L-IDEALS AND NUMERICAL RANGE PRESERVATION

BY

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1. Introduction

Let $B(H)$ and $C(H)$ denote respectively the algebras of bounded and compact operators on a complex separable Hilbert space H. Then $C(H)$ is a closed ideal in $B(H)$ and $B(H)/C(H)$, known as the Calkin algebra, is a C^*-algebra with identity in the quotient norm. One problem associated with this algebra is to find a compact operator K for a given $T \in B(H)$ such that $T + K$ reflects accurately in $B(H)$ the properties of $T + C(H)$ in $B(H)/C(H)$. Some progress on this question has already been made; it is well known that there exists $K \in C(H)$ such that $\|T + K\| = \|T + C(H)\|$, and Stampfli [6] has shown recently that there exists $C \in C(H)$ such that the spectrum of $T + C$ and the Weyl spectrum of $T + C(H)$ are equal. This paper arose from an attempt to answer the question of whether each element of $B(H)/C(H)$ has a representative in $B(H)$ which simultaneously preserves both the norm and the numerical range. The solution of this problem will appear as a consequence of a more general extension theorem, namely Theorem 3.3. The fact that $C(H)$ is an M-ideal in $B(H)$ is crucial to these results.

The notion of M-ideal in a Banach space has been formulated and studied in an important paper of Alfsen and Effros [2]. According to these authors a closed subspace J of a (real) Banach space X is an M-ideal in X if its annihilator $J^\perp$ is an L-ideal of the dual space X^*. This in turn means that $J^\perp$ is the range of an L-projection defined on X^*; that is, a projection $Q: X^* \to J^\perp$ with the property that $\|\phi\| = \|Q\phi\| + \|\phi - Q\phi\|$ for all $\phi \in X^*$. The following examples of M-ideals will be of particular interest:

(a) The ideal $\mathscr{C}(V)$ of compact operators on Banach spaces V that possess a special type of unconditional basis. Among such spaces are the sequence spaces c_0 and l^p, $1 < p < \infty$ [3].

(b) The ideals $I_K = \{f : f \in A(D), f|_K \equiv 0\}$ where $A(D)$ is the disk algebra and K is a compact set on the circle having Lebesgue measure zero.

(c) The set of continuous complex-valued affine functions vanishing on a closed split face of a compact convex set K [1].

In [2], an intrinsic characterization of M-ideals was given in terms of an intersection of balls property. The key to this proof was showing that for a given M-ideal J in a (real) Banach space X, w^*-continuous functionals on $J^\perp$ could be extended with preservation of norm to w^*-continuous functionals on X^*.

Received February 2, 1976.

The purpose of this paper is to show that under certain circumstances the w^*-continuous functionals may be extended in a certain range preserving manner. More specifically we show that there is an extension so that the set of images of the extended functional on the states of X^* is equal to the set of images of the functional on the states of $J^\perp$. As an application of our result we will show that for any bounded linear operator T on a complex separable Hilbert space, there is a compact perturbation $T + K$ of T, so that the essential numerical range of T is equal to the closure of the numerical range of $T + K$.

Some of our examples of M-ideals may occur in spaces X intrinsically defined over the complex field. In such cases, when applying any theorems from [2], we always consider that we are working in the real restriction of X. It is straightforward to verify that an M-ideal in a complex Banach space X remains an M-ideal in the real restriction of X.

2. Definitions and notation

We list here several important definitions and terms which will be needed for future use. The space X will always be understood to be a Banach space. An element $e \in X$ will be called a generalized unit if $\|e\| = 1$. The corresponding generalized states S will be $\{\phi \in X^*: \|\phi\| = 1 = \phi(e)\}$ and the generalized numerical range $W(x)$ of an element $x \in X$ is defined to be $W(x) = \{\phi(x): \phi \in S\}$. If J is an M-ideal in X, then the essential norm of $x \in X$, defined to be the quotient norm of $x + J$, will be denoted by $\|x\|_e$. The generalized essential numerical range $W_e(x)$ is the set

$$\{\phi(x): \phi \in J^\perp \cap S\}.$$

In the case that X is a Banach algebra with unit I, our definitions correspond to the usual definitions. (cf. [7]).

In keeping with [2], an M-ideal J in a Banach space X has an annihilator $J^\perp$ which will be denoted by N and its complementary L-summand denoted by $N^\perp$. Let $U(X)$ be the unit ball in X. We will at times set $K = U(X^*)$. Conv (A, B) will indicate the convex hull of A and B, and the distance of x to a set J will be $d(x, J)$.

3. The main theorem

We are now ready to prove our main result, namely, Theorem 3.3. The proof that under certain conditions the essential numerical range of $x \in X$ may be preserved falls naturally into two cases which are considered in Theorem 3.1 and Theorem 3.2. In the first case, the numerical range is taken with respect to a generalized unit.

THEOREM 3.1. *Let J be a closed M-ideal in a complex Banach space X, and let $x \in X$. If $W_e(x)$ contains three noncollinear points, then there exists a $j \in J$ so that $W_e(x) = W(x + j)$. In addition $\|x\|_e = \|x + j\|$.*

Proof. We first wish to derive a certain intersection of balls property, namely

$$(*) \qquad \bigcap_{\lambda \in \mathbf{C}} B(x + \lambda e, \rho(x + \lambda e)) \neq \emptyset$$

where e is our generalized unit and $\rho(x + \lambda e) = d(x + \lambda e, J)$. The preservation of numerical range will be an easy consequence of this relation. We proceed now to prove (*).

Since $W_e(x)$ is convex, the hypotheses of the theorem assure that $W_e(x)$ contains some ball relative to $\mathbf{C}$, and we call it $B(\alpha, \varepsilon)$. The next lemma is pertinent for what follows:

LEMMA 3.1. *Let* $\rho(x - \lambda e) = d(x - \lambda e, J)$. *Then*

$$\bigcap_{\lambda \in \mathbf{C}} B(x - \lambda e, \rho(x - \lambda e))$$

has nonempty interior.

Proof. We show that $B(x - \alpha e, \varepsilon) \subset \bigcap_{\lambda \in \mathbf{C}} B(x - \lambda e, \rho(x - \lambda e))$ where $B(\alpha, \varepsilon) \subset W_e(x)$. Let $\lambda \in \mathbf{C}$. It suffices to show that $|\alpha - \lambda| + \varepsilon \leq \rho(x - \lambda e)$. Now,

$$\begin{aligned} |\alpha - \lambda| + \varepsilon &\leq \sup_{\mu \in B(\alpha, \varepsilon)} |\mu - \lambda| \\ &\leq \sup_{\mu \in W_e(x)} |\mu - \lambda| \\ &\leq \sup_{\|\phi\| = 1} |\phi(x - \lambda e)| \\ &= d(x - \lambda e, J) \\ &= \rho(x - \lambda e). \end{aligned}$$

The next result appears as Theorem 5.8 in [2].

THEOREM [2, p. 120]. *Suppose J is a closed subspace of a Banach space X. Then the following two statements are equivalent*:

(i) *J is an M-ideal.*

(ii) *If $D_1, \ldots, D_n$ are closed balls with* $\operatorname{int}(D_1 \cap \cdots \cap D_n) \neq \emptyset$ *and $D_i \cap J \neq \emptyset$ for all i, then $D_1 \cap \cdots \cap D_n \cap J \neq \emptyset$.*

Our aim is to modify the above result for our particular case; namely, we wish to show that given the M-ideal J, then

(a) $\bigcap_{\lambda \in \mathbf{C}} B(x + \lambda e, \rho(x + \lambda e)) \neq \emptyset$ and

(b) $B(x + \lambda e, \rho(x + \lambda e)) \cap J \neq \emptyset$ for all x imply

$$\bigcap_{\lambda \in \mathbf{C}} B(x + \lambda e, \rho(x + \lambda e)) \cap J \neq \emptyset.$$

In order to motivate what we do next we give a heuristic proof of how (i) implies (ii). Let $v_1, \ldots, v_n$ denote the centers of the balls $D_1, \ldots, D_n$ and

$r_1, \ldots, r_n$ the corresponding radii, and where occasion demands we make the usual identification of an element of a Banach space X as w^*-continuous linear functional on X^*. In [2], an element $v \in D_1 \cap \cdots \cap D_n \cap J$ was produced as a w^*-continuous Hahn-Banach extension of a certain linear functional dominated by the w^*-lower semicontinuous concave function $g(\phi) = \inf (v_i + r_i)(\phi)$ where $v_i + r_i$ is now an affine functional on $U(X^*)$ for all i. Since g is the infinum of a finite number of w^*-continuous affine functionals it is automatically w^*-lower semicontinuous.

To prove our theorem, it is seen that the same argument as in [2] goes through except the verification that the functional

$$g(\phi) = \inf_{\lambda \in \mathbf{C}} \operatorname{Re} (\phi(x + \lambda e) + \rho(x + \lambda e))$$

is w^*-lower semicontinuous on $U(X^*) \equiv \{\phi \in X^*: \|\phi\| \leq 1\}$. As a pointwise infinum of affine w^*-continuous functionals, $g(\cdot)$ is automatically a concave w^*-upper semicontinuous function. To prove that $g(\cdot)$ is w^*-lower semicontinuous, and hence continuous, we need the following:

LEMMA 3.2. *For each* $\varepsilon > 0$, *there exists an* $M_0 < \infty$ *so that for all* λ, *with* $|\lambda| \geq M_0$,

$$(*) \qquad \sup_{\phi \in N \cap K} |\phi(x + \lambda e)| - \sup_{\phi \in N \cap S} |\phi(x + \lambda e)| \leq \varepsilon.$$

Proof. Let $\phi \in N \cap K$ with $\phi(x) = ae^{i\alpha}$, $\phi(e) = be^{i\beta}$, $\lambda = |\lambda| e^{i\theta}$. Then

$$|\phi(x + \lambda e)|^2 = (a \cos \alpha + b|\lambda| \cos (\theta + \beta))^2 + (a \sin \alpha + b|\lambda| \sin (\theta + \beta))^2.$$

Hence, $|\, |\phi(x + \lambda e)| - (b|\lambda| + a \cos (\theta + \beta - \alpha))| \leq C/|\lambda|$, so that

$$|\phi(x + \lambda e)| - |\lambda| - \sup_{\phi \in N \cap K} [(b - 1)|\lambda| + a \cos (\theta + \beta - \alpha)] \leq C/|\lambda|.$$

Set $f(\theta, |\lambda|, \phi) = (b - 1)|\lambda| + a \cos (\theta + \beta - \alpha)$ and note that

$$|f(\theta_1, |\lambda|, \phi) - f(\theta_2, |\lambda|, \phi)| = |a|\, |\cos (\theta_1 + \beta - \alpha) - \cos (\theta_2 + \beta - \alpha)| \leq D|\theta_1 - \theta_2|,$$

for some constant D. Now fix θ. There exists M_θ such that if $|\lambda| \geq M_\theta$,

$$\sup_{\phi \in N \cap K} f(\theta, |\lambda|, \phi) - \sup_{\phi \in N \cap S} f(\theta, |\lambda|, \phi) < \varepsilon.$$

To see this write $E_n = \{\phi: |\phi(e)| \geq 1 - 1/n\}$. Clearly $\bigcap_{n=1}^{\infty} E_n = S$. Since the E_n's are w^*-compact and $x(\phi)$ is a w^*-continuous function, we have $\bigcap_{n=1}^{\infty} x(E_n) = x(S)$. Thus, along any ray, $(*)$ is established.

We now show uniformity in θ. Pick an ε-net of θ's, call them $\{\theta_i\}_{i=1}^N$. Then

$$\sup_{\phi \in N \cap K} f(\theta, |\lambda|, \phi) - \sup_{\phi \in N \cap S} f(\theta, |\lambda|, \phi)$$

$$= \sup_{\phi \in N \cap K} f(\theta, |\lambda|, \phi) - \sup_{\phi \in N \cap K} f(\theta_i, |\lambda|, \phi) + \sup_{\phi \in N \cap K} f(\theta_i, |\lambda|, \phi)$$

$$- \sup_{\phi \in N \cap S} f(\theta_i, |\lambda|, \phi) + \sup_{\phi \in N \cap S} f(\theta_i, |\lambda|, \phi) - \sup_{\phi \in N \cap S} f(\theta, |\lambda|, \phi).$$

Picking the max of the M_i's corresponding to the θ_i's completes the proof.

Remark 1. We have actually shown that for fixed θ,

$$\lim_{|\lambda| \to \infty} \phi(x + \lambda e) - |\lambda| = \sup \operatorname{Re} W_e(e^{-i\theta}x).$$

To see this, note that in the previous proof we were concerned with

$$(**) \qquad \sup_{\phi \in N \cap K} [(b - 1)|\lambda| + a \cos(\theta + \beta - \alpha)]$$

where

$$\phi(x) = ae^{i\alpha}, \ \phi(e) = be^{i\beta}, \ \lambda = |\lambda|e^{i\theta}$$

or equivalently

$$\phi'(e^{-i\theta}x) = ae^{i(\alpha - \beta - \theta)}, \qquad \phi'(e) = b, \qquad \lambda = |\lambda|e^{i\theta}.$$

Since $(b - 1)|\lambda| \to -\infty$ as $|\lambda| \to \infty$, if $b \neq 1$, it follows that if $b \neq 1$, then

$$\lim_{|\lambda| \to \infty} \sup_{\phi \in N \cap K} [(b - 1)|\lambda| + a \cos(\theta + \beta - \alpha)] = \sup_{\phi \in N \cap S} \operatorname{Re} ae^{i(\alpha - \beta - \theta)}.$$

Now from the argument in Lemma 3.2, it is easy to see that for each $\varepsilon > 0$ there exists an M_ε so that

$$|\alpha(x + \lambda e) - |\lambda| - \sup \operatorname{Re} W_e(e^{-i\theta}x)| < \varepsilon \quad \text{for all } |\lambda| > M_\varepsilon.$$

Remark 2. The arguments in Lemma 3.2 also show that for $\lambda = re^{i\theta}$,

$$\lim_{|\lambda| \to \infty} \|x + \lambda e\| - |\lambda| = \sup \operatorname{Re} W(e^{-i\theta}x).$$

Returning to our proof, define

$$g(\phi) = \inf_{\lambda \in \mathbf{C}} \{\operatorname{Re} \phi(x) + \operatorname{Re} \lambda\phi(e) + \rho(x + \lambda e)\}$$

and

$$g_M(\phi) = \inf_{|\lambda| = M} \{\operatorname{Re} \phi(x) + \operatorname{Re} \lambda\phi(e) + \rho(x + \lambda e)\}.$$

Lemma 3.2 showed that if M_n is an increasing sequence diverging to infinity, then g_{M_n} converges uniformly to g on $U(X^*)$. Since it is easily seen that $g_{M_n}(\cdot)$

is w^*-continuous for each M_n, $g(\cdot)$, as a uniform limit of continuous functions, is itself a w^*-continuous function. Thus by the Alfsen-Effros theory

$$\bigcap_{\lambda \in \mathbf{C}} B(x + \lambda e, \rho(x + \lambda e)) \cap J \neq \emptyset.$$

We now complete the proof of Theorem 3.1. In [7], the following characterization of the numerical range of an element x of a Banach algebra was given:

THEOREM [7]. *Let p be a complex number. Then, $p \in W(x)$ if and only if $|p - \lambda| \leq \|x - \lambda e\|$ for all complex λ.*

The proof of this theorem depends on the fact that $\lim_{t\to\infty} (\|x + te\| - t) = \sup \operatorname{Re} W(x)$ which is what we established for our more general case. We now complete the proof of the theorem. Let $j_0 \in \bigcap_{\lambda \in \mathbf{C}} B(x + \lambda e, \rho(x + \lambda e)) \cap J$. Clearly $W_e(x) \subset W(x + j_0)$. Now let $p \in W(x + j_0)$. By the preceding argument, we have

$$|p - \lambda| \leq \|x + j_0 - \lambda e\| = \rho(x - \lambda e) = \|x - \lambda e\|_e \quad \text{for all } \lambda \in \mathbf{C},$$

which implies $p \in W_e(x)$. This completes the proof of the theorem.

We now consider the case in which $W_e(x)$ is a line segment. Here we must limit ourselves to certain Banach spaces. By an example to be discussed later, it will be seen that the conditions to be given are essentially necessary.

Let X be a complex Banach space with a norm closed real-linear subspace X_h. Further suppose that X has the following properties:

(i) $X = X_h \oplus iX_h$.
(ii) If $x, y \in X_h$ then $\|x + iy\| \geq \max(\|x\|, \|y\|)$.
(iii) X_h is a complete order unit space with order unit e.
(iv) If $x \in X_h$ and $\lambda \in R$ then $\|x + i\lambda e\|^2 = \|x\|^2 + \lambda^2$.

Examples of such spaces are

(i) a C^*-algebra with identity, and
(ii) the space $A_{\mathbf{C}}(K)$ of continuous complex-valued affine functions on a compact convex set K.

Under these assumptions we have the following:

THEOREM 3.2. *Suppose $x \in X$ has the property that $W_e(x)$ is a line segment in $\mathbf{C}$. Then there exists $z = x + j \in X$ such that $W(z) = W_e(x)$ and $\|z\| = \|x\|_e$.*

Proof. The proof proceeds in a series of lemmas.

LEMMA 3.3. $\{\phi \in X^*: \phi(e) = \|\phi\| = 1\} = \{\phi \in X^*: \|\phi\| = 1, \phi \geq 0\}$.

Proof. Suppose $\phi(e) = \|\phi\| = 1$ and that there exists $x \in X_h$ such that $\phi(x) = \alpha + i\beta$ with $\beta \neq 0$. Then for all $\lambda \in R$, we have

$$|\phi(x + i\lambda e)|^2 \leq \|x + i\lambda e\|^2 = \|x\|^2 + \lambda^2$$

and

$$|\phi(x + i\lambda e)|^2 = \alpha^2 + \lambda^2 + 2\lambda\beta + \beta^2$$

so that $2\lambda\beta \leq \|x\|^2 - \alpha^2 - \beta^2$, which is a contradiction. Hence $\phi(x)$ is real for $x \in X_h$. Let ψ be the real-linear restriction of ϕ to X_h. Then $\|\psi\| = \|\psi(e)\| = 1$ and hence $\psi \geq 0$ by Proposition II.1.3 of [1]. Thus $\phi \geq 0$ and $\|\phi\| = 1$.

Conversely, suppose that $\phi \geq 0$ and $\|\phi\| = 1$. Let ψ be the real-linear restriction of ϕ to X_h. Clearly, $\|\psi\| \leq 1$. Define $\bar{\psi}: X \to R$ by

$$\bar{\psi}(x + iy) = \psi(x) \quad \text{for all } x, y \in X_h.$$

$$|\bar{\psi}(x + iy)| = |\psi(x)| \leq \|\psi\| \, \|x\| \leq \|\psi\| \, \|x + iy\|$$

by (ii). Hence $\|\bar{\psi}\| = \|\psi\|$. Define $\theta: X \to C$ by $\theta(x) = \bar{\psi}(x) - i\bar{\psi}(ix)$ for $x \in X$. It is well known that θ is complex-linear and that $\|\theta\| = \|\bar{\psi}\|$. If $x \in X_h$ then $\theta(x) = \bar{\psi}(x) = \psi(x) = \phi(x)$ and hence $\theta = \phi$. Then $\|\phi\| = \|\theta\| = \|\bar{\psi}\| = \|\psi\|$ so that $\|\psi\| = 1$. Hence by Corollary II.1.5 of [1]. $\psi(e) = 1$ and so $\phi(e) = 1$. It follows that $\|\phi\| = \phi(e) = 1$.

Let $S = \{\phi \in X^*: \phi \geq 0 \text{ and } \|\phi\| = 1\}$.

LEMMA 3.4. *Let N be a w^*-closed L-ideal in X^* with complementary set N'. Then $S =$ conv $((N \cap S) \cup (N' \cap S))$.*

Proof. Suppose $s \in S$ with decomposition $s = n + n'$ such that $\|s\| = \|n\| + \|n'\|$. Then

$$1 = \|s(e)\| = s(e) = n(e) + n'(e) \leq \|n\| + \|n'\| = 1.$$

Hence $n(e) = \|n\|$ and $n'(e) = \|n'\|$. Assume n and n' are unequal to zero. Then it follows that

$$\frac{n}{\|n\|}(e) = \left\|\frac{n}{\|n\|}\right\| = 1 \quad \text{and} \quad \frac{n'}{\|n'\|}(e) = \left\|\frac{n'}{\|n'\|}\right\| = 1.$$

Thus $n/\|n\|$ and $n'/\|n'\|$ are in S, and s is the convex combination $\|n\|n'/\|n\| + \|n'\|n'/\|n'\|$. This completes the proof.

We now proceed with the proof of the theorem. There exist $\lambda \in \mathbf{C}$ and $\theta \in R$ so that $e^{i\theta}(x + \lambda e) = y$ has the property that $y(S \cap N) \subset R$ is a line segment symmetric about the origin. Let y have decomposition $y_1 + iy_2$. Then $y = y_1$ on $N \cap S$ and hence on N. In view of Lemmas 3.3 and 3.4, it follows that X_h^* can be isometrically identified with

$$[S] = \{\phi \in X^*: \phi(X_h) \subset R\},$$

and that $N \cap [S]$ and $N' \cap [S]$ are complementary real L-ideals in X_h^*. By the real Alfsen-Effros theorem there exists a $g \in X_h$ such that

$$\|g\| = \|y_1|_{N \cap U([S])}\| = \|y_1|_{N \cap S}\|$$

since $N \cap U([S]) = \operatorname{conv}(N \cap S) \cup (-(N \cap S))$ and $y_1(N \cap S)$ is symmetric. Now $g = y_1$ on N and $g(S) = y_1 (N \cap S)$. Then $g = y$ on N and $g(s) = y(N \cap S)$. Let $z = e^{-i\theta} g - \lambda e$. Then $z(S) = x(N \cap S)$ and $z = x$ on N. It remains to show that $\|z\| = \|x|_{N \cap X^*}\|$. By construction there exist $\phi \in N \cap S$ and $\psi \in -(N \cap S)$ such that $\|g\| = \phi(g) = -\psi(g)$. Let $e^{i\theta}\lambda = \alpha + i\beta$. Then

$$\|z\|^2 = \|g - \alpha e - i\beta e\|^2 = \|g - \alpha e\|^2 + \beta^2.$$

Clearly, depending on the sign of α, either $|\phi(g - \alpha e)|$ or $|\psi(g - \alpha e)|$ is equal to $\|g - \alpha e\|$. Without loss of generality suppose it is ϕ. Then $|\phi(z)| = \|z\|$. Thus $\|z\| = |\phi(z)| = |\phi(x)| \leq \|x|_{N \cap X^*}\|$ and the reverse inequality is clear.

Combining Theorems 3.1 and 3.2 leads us to the following:

THEOREM 3.3. *Let X satisfy the hypothesis of Theorem 3.2 and let $W_e(x)$, $W(x)$ be the numerical ranges corresponding to the units $e + j$ and e. Then there exists a perturbation of x, $x + j$ where $j \in J$, such that $W(x + j) = W_e(x)$, and furthermore $\|x + j\| = d(x, J)$.*

4. Remarks and open questions

In [6], Stampfli showed that for any bounded linear operator T on a complex separable Hilbert space H, there exists a compact operator K_0 so that

$$\sigma_w(T) \equiv \bigcap_{K \in C(H)} \sigma(T + K) = \sigma(T + K_0).$$

Now it is well known that $C(H)$ is a complex M-ideal in the C^*-algebra $B(H)$. In [7], it was shown that $\overline{W_0(T)} = W(T)$ where $W_0(T)$ is the usual numerical range. In addition it was shown that $W_e(T) = \bigcap_{K \in C(H)} \overline{W(T + K)}$. Combining these remarks with Theorem 3.3 gives the following

COROLLARY 4.1. *There exists a $K \in C(H)$ such that $\overline{W(T + K)} = W_e(T)$ and $\|T + K\| = \|T\|_e$.*

We next give an example to show that the hypotheses for Theorem 3.2 are not superfluous.

Example 4.1. There exists an M-ideal J in a Banach space X and elements $x \in X$ for which there exists no extension $x + j$ such that $W_e(x) = W(x + j)$.

Proof. Let $X = A(D)$, the disk algebra on the unit disk D. It is well known [4] that the M-ideals of X correspond to $\{x: x(z) \equiv 0, z \in K\}$ where K is closed set of Lebesgue measure zero on the unit circle. Let $K = \{z_1, z_2\}$ be a two point set on the unit circle. Let $x \in A(D)$ such that $x(z_1) \neq x(z_2)$. It is easily seen that the state space of X/J consists of convex combinations of $\delta_{z_1}(\ )$ and $\delta_{z_2}(\ )$ whereas the state space of $A(D)$ includes $\{\delta_z(\ ): z \in D\}$. Clearly $W_e(x)$ is a line segment. If there existed an $x = x + j$, $j \in J$, so that

$W(x + j) = W_e(x)$, then the fact that the nonconstant analytic functions are open maps would show that $x + j \equiv$ constant. But $x(z_1) \neq x(z_2)$, a contradiction.

Remark 1. By [5, prob. 10 p. 289], it was observed that $A(D)$ may be equipped with an involution $f^*(z) = \overline{f(\bar{z})}$. However assumption (iv) is seen to fail for the self-adjoint element $f(z) = z$.

Remark 2. By Theorem 3.2, if $W_e(x)$ has a nonempty interior then there exists a $j \in J$ so that $W(x + j) = W_e(x)$. Viewed as a result in approximation theory, this says that if $W_e(x)$ has nonempty interior then there exists a best approximation j to x from J, so that $x + j$ has minimal range among all possible functions of the form $x + j, j \in J$.

Remark 3. Let K be a compact convex set with closed split face F [1]. Then Theorem 3.3 implies that given $a \in A_{\mathbf{C}}(F)$, there exists $a \in A_{\mathbf{C}}(K)$ such that

(i) $a = \bar{a}$ on F and
(ii) $a(F) = \bar{a}(K)$.

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