) = m$ and $V(t) = V$. t is of *type* b_m or c_m if $m([V, t]) = m$, $m(V/V(t)) = 1$, and m is odd or even, respectively. By 3.5 the type of t determines its conjugacy class in H.

(3.6) *Let t be an involution in H and $U = [V, t]$. Then:*

(1) *If t is of type a_m then U is totally singular, and there exists $W \leq V$ with*

$$W = \bigoplus_{i=1}^{m/2} (W_i \oplus W_i^t)$$

where W_i is of dimension 2 and sign $-$, and $U = [W, t]$.

(2) *If t is of type b_m or c_m then $U = \langle v_0 \rangle \oplus U_0$ where $\langle v_0 \rangle$ is a nonsingular point and U_0 is totally singular.*

Proof. See Sections 7 and 8 in [3].

(3.7) *Let a be an involution of type a_2 in H, $D = a^H$, and $G = \langle D \rangle$. Then:*

(1) *D is a conjugacy class of $\{3, 4\}^+$-transpositions of G.*

(2) *Let $X \leq C_H(a)$ with $[C_V(a), X] \leq \langle u \rangle$ for some $u \in [V, a]$. Then X is abelian.*

(3) *If $b \in D$ with $ab \in D$ then $[V, a, b] = 0$ and $C_V(\langle a, b \rangle) = C_V(a) \cap V$, $v \in [V, b]$.*

(4) *If $b \in D$ with $\langle a, b \rangle \cong S_3$ then $[V, ab] = W_1 \oplus W_2$ with W_i of dimension 2 and sign $-$ and $[V, a] \cap [V, b] = 0$.*

Proof. See 11.9 in [3] for parts (1), (3) and (4). Part (2) follows from an easy calculation using the information in Sections 7 and 8 of [3].

(3.8) *Let* $G = U_n(2)$, $n \geq 4$, *D the class of transvections in G, t an involution in* Aut (G), *and* $W = \langle C_D(t) \rangle$. *Assume* $W \leq O_2(C_G(t))$. *Then:*

(1) $t \in W$ *and t is determined up to conjugacy in G.*

(2) *If* $x \in N_G(W) - C(t)$ *then* $\langle C_G(t), C_G(t^x) \rangle$ *contains a Sylow* 2*-subgroup of G.*

Proof. By 19.9 in [3] if $t \notin G$ then $W \cong Sp_n(2)$ or the centralizer of a transvection in $Sp_n(2)$. In particular W is not a 2-group. Hence $t \in G$. Represent G on a unitary space U. By Section 6 in [3], $[U, t]$ is a totally singular subspace of U and the class of t is determined by $m([U, t])$. Moreover $C_G(t)$ is exhibited and, by inspection, since W is a 2-group we have $m = [n/2]$. Let H be the stabilizer in G of $[U, t]$. Then H is a maximal parabolic of G, $O_2(H) = C_G(W)$, and H acts as $GL_m(4)$ on $[U, t]$ with $O_2(H)$ the kernel of this representation. t is the product of m members of D and hence lies in W. Moreover $C_G(t)/O_2(H) \cong GU_m(2)$ and corresponds to the centralizer in $H/O_2(H)$ of a graph-field automorphism of $H/O_2(H)$. Again this follows from the form of $C_G(t)$ exhibited in Section 6 of [3]. Hence by the main theorem of [4], $C_G(t)$ is maximal in H. Finally by 10.6.1 in [3], $\langle t \rangle = Z(C_G(t))$. Thus $H = \langle C_G(t), C_G(t^x) \rangle$ for $x \in H - C(t)$, and in particular contains a Sylow 2-subgroup of G.

(3.9) *Let* $G = U_n(2)$, $n \geq 4$, *and let a be a transvection in G. Then no element of* Aut (G) *induces a transvection on* $O_2(C(a))/\langle a \rangle$.

Proof. Let $Q = O_2(C(a))$ and suppose t induces a transvection on $Q/\langle a \rangle$. $C_G(a)$ is given explicitly in Section 6 of [3], and by inspection $t \notin G$. Now by Section 19 in [3], $C_G(t) \cong Sp_n(2)$ or the centralizer of a transvection in $Sp_n(2)$, so in particular $O_2(C_G(a)) \cap C(t)$ is abelian, a contradiction.

(3.10) *Let* $H =$ Aut $(V) = O_8^+(2)$ *and* $Sp_6(2) \cong G \leq H$. *Then:*

(1) $C_H(G)$ *is a* $3'$*-group.*

(2) *Either G stabilizes a nonsingular vector of V or an element of order* 5 *in G acts without fixed points on* $V^\#$.

Proof. Let x be an element of order 7 in G. $C_V(x)$ is of dimension 2 and sign $+$ and hence does not admit the faithful action of an element of order 3 in H. Moreover $[V, x]$ is of dimension 6 and sign $+$, so x does not centralize an element of order 3 in Aut $([V, x]) \cong S_8$. Hence $C_H(x)$ and then $C_H(G)$ is a $3'$-group.

Suppose (2) is false. G permutes the set v^H of 120 nonsingular vectors of V. From the table on page 113 of [15], G has orbits of length 120 or 1, 63, 56 or 36, 28, 56. As (2) does not hold we are in the last case. Then G has a 2-transitive orbit v^G of length 28. $K = H_v \cong Sp_6(2)$ and by [15, p. 113], $G_v \cong O_6^-(2)$ is determined up to conjugacy in both K and G. In particular we find G_v has orbits of length 1, 27, 2, 54, and 36. This forces $H = \langle G, K \rangle$ to act on the K orbit of length 56, a contradiction.

4. Extraspecial generalized Fitting groups

In this section G is a finite group, z is an involution in G, $M = C_G(z)$, and $Q = F^*(M)$ is an extraspecial 2-group of width at least 2.

(4.1) $\langle z \rangle = C_G(Q)$.

Proof. $\langle z \rangle = Z(Q)$, so $C_G(Q) = C_M(Q) = C_M(F^*(M)) = Z(F^*(M) = \langle z \rangle$.

(4.2) *If S is a 2-subgroup of G containing Q, then $\langle z \rangle = Z(S)$.*

Proof. This is a consequence of 4.1.

(4.3) *Let P be a subgroup of Q of index 2. Then P contains each involution in $C_G(P)$.*

Proof. See 13.62 in [11].

(4.4) *Let t be a noncentral involution in Q. Then:*

(1) *t is fused to z in G if and only if $C_Q(t) \leq M^g$ for some $g \in G - M$.*
(2) *If $F^*(C_G(t))$ is a 2-group then $z \in F^*(C_G(t))$.*

Proof. Set $P = C_Q(t)$. If $t = z^g$ then clearly $g \in G - M$ and $P \leq M^g$. Moreover by 2.4 in [2], $z \in Q^g$. Suppose $t \notin z^G$. tz is fused to t in Q and $\langle z, t \rangle = Z(P)$, so z is weakly closed in $Z(P)$, and hence by 4.3, also in $C_G(P)$. Let $P \leq T \in Syl_z(C_G(t))$. Then $z \in Z(T)$, so if $F^*(C_G(t))$ is a 2-group, we have z in $F^*(C_G(t))$. Moreover if $P \leq M^g$ then $z^g \in C_G(P) \cap z^G = \{z\}$, so $g \in M$.

(4.5) *Set $\overline{M} = M/Q$ and $\tilde{M} = M/\langle z \rangle$. Then:*

(1) *The maps $(\tilde{x}, \tilde{y}) = [x, y]$ and $f(\tilde{x}) = x^2$ are bilinear and quadratic forms on $\tilde{Q}$ preserved by M. Hence $\overline{M}$ is a subgroup of the orthogonal group on $\tilde{Q}$ preserving these forms.*
(2) *Let x be an involution in $M - Q$, $k = m(Q/C_Q(x))$, and $r = m(Z(C_Q(x))$. Then, in the notation of Section 3, either*

(i) *$\bar{x}$ is of type a_m, $k = m$, and $r = m + 1$ or*
(ii) *$\bar{x}$ is of type b_m or c_m, $k = m + 1$, $r = m$, and x is fused to xz or*
(iii) *$\bar{x}$ is of type a_m, $k = m + 1$, $r = m + 1$ or $m + 2$, and x is fused to zx.*

Proof. The first remark is well known. Set $m = m([\tilde{Q}, x])$. Then, in the notation of Section 3, $\bar{x}$ is of type a_m, b_m, or c_m. Let $\tilde{P} = C_{\tilde{Q}}(x)$. Then $m(Q/P) = m$. If $\bar{x}$ is of type b_m or c_m, then by 3.6 there exists a nonsingular vector $\tilde{u}$ in $[\tilde{Q}, x]$. Thus u is of order 4 and as xu is fused to u, x inverts u. Hence x is fused to xz in $\langle x, u \rangle$. Also $|P: C_P(x)| = 2$, so $k = m + 1$. Finally $r = m(Z(P)) = m$ by 3.6.2.

So take $\bar{x}$ of type a_m. Then $m(Z(P)) = m + 1$ by 3.6. Hence if $[P, x] = 1$ then (i) holds. Otherwise $\langle x, u \rangle \cong D_8$ for some $u \in P$, so x is fused to xz in $\langle x, u \rangle$, and $k = m + 1$. By 3.6, there is a subgroup R of Q with

$$R = Q_1 * Q_1^x * \cdots * Q_s * Q_s^x, \quad s = m/2,$$

$Q_i \cong Q_8$, and $[Q, x] = [R, x]$. Thus $Z(P) = P \cap R \leq C(x)$, and is of 2-rank $m + 1$. Let $P = WZ(P)$ with $Z(P) \cap W = \langle z \rangle$ and $\langle v, z \rangle = Z(C_P(x))$. Then $\langle v \rangle Z(P) = Z(C_Q(x))$ is of 2-rank $m + 1$ or $m + 2$, completing the proof.

(4.6) *Assume Q has width 2 and $L = F^*(G)$ is simple. Then:*

(1) $L \cong U_3(3)$, $U_4(2)$, $L_4(3)$, $U_4(3)$, $G_2(3)$, A_8, A_9, M_{12}, J_2, or J_3.
(2) *If z is weakly closed in Q then $G \cong U_4(2)$, or $L_4(3)$.*

Proof. First we claim G is of sectional 2-rank 4. By 4.2, M contains a Sylow 2-subgroup T of G. By 4.5, M/Q acts as a subgroup of $O_4^{\varepsilon}(2)$ on $Q/\langle z \rangle$. Hence T/Q is of sectional 2-rank at most 2. So if A/B is an elementary section of T of rank 5 then $(BQ \cap A)/B$ is of rank at least 3 and is centralized by AQ/BQ. But no 4-group in $O_4^{\varepsilon}(2)$ centralizes a section of rank 3 in its corresponding orthogonal space, so $B = 1$ and $A \cap Q \cong E_8$. Now by 4.5, AQ/Q is a 4-group in which each involution is of type a_2. But no such 4-group exists.

So G is of sectional 2-rank 4. Now L is determined by the main theorem of [7]. By 4.2, z^G is the unique class of 2-central involutions of G so $z \in L$. Of course $F^*(C_L(z)) \leq Q$ is a 2-group of order at most 32. Hence by inspection, L is described in (1) or $L \cong L_3(3)$, $L_2(16)$, A_{10}, A_{11}, M_{22}, or M_{23}. Further inspection establishes (1) and (2).

(4.7) *Assume $z \in O_2(C(t))$ for each involution $t \in Q$. Then one of the following hold:*

(1) $G = M$.
(2) $\langle z^G \rangle^{\#} = z^G$.
(3) $F^*(G) = \langle z^G \rangle$ *is simple.*

Proof. Assume not. Let U be a 4-group in Q and $X = O(G)$. For $u \in U^{\#}$,

$$[z, C_X(u)] \leq O_2(C(u)) \cap X = 1,$$

so $X = \langle C_X(u) : u \in U^{\#} \rangle \leq C(z) \cap X \leq O(C(z)) = 1$. So $X = 1$.

Let $Q \leq T \in Syl_2(G)$. By 4.2, $\langle z \rangle = Z(T)$. Set $H = \langle z^G \rangle$ and $C = C_G(H)$. If $z \in C$ then H is abelian, so $H \leq Q$. Hence zz^g is fused to z^g in Q, so $H^{\#} = z^G$, and (2) holds. Therefore $z \notin C$. But z is contained in every non-trivial normal subgroup of M, so $C = 1$.

Suppose $Y = O_2(G) = 1$. Then $z \in Z(Y)$, so $H \leq C = 1$, a contradiction. Thus $Y = 1$ and G contains a component L. $\langle z \rangle = C_G(Q) \geq \langle L^Q \rangle \cap C(Q) \neq 1$, so $z \in \langle L^Q \rangle$. Thus $H \trianglelefteq \langle L^G \rangle$, so as $C = 1$, $H = \langle L^G \rangle$. Let t be a non-central involution in Q. If $L \neq L^t$ then $K = LL^t \cap C(t) \cong L$. But as $H = \langle L^G \rangle$ we may pick L so that $L = [L, z]$ and hence

$$K = [K, z] \leq [K, O_2(C(t))] \leq O_2(C(t)),$$

a contradiction. Thus t fixes L. Hence $Q = \Omega_1(Q) \leq N(L)$, so $z \in \langle L^Q \rangle = L$. Let J be the product of all components of H distinct from L. Then $J \trianglelefteq M$, so as $Q = F^*(M)$, $J = 1$. Thus $H = L$. As $C = 1$, $H = F^*(G)$.

(4.8) *Let N be a minimal subject to $Q \leq N \nleq M$ and $O_2(N) \neq 1$. Set $V = \langle z^N \rangle$ and let K be the largest normal subgroup of N contained in M. Then either*

(1) $|V| = 4$, $N/K \cong S_3$, *and* $V \leq Q$, *or*
(2) $N/K \cong A_5$ *and V is the natural module for* $O_4^-(2)$.

Proof. Let $Q \leq T \in Syl_2(N)$. By 4.2, $\langle z \rangle = Z(T)$. $O_2(N) \neq 1$, so $1 \neq Z(T) \cap O_2(N) \leq \langle z \rangle$ and hence $z \in O_2(N)$. By minimality of N, $M \cap N$ is the unique maximal subgroup of N containing Q, and $N = \langle Q^N \rangle$. As $\langle z \rangle = Z(T)$ and $N \nleq M$, $C_V(G) = 1$. Thus by 2.2 it remains to show $V \leq Q$ if $|V| = 4$. But in this case V centralizes a subgroup of index 2 in Q, so the remark follows from 4.3.

(4.9) *Assume Q is of width at least 3 and Q is a TI-set in G. Then $M = G$.*

Proof. Assume $G = M$. Then $C_G(t) \leq M$ for all $t \in Q^\#$ and z is weakly closed in Q, so by 4.7 and the Z^*-theorem, $F^*(G)$ is simple and there exists $g \in G - M$ with $z^g \in M$. Let $P = Q^g \cap M$, $R = Q \cap M^g$, and $S = N_Q(RP)$. By 4.2, $P \neq Q^g$, so by Theorems 2 and 3 and 4.5 in [1], $R \cong P$ is abelian, $|Q: S| \leq 2$ with $\Omega_1(S) = R$ and $|S| = |R|^2$ in case of equality, and $Z = \Omega_1(R) \leq Z(S)$. Therefore $|Q: C_Q(Z)| \leq 2$, so $Z \cong E_4$ and $S \neq Q$. Hence $R = \Omega_1(S) = Z$ and $|Q| = 2^5$. This is impossible as Q is of width at least 3.

5. $D(z)$

We continue the hypothesis of Section 4. In addition assume $z \in O_2(C_G(x))$ for each $x \in Q^\#$, Q has width $n \geq 3$, $z \notin O_2(G)$, and $G = \langle Q^G \rangle$. Set $\bar{M} = M/Q$ and $\tilde{M} = M/\langle z \rangle$. By 4.5, $\bar{M}$ is faithfully represented as a subgroup of the orthogonal group on $\tilde{Q}$ defined by the forms $(\tilde{x}, \tilde{y}) = [x, y]$ and $f(x) = x^2$. Hence we may use the notation and results of Section 3.

Let $\mathscr{H}$ be the set of all subgroups H of G containing Q such that $O_2(H) \neq 1$ and $H \nleq M$. Let $\mathscr{H}^*$ be the set of minimal members of $\mathscr{H}$ under inclusion. Let $E = E(z)$ be the set of all conjugates z^h of z such that $\langle Q, Q^h \rangle \in \mathscr{H}^*$ and $z^{\langle Q, Q^h \rangle} = z * z^h$ is of order 5. Let $D = D(z)$ be the set of $z^h \in E$ such that $zz^h \notin z^G$.

Throughout this section we assume $t = z^g \in E(z)$ and set $H = \langle Q, Q^g \rangle$ and $V = \langle z * t \rangle = \langle z^H \rangle$. Let K be the largest normal subgroup of H contained in M. Set $P = Q^g \cap M$.

(5.1) (1) $H/K \cong A_5$ *and V is the natural module for* $O_4^-(2)$.
(2) *t induces a_2 on $\tilde{Q}$,* $C_{\tilde{Q}}(t) = C_Q(t)/\langle z \rangle$, *and* $|Q: C_Q(t)| = 4$.
(3) $[Q, t] = (Q \cap P)\langle z \rangle$ *with* $|Q \cap P| = 4$.
(4) $z^G \cap V = z^H = z * t$ *if* $t \in D$.

Proof. Part (1) follows from 4.8 and the fact that $|z * t| = 5$. By part (1),

$$|Q: C_Q(t)| = 4 \quad \text{and} \quad Q \cap V = Z(C_Q(t)) = \langle z \rangle \otimes (Q \cap Q^g) \cong E_8$$

Now (2) and (3) hold with 4.5. H has orbits z^H and $(tz)^H$ on $V^\#$, yielding (4).

(5.2) *$F^*(G) = \langle z^G \rangle$ is simple and $G = F^*(G)Q$.*

Proof. The first remark follows from 4.7 and the hypothesis that $z \notin O_2(G)$. Now $G = \langle Q^G \rangle = F^*(G)Q$.

(5.3) $\langle E \cap tQ \rangle = V$.

Proof. Assume $tx \in E$, $x \in Q$. Let $y \in C_Q(tx)$. Then $[\tilde{x}, \tilde{y}] = [\tilde{t}\tilde{x}, \tilde{y}] = 1$, so $[\tilde{t}, \tilde{y}] = 1$. Hence by 5.1.2, $[t, y] = 1$, so $[t, x] = 1$. Thus $C_Q(tx) = C_Q(t) \cap C_Q(x)$. But $tx \in D$, so $|C_Q(tx)| = |C_Q(t)|$ and hence

$$C_Q(tx) = C_Q(t) = C_Q(x).$$

Thus $x \in V \cap Q$, so $tx \in V$.

(5.4) *$C_{\bar{M}}(\bar{t}) = C_M(t)Q/Q$ if $t \in D$. In any case $|C_{\bar{M}}(\bar{t})\colon C_M(t)Q/Q| \leq 2$.*

Proof. 5.2, 5.3, and a Frattini argument.

(5.5) *Let $x \in M$ with $[Q, t, x] = 1$. Then $[t, x] = 1$.*

Proof. As $[Q, t, x] = 1$, $[\bar{t}, \bar{x}] = 1$ and x centralizes the hyperplane $V \cap Q$ of V. As $[\bar{t}, \bar{x}] = 1$, x acts on V by 5.3. Thus if $[t, x] \neq 1$ then x induces a transvection on V with axis $V \cap Q$. This is impossible since transvections in $O_4^-(2)$ have axes not conjugate to $V \cap Q$.

(5.6) *Let $s \in D$ with $\langle \bar{s}, \bar{t} \rangle \cong S_3$. Then $\langle s, t \rangle \cong S_3$.*

Proof. $x = (st)^3 \in Q$ and xt is fused to s or t in $\langle s, t \rangle$, so by 5.3, $x \in V$. Similarly $x \in \langle z * s \rangle$. By 5.1 and 3.7.4, $\langle z * s \rangle \cap V = \langle z \rangle$. If $x = z$ then $tx = tz \in z^G$, against $s \in D$. So $x = 1$.

Let $\bar{B}$ be the set of involutions of $\bar{M}$ of type a_2. (See Section 3). Then $\bar{E} \subseteq \bar{B}$. By 3.7, $\bar{B}$ is a set of $\{3, 4\}^+$-transpositions of $\bar{M}$.

(5.7) *Suppose $\bar{t}$ inverts a subgroup $\bar{X}$ of odd order. Then $|\bar{X}| \leq 3$.*

Proof. As $\bar{B}$ is a set of $\{3, 4\}$-transpositions, $\bar{X}$ is an elementary abelian 3-group. Let X be a Sylow 3-group of the preimage of $\bar{X}$, let $y \in X^\#$, and let $Y = \langle y \rangle$. Then by 3.7.4, $[Q, Y] \cong Q_8 * Q_8$ and $[Q, t] \leq [Q, Y]$. Thus if $x \in X^\#$, $0 \neq [\tilde{Q}, t] \leq [\tilde{Q}, Y] \cap [\tilde{Q}, x]$, so as x acts on $[Q, Y]$, $[Q, x] = [Q, t, x] = [Q, Y]$. It follows that $X = Y$ is of order 3.

(5.8) *Let $\bar{X}$ be a $\bar{P}$-invariant subgroup of $C_{\bar{M}}(t)$ with $\bar{X} = O(\bar{X})E(\bar{X})$. Then $|\bar{X}| \leq 3$, $C(t)$ contains a Sylow 3-group X of the preimage of $\bar{X}$, and if $|X| = 3$ then $[Q, t, X] = Q \cap P$.*

Proof. By 5.4 we may choose $X \leq C(t)$ to cover $\bar{X}$. Then $[\bar{X}, \bar{P}] \leq \bar{P}$, so as $\bar{X} = O(\bar{X})E(\bar{X})$ is P-invariant, $[\bar{X}, \bar{P}] = 1$. Let x be an element of $X^\#$ of odd order. Then $[x, P] \leq Q \cap Q^g \cong E_4$, so either $[P, x] = 1$ or $[P, x] =$

$Q \cap P$. Also $|Q^g : P| = 4$ and x acts faithfully on Q^g, so if $[P, x] = 1$ then $P = Q^g \cap C(x)$. This is impossible as $Q^g \cap C(x)$ is extraspecial while $Q \cap P \leq Z(P)$. So $Q \cap P = [Q, t, x]$. As this holds for each element x of odd order in $X^\#$, $|\bar{X}| \leq 3$.

(5.9) *$z \in O_2(C_G(zt))$ if $t \in D$. If also $\langle \bar{t} \rangle$ is the only elementary normal 2-subgroup of $C_{\bar{M}}(\bar{t})$ then $\langle t, z \rangle \trianglelefteq C_G(tz)$.*

Proof. Let $\Delta = z * t - \{t, z\}$. Then $zt \in Q^h$ for each $z^h \in \Delta$. Thus by hypothesis $z^h \in O_2(C(zt))$. Also zt is fused to ztz^h in Q^h, so z^h is weakly closed in $Z(Q^h \cap C(tz))$ and then in $C(Q^h \cap C(tz))$ by 4.3. So z^h is in the center of a Sylow 2-subgroup of $C(tz)$. Let Γ be the set of conjugates z^x such that $zt \in Q^x$. It follows that $\Delta \subseteq \Gamma \subseteq Z(O_2(C(tz)))$. Let $U = \langle \Gamma \rangle$. Then $U \leq C(\Delta) \leq N(V)$, so $[U, z] \leq \langle tz \rangle$. As z^x is weakly closed in $\langle tz, z^x \rangle$ for $z^x \in \Gamma$, $[U, z] = 1$. Set $X = \langle z^{C(tz)} \rangle$. Then $[U, X] = 1$. Hence $X \leq C(\Delta) \leq N(V)$, so $[X, z] \leq \langle tz \rangle$. In particular $z \in O_2(C_G(tz))$. If also $C_{\bar{M}}(\bar{t})$ is as hypothesized above then as $\bar{U}$ is an abelian normal subgroup of $C_{\bar{M}}(\bar{t})$, $\bar{U} \leq \langle \bar{t} \rangle$. By symmetry $U \leq C_Q(t)\langle t \rangle \cap P\langle z \rangle = V$. So $C(zt) = N(V) \cap C(tz) = N(\langle z, t \rangle)$.

(5.10) *Assume $t \in E - D$ and let $I = \langle Q^G \cap N(V) \rangle$ and $C = C_I(V)$. Then:*

(1) *$I/C \cong A_6$.*
(2) *I is transitive on $V^\#$.*
(3) *$t^x = tz$ for some $x \in I \cap M$, $[\bar{P}, \bar{P}^x] = 1$ and $\bar{P} \cap \bar{P}^x = \langle \bar{t} \rangle$.*

Proof. Let $\Delta = z * t - \{z, t\}$. Calculating in H, $zt \in Q^h$ for $z^h \in \Delta$. Let $zt = z^y$. As $zt \notin Q$, 2.4 in [2] implies $Q^y \cap V = \langle \Delta \rangle$. Thus $P \cap Q \leq \langle \Delta \rangle \leq Q^y$. So

$$[P \cap Q, Q^y \cap M] \leq \langle tz \rangle \cap Q = 1.$$

That is $P \cap Q \leq Z(Q^y \cap M)$. We conclude from 3.5 that there exists $1 \neq v \in \langle tz \rangle (P \cap Q)$ and $r \in Q^y$ with $[z, r] = v$. Then r acts on $\langle [Q, t], tz \rangle = V$ and $z^r = zv$. As $zv \in (zt)^H$ we conclude $\langle H, r \rangle \leq N(V)$ is transitive on $V^\#$. Thus $r \in Q^y \leq I$ and I is transitive on $V^\#$. As $H \leq I$ and $C_I(z)$ acts on $Q \cap V$ we conclude $I/C \cong A_6$ and $t^x = tz$ for some $x \in I \cap M$. $P \trianglelefteq C$, so $[P, P^x] \leq P \cap P^x \leq [Q^g, tz] = [Q, t]$, so $[\bar{P}, \bar{P}^x] = 1$. Also $\langle \bar{t} \rangle = Z(\bar{P})$, so $\bar{P} \cap \bar{P}^* = \langle \bar{t} \rangle$.

(5.11) *Assume $1 \neq \bar{X} = [\bar{X}, t]$ is a $\bar{P}$-invariant subgroup of odd order. Then:*

(1) *$\bar{X}$ is extraspecial of order 27 and $\overline{PZ^g} \cong SL_2(3)$. $n = 3$.*
(2) *If $\bar{X} \trianglelefteq \bar{M}$ then $G \cong U_5(2)$.*

Proof. As $\bar{t}$ is contained in every nontrivial normal subgroup of $\bar{P}$ and $\bar{t}$ acts faithfully on $\bar{X}$, $\bar{P}$ acts faithfully on $\bar{X}$. Hence $\bar{X}$ is not cyclic, so by 5.7, t does not invert $\bar{X}$. As t inverts the $\bar{P}$-invariant subgroup $[Z(\bar{X}), t]$ we conclude $[Z(\bar{X}), t] = 1$. So by 5.8, $\bar{Z} = Z(\bar{X}) = C_{\bar{X}}(t)$ is of order 3. Now t inverts

$\bar{X}/\bar{Z}$, so as $\langle \bar{B} \rangle$ is a set of $\{3, 4\}$-transpositions, $\bar{X}/\bar{Z}$ is an elementary abelian 3-group. Thus $\bar{Z} = \Phi(\bar{X})$ and $\bar{X}$ is extraspecial.

Let X be a Sylow 3-subgroup of the preimage of $\bar{X}$. Let $Z \leq Y \leq X$ with $|Y| = 9$. Then $Y = Z \otimes [Y, t]$. Set $Y_1 = [Y, t]$. Then $Q = [Q, Y_1] * C_Q(Y_1)$ and by 3.7, $[Q, Y_1] \cong Q_8 * Q_8$. By 5.8, $[Q, t, Z] = Q \cap P$, so as $[Q, Y_1] = [Q, t, Y_1]$, $[Q, Y_1] = [Q, Y_1, Z]$. X is transitive on the three subgroups Y_i of order 3 in Y distinct from Z, so $[Q, Y_i] \cong Q_8 * Q_8$ for each i. As $[Q, Y_1] = [Q, Y_1, Z]$, $[Q, Y_1, Y_i] \cong Q_8$ for $i = 3, 2$. Thus

$$[Q, Y] = \langle [Q, Y_i]: 1 \leq i \leq 3 \rangle = [Q, Y_1][Q, Y_2] \cong Q_8 * Q_8 * Q_8.$$

Moreover $[Q, Y]$ is X-invariant, so as Z acts faithfully on $[Q, Y]$, so does X. Hence $|X| = 27$. As P acts faithfully on $\bar{X}$, $|P| \leq 8$, so $n = 3$. Thus $Q = [Q, Y]$ and then $P \cong Q_8$.

Next $C_Q(t)Z/Q \cap P \cong SL_2(3)$ so $\bar{Z}^g\bar{P} \cong SL_2(3)$. Let $M_0 = QXPZ^g$. Then the isomorphism class of M_0 is determined and M_0 contains a Sylow 2-subgroup T of H. X acts transitively on the noncentral involutions of Q. Moreover all involutions in tQ are in V. Hence all involutions in T are fused into V in M. As there are two H classes of involutions in V with representatives z and zt, these are representatives for the G classes of involutions in T.

Assume $X \trianglelefteq M$. X does not admit $Q_8 * Q_8$ so by 5.10.3, $t \in D$. Also as the normalizer of $\bar{X}$ in $O_6^-(2)$ contains $\bar{M}_0$ as a subgroup of index 2, $|M: M_0| \leq 2$. X acts irreducibly on $\tilde{Q}$, so by 5.2, G is simple. If $M = M_0$ then M is isomorphic to the centralizer of a transvection in $U_5(2)$, so by [10], $G \cong U_5(2)$, a contradiction. So $|M: M_0| = 2$.

Let $\bar{a}$ be an involution in $\bar{M} - \bar{M}_0$. $\bar{M}$ is transitive on involutions in $\overline{aM_0}$ and $[\tilde{Q}, a] = C_{\tilde{Q}}(a)$, so Q is transitive on involutions in $\tilde{a}\tilde{Q}$. Thus we pick $\tilde{a}$ to be an involution and $\tilde{M}$ is transitive on involutions in $\tilde{a}\tilde{M}_0$. $N_{\tilde{M}}(\tilde{X}) = \tilde{X}\tilde{P}\tilde{Z}^g\langle \tilde{a} \rangle$, so we may pick a to act on X and centralize t. Next a centralizes an element x of order 3 in $X - Z$ and

$$W = C(a) \cap [x, Q]\langle x, t \rangle \cong Z_2 \times S_4.$$

Moreover $C_Q(x)\langle a \rangle \cong C_Q(x)\langle a \rangle O_{2,3}(M^g)/O_{2,3}(M^g)$ is a Sylow 2-group of $M^g/O_{2,3}(M^g)$ and hence semidihedral. Thus we may pick a to be an involution, M is transitive on involutions in aM_0, and $W = C(a) \cap M_0$. As G is simple and all involutions in T are fused to z or zt, by Thompson transfer, a is fused to z or zt in G.

Next $O_2(C_{\bar{M}}(\bar{t})) = \bar{P}$ with $C_M(t)$ irreducible on $\bar{P}/\langle \bar{t} \rangle$ so by 5.9, $C(zt) \leq N(\langle z, t \rangle) \leq H\langle a \rangle$. In particular for $u = z$ or zt, if $U \leq C(u)$ with $U \cong A_4$, then $O_2(U) \leq O_2(C(u))$. Therefore $[C_Q(a), x] \leq O_2(C(a))$.

Now a acts on $z * t$ as a transposition (s, r). Let B be a Sylow 3-group of $C_H(\langle s, r \rangle)$. Then a Sylow 2-subgroup of $H\langle a \rangle \cap N(B)$ is of the form $\langle a_1, s \rangle \otimes \langle v \rangle$ where $\langle a_1, s \rangle \cong D_8$ and $Ka = Ka_1$. Thus we may choose $a = a_1 \in N(B)$, and a centralizes an element $b \in B^\#$ acting as (z, t, t') on $z * t$. But now $[C_Q(a), x, b]$ is not a 2-group, contradicting $[C_Q(a), x] \leq O_2(C(a))$.

(5.12) *Let* $\bar{L} = \langle \bar{E} \rangle$. *Then* $F^*(C_{\bar{M}}(\bar{L}))$ *has order at most* 3 *and one of the following hold*:

(1) $G \cong U_5(2)$.
(2) $G \cong Sz$ *the sporadic Suzuki group or* Co_1, *the largest Conway group.*
(3) $\bar{L}/Z(\bar{L}) \cong U_n(2)$, $n \geq 4$, $P \leq L$, *and* $D = E$.
(4) $\bar{L}/Z(\bar{L}) \cong \Omega_{n+2}^{\varepsilon}(2)$. *Moreover* $D = E$, $\bar{P} = O_2(C_{\bar{L}}(\bar{t}))$, *and* $D \cap P \not\subseteq \{t\}$.
(5) $\langle \bar{P}^M \rangle \cong G_2(2)$, $\bar{P} \cong Q_8 * Q_8$, $n = 4$, $D = E$ *and* $D \cap P \not\subseteq \{t\}$.
(6) $G \cong Co_2$.

Proof. By 3.1, $\bar{B}$ is partitioned into subsets $\bar{B}_i$ such that $[\bar{B}_i, \bar{B}_j] = 1$ for $i \neq j$, $\langle \bar{B}_i \rangle$ is transitive on $\bar{B}_i$ and if $\bar{a}, \bar{b} \in \bar{B}$ with $\bar{a}\bar{b} \in \bar{B}$ then $\bar{a}\bar{b}$ is conjugate to $\bar{a}$ in $\langle \bar{B} \rangle$ and hence lies in the same orbit $\bar{B}_i$ as $\bar{a}$. Choose notation so that

$$\bar{E} = \bar{B}_1 \cup \cdots \cup \bar{B}_r \quad \text{and} \quad \bar{t} \in \bar{B}_1.$$

Let $\bar{L}_i = \langle \bar{B}_i \rangle$.

By 5.8, $|F^*(C_{\bar{M}}(\bar{L}_1))| \leq 3$. By 5.11, $|F^*(\bar{L}_1)| > 3$. We conclude $\bar{L} = \bar{L}_1$ and $\bar{E} = \bar{B}_1$. Then by 5.11 either $\bar{L} = \bar{E}(\bar{L})$ or $G \cong U_5(2)$, and we may assume the former. If $t \in D$ let $R = P$. If $t \notin D$ then by 5.10 there exists $x \in M$ with $t^x = tz$. In this case let $R = PP^x$. By 5.10, $[\bar{P}, \bar{P}^x] = 1$ and $\bar{P} \cap \bar{P}^x = \langle \bar{t} \rangle$. Thus in any case with 5.4, $\bar{R} \trianglelefteq C_{\bar{M}}(\bar{t})$ and $\bar{R}$ is extraspecial. As $|F^*(C(\bar{L}))| < 3$, $\bar{L} = F^*(\bar{L}\bar{R})$. Hence one of the conclusions of 3.4 holds. Notice $Q^g \cong Q_8 * Q_8 * P$ by 3.7.4.

Assume first that $t \notin D$. Then $\bar{R} = \bar{P} * \bar{P}^x$, $x \in C_{\bar{M}}(\bar{t})$. In particular $\bar{R}$ is of width at least 2 so neither 3.4.3 nor 3.4.4 hold. In 3.4.5, $n = 3$ while $Sp_6(2) \not\leq O_6(2)$. Moreover $O^2(C_{\bar{M}}(\bar{t}))$ leaves $\bar{P}$ and $\bar{P}^x$ invariant and hence does not act irreducibly on $\bar{R}/\langle \bar{t} \rangle$. This forces $\bar{R}\bar{L}$ isomorphic to $L_m(2)$, $G_2(2)$, $U_4(2)$, or $\Omega_8^+(2)$. If $\bar{R}\bar{L} \cong L_m(2)$, $m > 4$, then $C_{\bar{L}}(\bar{t})/\bar{R}$ has 2 nonequivalent irreducible submodules $\bar{R}_i/\langle \bar{t} \rangle$ on $\bar{R}/\langle \bar{t} \rangle$ and $\bar{R}_i$ is abelian. Hence $C_{\bar{L}}(\bar{t}) = O^2(C_{\bar{L}}(\bar{t}))$ does not act on $\bar{P}$. If $\bar{L}\bar{R} \cong G_2(2)$ then there are precisely two proper $O^2(C_{\bar{L}}(\bar{t}))$ invariant extraspecial subgroups of $\bar{R}$ and both are invariant under $C_{\bar{M}}(\bar{t})$. So this case is out. We are left with (i) $\bar{L}\bar{R} \cong U_4(2)$ or $L_4(2)$ and $\bar{P} \cong Q_8$, or (ii) $\bar{L}\bar{R} \cong \Omega_8^+(2)$ and $\bar{P} \cong Q_8 * Q_8$. As $Q^g \cong Q_8 * Q_8 * P$, $Q \cong (Q_8)^k$, $k = 3$ or 4 in (i) or (ii), respectively. Now in (i), $\bar{L}$ is a subgroup of $O_6^-(2) \cong \text{Aut}\,(U_4(2))$, so $\bar{L} \cong U_4(2) \cong \Omega_6^-(2)$. Thus in either case $\bar{L} \cong \Omega_{2n}^{\varepsilon}(2)$. $\bar{L}$ acts irreducibly on $\tilde{Q}$ so by 5.2, G is simple. By Theorem 2 in [17], $\bar{M} = \bar{L}$. Now by [8] and [9], G is isomorphic to Sz or Co_1 in (i) or (ii), respectively.

This completes the case $t \notin D$. In the remaining cases, $E = D$. Let X be a subgroup of order 3 in $C_H(\langle t, z \rangle)$. If $[\bar{P}, \bar{X}] = 1$ then as z and t are interchanged in $N_H(X)$, $[P, X] = P \cap Q$ so $[Q, X] \cong Q_8 * Q_8$. In particular $[Q, X, L] = 1$ whereas $[Q, X, t] \neq 1$. So $[\bar{P}, \bar{X}] \neq 1$. Therefore

(5.12.7) $$O_3(C_{\bar{M}}(\bar{t})/\bar{P}C(\bar{L})) \neq 1.$$

This eliminates 3.4.4. If 3.4.3 holds, $Q \cong (Q_8)^3$ whereas 7 divides the order of $U_3(3)$ but not $O_6^-(2)$. In 3.4.5, $n = 4$, so (6) holds by [14] and 3.10. Here use Corollary 1 in [17] to show M does not fix a singular point of $\tilde{Q}$.

Finally assume 3.4.1 holds. 5.12.7 implies $\bar{L}/Z(\bar{L}) \cong \Omega_n^\varepsilon(2)$ or $\bar{M} \cong G_2(2)$. P has width $n - 2$ while the isomorphism class of P is determined by 3.4.1. This allows us to calculate n by inspection. By 3.4.1 there exists $\bar{b} \in \bar{D} \cap \bar{P}$ distinct from $\bar{t}$. It remains to show $(z * b) \cap P$ is nonempty. $\bar{b}\bar{t} \in \bar{D}$ so by 3.7.3, $[\tilde{Q}, t, b] = 0$. Thus by 5.1.2, $[Q, t, b] = 1$, so by 5.5, $[b, t] = 1$. Hence $b = xy$, $x \in C_Q(t)$, $y \in P$. $\bar{b} = \bar{y}$ is an involution so as $\Phi(P) \cap Q = 1$, y is an involution. Similarly $x^2 = 1$, so $[x, y] = 1$. If $u \in C_Q(\langle b, t\rangle) - C(x)$ then $z = [x, u] = [y, u] \in Q^g$, a contradiction. So $C_Q(\langle b, t\rangle) \leq C_Q(x)$. But by 3.7.3, $C_Q(\langle b, t\rangle) = C_Q(\langle b, v\rangle)$, $v \in [Q, t]$, so $x \in [Q, t][Q, b]$. As $[Q, t] \leq P[Q, b]$ we may take $x \in [Q, b]$ and then $y = bx \in b[Q, b] \leq \langle z * b\rangle$. Thus either $y \in D \cap P$ or $y = zd$, $d \in z * b$, and we may assume the latter. Now if $\bar{L}$ is not $L_4(2)$, there is no normal elementary abelian 2-subgroup of $C_{\bar{M}}(\bar{t})$ properly containing $\bar{t}$, so by 5.9, $\langle z, d\rangle \trianglelefteq (Q^g \cap C(y))\langle z\rangle$, a contradiction. So $\bar{L} \cong L_4(2)$. As $[X, P] = 1$, X is semiregular on $\tilde{Q}$. This is impossible as a conjugate of $\bar{t}$ inverts $\bar{X}$ whereas $m([\tilde{Q}, \bar{t}]) = 2$. (Argue as in 3.10 to get $F^*(\bar{M}) = \bar{L}$ so that $X \leq L$.)

(5.13) *Let $s \in D$ with $[\bar{s}, \bar{t}] = 1$. Then $[Q, t, s] = [t, s] = 1$.*

Proof. If $[Q, t, s] = 1$ then $[t, s] = 1$ by 5.5, so assume $[Q, t, s] = \langle u\rangle \neq 1$. Then $u = sr$, $r \in z * s$. But now we choose u with

$$s \notin O_2(C(u) \cap H\langle s\rangle),$$

against 5.9.

6. Proof of the main theorem

In this section, G is a counter example of minimal order to the main theorem. Thus z is an involution in G, $M = C_G(z)$, $Q = F^*(M)$ is extraspecial of width $n \geq 2$, z is weakly closed in Q with respect to G, $G \neq M$, and $z \in O_2(C_G(t))$ for each $t \in Q^\#$. We continue the notation of Section 5. By 4.7 and minimality of G:

(6.1) *$F^*(G) = \langle z^G\rangle$ is simple and $G = F^*(G)Q$.*
(6.2) *$n \geq 3$.*

Proof. See 4.6.2.

(6.3) *$\mathscr{H}$ is nonempty.*

Proof. Assume $\mathscr{H}$ is empty. By 4.9 there exists $g \in G - M$ with $Q \cap Q^g \neq 1$. Let u be an involution in $Q \cap Q^g$. As z is weakly closed in Q, $u \notin z^G$. Thus $P = C_Q(u)$ is of index 2 in Q and by 4.4, $C_G(u) \not\leq M^g$. By symmetry, $C_G(u) \not\leq M$. Let N be a minimal subject to $P \leq N \leq C(u)$ and $N \not\leq M$. Let $V = \langle z^N\rangle$ and let K be the largest normal subgroup of N contained in M. As z is weakly closed in Q, z is weakly closed in $Z(P) = \langle u, z\rangle$ and hence also in $C_G(P)$ by 4.3. Let $P \leq T \in Syl_2(N)$. We conclude $z \in Z(T)$. By minimality of N, $N = \langle P^N \rangle$ and $M \cap N$ is the unique maximal subgroup of N containing P. We may choose $g \in N - M$. Set $t = z^g$. By 2.2 either $|V: C_V(N)| = 4$ and $N/K \cong S_3$ or $N/K \cong A_5$ and $V/C_V(N)$ is the natural module for $O_4^-(2)$. If $x \in C_Q(t) - P$, then as $u \in Q^g$, $z = [x, u] \in Q^g$, a contradiction. So $C_Q(t) \leq P$.

Suppose $N/K \cong S_3$. Then $|Q: C_Q(t)| = 4$ and $\langle u, z \rangle \leq Z(C_Q(t))$, so by 4.5, t induces a_2 on $\tilde{Q}$ and $Z(C_Q(t)) = Z \cong E_8$ with $[Q, t] \leq Z$. Suppose $Z = \langle z \rangle(Q \cap Q^g)$, and set $U = \langle t \rangle Z$. Then U contains $[Q, U]$ and $[Q^g, U]$, so $H = \langle Q, Q^g \rangle = \leq N(U)$. Hence $H \in \mathscr{H}$, contrary to assumption. So $Q \cap Q^g = \langle u \rangle$. Then $(Q^g \cap M)/\langle u \rangle \cong (Q^g \cap M)Q/Q$ acts on $\tilde{Q}$ with $[C_{\tilde{Q}}(t), Q^g \cap M] \leq \langle \tilde{u} \rangle$, so by 3.7.2, $(Q^g \cap M)/\langle u \rangle$ is abelian. This is impossible as Q has width $n \geq 3$ and $|Q^g: Q^g \cap M| = 4$.

So $N/K \cong A_5$. Then $\langle u \rangle[V, P] = W \leq Z(C_Q(t))$ with $m(W) \geq 4$, so by 4.5, t induces a_2 on $\tilde{Q}$, $W = Z(C_Q(t))$ is of rank 4, and t is fused to tz. Now $W = \langle u \rangle V \cap Q$ and tz is fused into $W - \langle z \rangle$ in N, contradicting z weakly closed in Q.

(6.4) *Let $H \in \mathscr{H}^*$, $g \in G - H$, and $t = z^g$. Then $H = \langle Q, Q^g \rangle$, z^H is of order 5 and $tz \notin z^G$.*

Proof. Let K be the largest normal subgroup of H contained in M and $V = \langle z^G \rangle$. As $t \notin Q$, 4.8 implies $H = \langle Q, Q^g \rangle$, $H/K \cong A_5$, and V is the natural module for $O_4^-(2)$. Now tz is fused into $Q - \langle z \rangle$ in H, so $tz \notin z^G$.

6.4 establishes the hypothesis of Section 5. Continuing the notation established there we take $t = z^g \in D$, $H = \langle Q, Q^g \rangle$, $V = \langle z * t \rangle$, and $P = Q^g \cap M$. By 5.12.

(6.5) $\langle \bar{D} \rangle \cong U_n(2)$, $n \geq 4$, $\bar{P} \leq \langle \bar{D} \rangle$, *and* $|F^*(C_{\bar{M}}(\bar{D}))| \leq 3$.

(6.6) *D is a set of 3-transpositions of $\langle D \rangle$.*

Proof. By 6.5, $\bar{D}$ is a set of 3-transpositions of $\langle \bar{D} \rangle$, so the result follows from 5.6 and 5.13.

(6.7) *Let $\langle s, t \rangle \cong S_3$ and $X = \langle z^{C(st)} \rangle$. Then $X/Z(X) \cong U_n(2)$.*

Proof. Let $x = st$ and $Y = \langle C_D(x) \rangle$. Then $F^*(y) = C_Q(x)$ is extraspecial of width $n - 2 \geq 2$ and $Y/C_Q(x) \cong SU_{n-2}(2)$ using 6.5 and 6.6. By 6.5, $|F^*(C_{\bar{M}}(\bar{D}))| \leq 3$ and as $\langle \bar{D} \rangle$ acts irreducibly on $\tilde{Q}$, $C_{\bar{M}}(\bar{D})$ acts without fixed points on $\tilde{Q}$ if $C_{\bar{M}}(\bar{N}) \neq 1$. Hence $F^*(Y)\langle x \rangle/\langle x \rangle = F^*(C_M(x)/\langle x \rangle)$. Thus by minimality of G it suffices to show $Y \neq X$. Let $z^k \in C_D(\langle s, t \rangle)$. Then $x \in \langle s, t \rangle$ centralizes $z * z^k$, so $[Q, x]$ centralizes $z * z^k$. Thus

$$C_Q(\langle x, z^k \rangle)Q^k/Q^k \cong C_Q(\langle x, z^k \rangle)/Q \cap Q^k$$

is the central product of $n - 4$ quaternion groups, so with 6.6 the action of x on Q^k is determined and in particular $Q^K \cap C(x) \not\leq M$.

(6.8) (1) $C_D(t) \subseteq D(t)$.

(2) *If* $r \in D$ *and* $\langle r, t\rangle \cong S_3$ *then* $\langle z^{C(rt)}\rangle = \langle D(r) \cap D(t)\rangle$ *is a complement to* Q^g *in* $\langle D(t)\rangle$.

Proof. Of course $z * t - \{t\} \subseteq D(t)$, so take $s \in C_D(t) - z * t$. There exists $r \in C_D(s)$ with $\langle r, t\rangle \cong S_3$. Set $X = \langle z^{C(rt)}\rangle$. Then $X/Z(X) \cong U_n(2)$ by 6.7. Moreover t centralizes $Q \cap X = F^*(C_X(z))$, so as $F^*(C_X(z))$ is self centralizing in Aut (X), t or tz centralizes X. As $t \in O_2(C(tz))$ by 5.9, $[t, X] = 1$. Thus $s \in z^X \subseteq D(t)$ and $X \cong \langle D\rangle/Q$ so X is a complement to Q^g in $\langle D(t)\rangle$.

Define $B(z) = \bigcup_{d \in D} D(d)$.

(6.9) *The relation* $z \sim a$ *if and only if* $a \in B(z)$ *is an equivalence relation on* z^G.

Proof. Suppose $z \sim a$. Then $a \in D(d)$ for some $d \in D$. Thus $d \in D(a)$ and $z \in D(d)$ so $a \sim z$.

Suppose $z \sim a \sim b$. We must show $z \sim b$. Let $a \in D(d)$, $d \in D$. Then a, $z \in D(d)$, so by 6.6 and 6.7, either $a \in D(z)$ or $\langle a, z\rangle \cong S_3$. In the first case clearly $z \sim b$, so assume $\langle a, z\rangle \cong S_3$. By symmetry we may take $\langle a, b\rangle \cong S_3$. By 6.7 there exists $e \in D(b) \cap a * d$. Then $e \in D(z)$ and $b \in D(e)$, so $z \sim b$.

(6.10) $B(z)$ *is a set of* 3-*transpositions of* $\langle B(z)\rangle = L$. *If* $z^x \in B(z)$ *then* $x \in N(L)$.

Proof. This follows from 6.6 and 6.9.

(6.11) $B(z) = z^G$.

Proof. Assume not. Set $L = \langle B(z)\rangle$ and $X = N_G(B)$. Then $X \neq G$ and by 6.10, $M \leq X$, so by minimality of G, $L \cong U_{n+2}(2)$ and $L = F^*(X)$. We first show $z^G \cap X = z^X$. Assume not.

Then there exists $a = z^k \in X$ with $k \in G - X$. Let $Y = X \cap X^k$ and $a \in T \in Syl_2(Y)$. Pick k so that $|T|$ is maximal. Let $T \leq S \in Syl_2(G)$. Without loss we take $T \leq S \in Syl_2(G)$ with $\langle z\rangle = Z(S)$. $C_X(a) \leq Y$, so $z \in C_S(a) \leq T$ and hence $z \in Z(T)$. By symmetry we may take $a \in Z(T)$. Thus as $\langle z\rangle = C_S(Q)$, $Q \nleq T$ and $S \neq T$. Let Γ be the collection of sets $\Delta(z^h) = B(z^h) \cap T$ for which $T \leq X^h$. Then $N_S(T)/T$ acts semiregularly on $\Gamma - \{\Delta(z)\}$ by maximality of $|T|$. As this holds for each $\Delta \in \Gamma$ we conclude $N_G(T)$ is transitive on Γ and $N_X(T)/T$ is strongly embedded in $N_G(T)/T$. Thus we may pick $k \in N_G(T)$ with $k^2 \in X$.

Claim a centralizes a pair of noncommuting members of $B(z)$. Assume not. Then

$$W = \langle B(z) \cap C(a)\rangle \leq O_2(C_X(a)).$$

So $W \trianglelefteq C_X(a)$. $a \in Z(T)$ so $W = \langle T \cap B(z)\rangle$. Let $x \in N_S(T) - T$. Then $x \in N(W)$. By 3.8, $a \in W$, so $a \neq a^x \in W$ and $X_0 = \langle C_X(a), C_X(a^x)\rangle \leq Y$. But by 3.8, X_0 contains a Sylow 2-subgroup of X, contradicting $S \neq T$.

So a centralizes noncommuting members of $B(z)$. Conjugating by k, there is b, $c \in B(a) \cap C(z)$ with $\langle b, c\rangle \cong S_3$. Let $x = (bc)^k$. $C_L(x)/Z(C_L(x)) \cong U_{n+2}(2)$ by 6.7, so if $u \in B(z) \cap C(x)$ then $F^*(C(\langle u, x\rangle)/\langle x\rangle$ is extraspecial. Hence by minimality of G, $C_L(x) \trianglelefteq C_G(x)$, so $C_G(x) \leq X$. Thus $C_G(bc) \leq X^g$. Now $Q = [Q, bc] * C_Q(bc)$ and $[\tilde{Q}, bc] = \tilde{R} = C_{\tilde{R}}(b) \otimes C_{\tilde{R}}(c)$ with

$$|C_{\tilde{R}}(b)\colon C_R(b)/\langle z\rangle| \leq 2.$$

Hence $Q \not\leq Y$, we conclude $|Q\colon Q \cap X^g| = 4$. We may pick $Q \cap X^g \leq T$, so a centralizes $Q \cap X^g$. Let $\tilde{U}$ be a bc-invariant complement to $(Q \cap X^g)/\langle z\rangle$. Then $C_Q(Q \cap X^g) = U \cong Q_8$, so by 4.4, t induces a transvection on $\tilde{Q}$. However no element of Aut $(U_{n+2}(2))$ induces a transvection $\tilde{Q}$ by 3.9. So we have shown $z^G \cap X = z^X$.

If $z^h \in C(z) - \{z\}$ then $C(zz^h) = N(\langle z, z^h\rangle) \leq X$ by 5.9. Now 3.3 in [1] implies X is strongly embedded in G, a contradiction. This completes the proof of 6.11.

(6.12) $F^*(G) \cong U_{n+2}(2)$.

Proof. By 6.10 and 6.11, z^G is a set of 3-transpositions of $L = F^*(G)$. $F^*(C_L(z))$ is a 2-group and $C_L(z)$ has a $U_n(2)$ section, so by the main theorem of [5], $L \cong U_{n+2}(2)$.

This completes the proof of the main theorem.

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