IDEAL NORMS ON $E \otimes L_p$

BY

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Introduction

Let E and F be two Banach spaces. There is a natural 1-1 canonical map from $E \otimes F$ to $L(E', F)$ the space of all bounded operators from E' to F. In this paper, we identify $E \otimes F$ as a subspace of $L(E', F)$. Then we can define any ideal norm α on the elements of $E \otimes F$. Let Ω be a locally compact topological space, μ a positive Radon measure on Ω, p a real number, $1 \leq p \leq \infty$, $L_p = L_p(\Omega, \mu)$ the Banach space of equivalence classes of scalar valued pth power integrable functions defined on Ω. It is also possible to identify canonically $E \otimes L_p$ as a subspace of the Banach space $L_p(\Omega, \mu, E)$ consisting of the equivalence classes of E-valued pth power integrable functions defined on Ω. Let Δ_p be the norm of the space $L_p(E) = L_p(\Omega, \mu, E)$, $E \otimes_{\Delta_p} L_p$ be the subspace $E \otimes L_p$ normed by Δ_p, and let $E \hat{\otimes}_{\Delta_p} L_p$ be the completion of $E \otimes_{\Delta_p} L_p$. We know that $L_p(E) = E \hat{\otimes}_{\Delta_p} L_p$.

Δ_p is a natural norm to consider on $E \otimes L_p$ and in general if $1 < p < \infty$ there is no ideal norm γ such that $\gamma = \Delta_p$ on $E \otimes L_p$. But if ν_p denotes the ideal norm associated with the class of p-nuclear operators, π_p the ideal norm of the class of p-absolutely summing operators, and π'_p the dual norm of π_p, we have on $E \otimes L_p$ the following inequalities for all $1 \leq p \leq \infty$:

$$(*) \qquad \pi_p \leq \nu_p \leq \Delta_p \leq \pi'_p \leq \nu'_p \qquad \text{(cf. [17] and [20]).}$$

If $p = 1$, on $E \otimes L_1$, $\pi_1 \leq \nu_1 = \Delta_1 = \pi'_1 = \nu'_1$, and if $p = \infty$, on $E \otimes L_\infty$, $\pi'_\infty = \nu_\infty = \Delta_\infty = \pi'_\infty \leq \nu'_\infty$ [18].

In Section 1 we prove that in a certain sense the inequality $\nu_p \leq \Delta_p \leq \pi'_p$ on $E \otimes L_p$ is the best possible. To obtain this result, we prove at the beginning another version of the duality theorem of L. Schwartz [21]. Later in Section 1 we improve some characterizations obtained by S. Kwapien [14] on the subspaces of $L_p(SL_p)$, quotients of $L_p(QL_p)$ and subspaces of quotients of $L_p(SQL_p)$, and give some applications of the preceding results to the theory of p-absolutely summing operators between L_q-spaces, which extend some results obtained by L. Schwartz [21] for diagonal radonifying operators to the class of all operators.

Section 2 introduces a technique of interpolation which is useful in estimating, for example, the (p, q)-absolutely summing norms. The following result follows immediately from Proposition 2.2: If u is a linear map from $L_p(\mu)$ to

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$L_{p_0}(\nu) \cap L_{p_1}(\nu)$ and if u is (r, q_j)-absolutely summing when considered as a map, denoted by u_j, from $L_p(\mu)$ to $L_{p_j}(\nu)$ for $j = 0$ and 1, then for any $0 \leq \alpha \leq 1$ the operator u is (r, q_α)-absolutely summing as a map from $L_p(\mu)$ to $L_{p_\alpha}(\nu)$, where $1/p_\alpha = (1 - \alpha)/p_0 + \alpha/p_1$, $1/q_\alpha = (1 - \alpha)/q_0 + \alpha/q_1$. Moreover, the (r, q_α)-absolutely summing norm of $u\colon L_p \to L_{p_\alpha}$ satisfies the inequality

$$\pi_{r, q_\alpha}(u\colon L_p \to L_{p_\alpha}) \leq \pi_{p_0}^{1-\alpha}(u_0)\pi_{p_1}^{\alpha}(u_1).$$

S. Kwapien used a very specific method of interpolation when proving in [12] that for every $1 \leq p \leq \infty$, $L(l_1, l_p) = \prod_{1,r(p)}(l_1, l_p)$ where $1/r(p) = 1 - |\frac{1}{2} - 1/p|$. Our method cannot yield his result exactly, however, it is general, and applicable to many other cases.

The technique of interpolation is extended to include another class of operators introduced by H. P. Rosenthal [19] and B. Maurey [16], called the q-cylindrical operators of type p. These results are derived from Proposition 2.1 which tells us when the intersection $L_{p_0}(\Omega, \mu, E_0) \cap L_{p_1}(\Omega, \mu, E_1)$ admits interpolation and describes the interpolating norm.

In Section 3 we consider an arbitrary ideal norm assigned to the space of linear operators from l_1^n to l_2^n. We show that if α is not uniformly equivalent to the ordinary operator norm $\|\cdot\|$ on the spaces of operators from l_1^n to l_2^n, then the unconditional structure of the spaces $L(l_1^n, l_2^n)$ normed by α is not good; for example, these spaces are not uniformly complemented in any infinite-dimensional Banach space X which has an unconditional Schauder decomposition into finite-dimensional spaces all having the same dimension p. This implies that the local unconditional constant (defined in [7]) of $(L(l_1^n, l_2^n), \alpha)$ tends to ∞.

Our notations in general will be similar to [6]. We denote by $\mathscr{F}(E, F)$ the subspace of $L(E, F)$ consisting of all finite-rank operators from E to F. Given any sequence $\{x_i\}_{i \geq 1}$ in E and $1 \leq p \leq \infty$, we write

$$\begin{aligned} \varepsilon_p(x_i) &= \sup\{(\textstyle\sum |\langle x_i, x'\rangle|^p)^{1/p};\ x' \in E',\ \|x'\| = 1\} &&\text{if } 1 \leq p < \infty,\\ &= \sup_i \|x_i\| &&\text{if } p = \infty. \end{aligned}$$

If $[A, \alpha]$ is a normed ideal, α' will be the dual norm, α^Δ the adjoint norm, α^* the conjugate norm. $[A, \alpha]$ is called perfect if $\alpha = \alpha^{**}$. We denote by $(\prod_p, \pi_p)$ the ideal of p-absolutely summing operators, by (N_p, ν_p) the ideal of p-nuclear operators and by (I_p, i_p) the ideal of p-integral operators. For any $T \in L(E, F)$, the ideal norm ε is defined by $\varepsilon(T) = \|T\|$. $E \otimes_\alpha F$ will denote the space $E \otimes F$ (subspace of $L(E', F)$) equipped with the norm α, and $E \hat{\otimes}_\alpha F$ the completion of $E \otimes_\alpha F$.

An L_p space will always be some $L_p(\Omega, \mu)$ $(1 \leq p \leq \infty)$. An operator $A \in L(E, L_p)$ is said to be p-decomposed [23] if there exists a function $f \in L_p(\Omega, \mu, E')$ such that, for any $x \in E$, $A(x)(t) = \langle x, f(t)\rangle$ μ almost everywhere. We have to note that this definition of p decomposed operator is, in general, stronger than the definition of p-decomposable operators of S. Kwapien (cf. [12]). It is easy to verify that f is uniquely determined. We set $\Delta_p(A) = \Delta_p(f)$ and we denote by $\Delta_p(E, L_p)$ the Banach space of all p-decomposed operators from E to L_p normed by Δ_p. With this identification $\Delta_p(E, L_p(\Omega, \mu)) = L_p(\Omega, \mu, E')$.

Let $u \in L(E, F)$, u is called p-decomposing [23] if for any L_p space and for any $A \in L(F, L_p)$, Au is p-decomposed. If u is p-decomposing, then for any L_p space there exists a number $m > 0$ such that for all $A \in L(F, L_p)$, $\Delta_p(Au) \leq m\|A\|$. The smallest possible m in this inequality is denoted by $m(u, L_p)$.

1. Relations between Δ_p and ideal norms

In this paragraph we obtain some inequalities between Δ_p and other ideal norms on $E \otimes L_p$; these are useful, for example, in estimating the ideal norms involved, via Δ_p.

THEOREM 1.1. *Let E and F be two Banach spaces, $u \in L(E, F)$ and $1 < p < \infty$.*

(1) *The operator u is p-decomposing if and only if u' is p-absolutely summing, and for any infinite dimensional L_p-space, $m(u, L_p) = \pi_p(u')$.*

(2) *For $p = 1$ the analogous assertion is true if E' has the Radon Nikodym property (R.N.P.). If E' does not have the R.N.P. there exists a Banach space F and $u \in L(E, F)$ such that $\pi_1(u') < \infty$ and u is not 1-decomposing.*

Proof. It was proved in [20] that $m(u, l_p) = \pi_p(u')$ if $1 < p < \infty$. (1) can be proved in a similar manner.

If $p = 1$ and $m(u, L_1) < \infty$, embed

$$l_1 \xrightarrow{j} L_1$$

isometrically. Let $A \in L(F, l_1)$ then there exists a sequence $\{x_i'\} \subset F'$ such that $Au(x) = (\langle x, u'(x_i')\rangle)_{i \geq 1}$. It follows that

$$\Delta_1(jAu) = \sum \|u'(x_i')\| \leq m(u, L_1)\|A\| = m(u, L_1)\varepsilon_1(\{x_i'\})$$

implying $\pi_1(u') \leq m(u, L_1)$.

Suppose now E' has the R.N.P. and $\pi_1(u') < \infty$. Let L_1 be any infinite-dimensional $L_1(\Omega, \mu)$ space and $A \in L(F, L_1)$. Since $(L_1)' = L_\infty$, $u'A'$ is 1-integral, and since E' has the R.N.P., $u'A'$ is 1-nuclear [24] and therefore Au is 1-nuclear. The inequality

$$\nu_1(Au) = \Delta_1(Au) = \pi_1'(Au) \leq \|A\|\pi_1(u')$$

implies that u is 1-decomposing and that $m(u, L_1) \leq \pi_1(u')$.

We know by [24] that there exists a Banach space G such that $N_1(E, G') \neq I_1(E, G')$. Let T be an integral operator from E to G' which is not nuclear, let K be the unit ball of E' equipped with the weak star topology, i the canonical map from E to the Banach space of continuous functions on K, $C(K)$. We know that there exists a positive measure μ on K such that T has the following factorization:

$$T\colon E \xrightarrow{i} C(K) \xrightarrow{j} L_1(K, \mu) \xrightarrow{A} G',$$

j being the canonical map from $C(K)$ to $L_1(K, \mu)$ and A linear and continuous. Take here $F = L_1(K, \mu)$, and $u = ji$. It is clear that u is integral and not nuclear. Then, $\nu_1(u) = \Delta_1(u) = +\infty$ and therefore u is not 1-decomposing. But, since u is integral, so is u', and it follows that u' is also absolutely summing.

Remark. The equality $\pi_p(u') = m(u, L_p)$ for $1 < p \leq \infty$, and $p = 1$ if E' has R.N.P., adds to related results of L. Schwartz [21] and S. Kwapien [13].

LEMMA 1.2. *Let E be a normed space, $1 \leq p < \infty$, α be an ideal norm, (Ω, μ) a measure space, and assume that on $E \otimes L_p(\Omega, \mu)$ $a\alpha \leq \Delta_p$ for some positive constant a. Then on $E' \otimes L_{p'}$, $a\Delta_{p'} \leq \alpha^{\Delta'}$.*

Proof. Let $u \in E \otimes L_p$ and $v \in E' \otimes L_{p'}$. By [26], the norm $\Delta_{p'}$ on $E' \otimes L_{p'}$ is known to be equal to the norm induced by $(L_p(E))'$. It follows therefore that

$$\Delta_{p'}(v) = \sup_{u \in E \otimes L_p} \frac{|\text{trace }(v'u)|}{\Delta_p(u)}$$

and by the hypothesis of the lemma

$$a\Delta_{p'}(v) \leq \sup_{u \in E \otimes L_p} \frac{|\text{trace }(v'u)|}{\alpha(u)} \leq \sup_{w \in \mathscr{F}(E', L_p)} \frac{|\text{trace }(v'w)|}{\alpha(w)} = \alpha^{\Delta}(v') = \alpha^{\Delta'}(v).$$

PROPOSITION 1.3. *Let E be a Banach space, $1 \leq p \leq \infty$, (Ω, μ) be a measure space, with μ not supported on a finite set of points, α a perfect ideal norm and β be any ideal norm, and let a, b be two positive numbers. Then on $E \otimes L_p$ we have*

(1) $\Delta_p \leq b\beta$ *iff* $\pi'_p \leq b\beta$,
(2) $a\alpha \leq \Delta_p$ *iff* $a\alpha \leq \nu_p$.

Proof. Suppose that $1 < p < \infty$. For (1) it is sufficient to prove $\Delta_p \leq b\beta$ implies $\pi'_p \leq b\beta$. Let $u \in E \otimes L_p$, $A \in L(L_p, L_p)$ and suppose $\Delta_p \leq b\beta$. Then $\Delta_p(Au) \leq b\beta(Au) \leq b\|A\|\beta(u)$, but by Theorem 1.1, $\pi_p(u') = m(u, L_p) = \sup_A \Delta_p(Au)/\|A\|$. It follows that $\pi'_p(u) \leq b\beta(u)$, which proves (1).

To prove (2) it is sufficient to show that $a\alpha \leq \Delta_p$ implies $a\alpha \leq \nu_p$. Suppose that $a\alpha \leq \Delta_p$ on $E \otimes L_p$; then by Lemma 1.2 we have on $E' \otimes L_{p'}$, $a\Delta_{p'} \leq \alpha^{\Delta'}$. By (1) it follows that on $E' \otimes L_{p'}$, $a\pi'_{p'} \leq \alpha^{\Delta'}$, hence if $w \in \mathscr{F}(E, L_{p'})$, then $w'' \in E' \otimes L_{p'}$ maps E'' to $L_{p'}$, so $a\pi'_{p'}(w) = a\pi'_{p'}(w'') \leq \alpha^{\Delta'}(w'') = \alpha^{\Delta'''}(w)$.

Therefore, for every $v\colon L_{p'} \to E$, since α is perfect,

$$\alpha'(v) = \alpha^{\Delta'''*}(v) \leq \alpha^{\Delta'''\Delta}(v) \leq a^{-1}\pi'^{\Delta}_{p'}(v) = a^{-1}\pi^{\Delta'}_{p'}(v) = a^{-1}i'_p(v).$$

Let $w \in E \otimes L_p$; $w = \sum_{i=1}^n x_i \otimes y_i$ maps E' to L_p. Let $v = \sum_{i=1}^n y_i \otimes x_i$ map $L_{p'}$ to E. Then $v' = w$ and so

$$\alpha(w) = \alpha(v') = \alpha'(v) \leq a^{-1}i'_p(v) = a^{-1}\nu_p(w)$$

which concludes the proof for $1 < p < \infty$.

If $p = 1$ the assertions are clear since on $E \otimes L_1$, $\nu_1 = \Delta_1 = \pi'_1$. If $p = \infty$ the result is again obvious since on $E \otimes L_\infty$, $\nu_\infty = \Delta_\infty = \pi'_\infty$.

The following three corollaries improve on some known results of S. Kwapien [14].

COROLLARY 1.4. *Let E be a Banach space, $1 \leq p < \infty$ and $b \geq 1$. The following conditions are equivalent:*

(1) *There exists an infinite-dimensional L_p-space such that on $E \otimes L_p$, $\Delta_p \leq b\pi_p$.*

(2) *There exists an infinite-dimensional L_p-space such that on $E \otimes L_p$, $\pi'_p \leq b\pi_p$.*

(3) *E is an SL_p-space.*

Proof. The equivalences of (2) and (3) are due to [14] or [9]. (1) and (2) are equivalent by Proposition 1.3.

Remark. It can be shown that the least constant b appearing in (1) is equal to the least distance of E from the subspaces of the L_p-spaces. The same method establishes also:

COROLLARY 1.5. *Let E be a Banach space, $1 \leq p < \infty$ and $a \geq 1$. The following conditions are equivalent:*

(1) *There exists an infinite-dimensional L_p-space such that on $E \otimes L_p$, $\nu'_p \leq a\Delta_p$.*

(2) *There exists an infinite-dimensional L_p-space such that on $E \otimes L_p$, $\nu'_p \leq a\nu_p$.*

(3) *E is a QL_p-space.*

COROLLARY 1.6. *Let E be a Banach space, $1 < p < \infty$. E is a SQL_p-space iff there exists an infinite-dimensional L_p-space and an ideal norm α such that on $E \otimes L_p$ α is equivalent to Δ_p.*

Proof. Suppose that on $E \otimes L_p$, $a\alpha \leq \Delta_p \leq b\alpha$. Then by (1) of Proposition 1.3, $a\alpha \leq \Delta_p \leq \pi'_p \leq b\alpha$ on $E \otimes L_p$. But π'_p is perfect, so by (2) of Proposition 1.3 we have on $E \otimes L_p$, $(a/b)\pi'_p \leq \nu_p \leq \Delta_p \leq \pi'_p$. The equivalence of π'_p and ν_p is equivalent to E being a SQL_p-space [14]. The converse is obvious from the inequality $\nu_p \leq \Delta_p \leq \pi'_p$ on $E \otimes L_p$.

COROLLARY 1.7. *Let $1 \leq s < p' < r \leq 2$. Then for any L_p, L_r, and L_s spaces and $\varepsilon > 0$ we have:*

(1) *On the classes of operators from L_s to L_p, $\prod_p \neq \prod_{p+\varepsilon} = \prod_{s'} = I_\infty = I_{p+\varepsilon} \neq I_p$.*

(2) *On the classes of operators from L_p to L_s, $\prod_{p'} \neq \prod_{p'-\varepsilon} = \prod_1 = I_s = \Delta_s = I_{p'-\varepsilon} \neq I_{p'}$.*

(3) *On the classes of operators from L_p to L_r, $\prod_1 = \prod_{p'} = \Delta'_{p'}$ (where $\Delta'_{p'}(L_p, L_r) = \Delta_{p'}(L_{r'}, L_{p'})$).*

Proof. (1) It is sufficient to prove the results for the spaces $L_p[0, 1]$ and $L_s[0, 1]$ with the Lebesgue measure of $[0, 1]$. Since $L_{p'}$ is isomorphic to a subspace of L_s, it follows by Corollary 1.4 that on $L_{p'} \otimes L_s$, Δ_s, ν_s, and π_s are equivalent. By duality, $\prod_{s'}(L_s, L_p) = I_{s'}(L_s, L_p)$. Let $T \in I_{s'}(L_s, L_p)$; then T factors through some L_∞-space, so there exist $A \in L(L_s, L_\infty)$, $B \in \prod_{s'}(L_\infty, L_p)$ such that $T = BA$. By Kwapien [13] $B \in \prod_{p+\varepsilon}(L_\infty, L_p) = I_{p+\varepsilon}(L_\infty, L_p)$, hence so also is T. All the equalities in (1) are obtained.

Let u be a diagonal operator from l_p to $l_{p'}$ defined by $u(e_i) = \lambda_i e'_i$. By L. Schwartz [22], u is 1-absolutely summing iff $\sum |\lambda_i|^{p'}(1 + \ln (1/|\lambda_i|)) < \infty$. By Corollary 1.4, u will be p'-nuclear, or p'-absolutely summing, iff $\sum |\lambda_i|^{p'} < \infty$. Therefore $\prod_{p'}(l_p, l_{p'}) = N_{p'}(l_p, l_{p'}) \neq \prod_1(l_p, l_{p'})$. It is clear that this relation is maintained for $(L_p, L_{p'})$ as well.

Let i be an isomorphic embedding of $L_{p'}$ in L_s and T map L_p to $L_{p'}$ such that $\nu_{p'}(T) = \pi_{p'}(T) < \infty$ and $\pi_1(T) = \infty$. Then $\pi_{p'}(iT) = \pi_{p'}(T) = \nu_{p'}(T) \geq \nu_{p'}(iT)$ and $\pi_1(iT) = \pi_1(T) = \infty$, so on $L_{p'} \otimes L_s$, $\pi_{p'}$ and π_1 are not equivalent. Using duality we get $I_\infty(L_s, L_p) \neq I_p(L_s, L_p)$. We see also that on $L_{p'} \otimes L_s$, $\nu_{p'}$ and π_1 are not equivalent, hence $I_\infty(L_s, L_p) = \pi_{p+\varepsilon}(L_s, L_p) \neq \pi_p(L_s, L_p)$.

(2) This follows from (1) by duality and Corollary 1.4.

(3) It is sufficient to prove the result for $L_p[0, 1]$ and $L_r[0, 1]$. This is equivalent to proving that on $L_{p'} \otimes L_r$, π_1, $\pi_{p'}$ and $\Delta'_{p'}$ are equivalent, that is, on $L_r \otimes L_{p'}$, the norms π'_1, $\pi'_{p'}$, and $\Delta_{p'}$ are equivalent. But L_r is isomorphic to a subspace of $L_{p'}$ and by Corollary 1.4 on $L_r \otimes L_{p'}$ the above norms are indeed equivalent.

Let $T \in \prod_{p'}(L_{r'}, L_{p'})$ and $A \in L(L_{p'}, L_1)$. We know that (see [20]) $\prod_{p'}(L_{r'}, L_{p'}) = \prod_1(L_{r'}, L_{p'})$, so $AT \in \prod_1(L_{r'}, L_1)$. But $L_{r'}$ is isomorphic to a subspace of an L_1-space, so $\prod_1(L_{r'}, L_1) = \Delta_1(L_{r'}, L_1)$ by Corollary 1.4. By Theorem 1.1, since $L_{r'}$ is reflexive, we deduce that $\pi'_1(T) < \infty$. Hence $\pi'_{p'}$ and π'_1 are equivalent on $L_r \otimes L_{p'}$, and the result follows.

Problem 1. Is there a number $a > s'$ such that $\prod_a(L_s, L_p) = \prod_{s'}(L_s, L_p)$?

Remark. The formulas $\prod_1(L_p, L_s) = \Delta_s(L_p, L_s)$ and $\prod_1(L_p, L_r) = \Delta'_{p'}(L_p, L_r)$ generalize a similar result obtained by L. Schwartz [21] on the representation of diagonal radonifying operators (1-absolutely summing operators by our notation) from l_p to l_s and l_p to l_r.

2. Interpolating some classes of operators between $L_p(\Omega, \mu, E)$ spaces

Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms defined on a given complex linear space E, and denote by $\beta(E)$ the family of all E valued functions F holomorphic and bounded with respect to both norms in a neighborhood D of the strip $S = \{z; 0 \leq \text{Re}(z) \leq 1\}$. As in [11, Chapter IV] we norm $\beta(E)$ as follows:

$$\|F\| = \sup_{-\infty < t < \infty} \{\|F(it)\|_0, \|F(1 + it)\|_1\}.$$

For $0 < \alpha < 1$ let $\beta_\alpha(E) = \{F \in \beta(E); F(\alpha) = 0\}$. $\beta_\alpha(E)$ is a linear subspace of $\beta(E)$. The norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are said to be consistent on E if $\beta_\alpha(E)$ is closed in $\beta(E)$ for all $0 < \alpha < 1$, in which case E can be renormed by

$$\|x\|_\alpha = \inf\{\|F\|; F \in \beta(E), F(\alpha) = x\} \qquad (x \in E).$$

$\|\cdot\|_\alpha$ is then called the interpolating norm and we denote by E_α the completion of the space $(E, \|\cdot\|_\alpha)$, $0 \le \alpha \le 1$.

Corresponding to the pair of norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on E there is the pair of conjugate norms, which we shall also continue to denote by $\|\cdot\|_0$ and $\|\cdot\|_1$, on the space of linear functionals defined on E. Denoting by E^* the space

$$(E_0', \|\cdot\|_0) \cap (E_1', \|\cdot\|_1)$$

of linear functionals on E continuous in both norms, we may also talk about the consistency of the (conjugate) norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on E^*.

By [11], $\|\cdot\|_0$ and $\|\cdot\|_1$ are consistent on E if given any $0 \ne x \in E$ there exists $x^* \in E^*$ such that $x^*(x) \ne 0$. It follows therefore that $\|\cdot\|_0$ and $\|\cdot\|_1$ are always consistent on E^*, since $E \subseteq E^{**}$.

If $0 < \alpha < 1$ and both norms are consistent on E, the inclusion map $(E^*)_\alpha \to (E_\alpha)'$ has norm ≤ 1. We shall say E^* is a consistent norming space for E if in addition to the consistency of the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on E, for every $0 < \alpha < 1$ and $x \in E$ the following equality is satisfied:

$$(**) \qquad \|x\|_\alpha = \sup\{|\langle x, x'\rangle|/\|x'\|_\alpha; x' \in (E^*)_\alpha\}.$$

A well-known classical example is the following: let (Ω, μ) be a measure space, $1 \le p_0, p_1 < \infty$, and E be the space $L_{p_0}(\mu) \cap L_{p_1}(\mu)$ which is equipped with the two norms

$$\|\cdot\|_j = \|\cdot\|_{L_{p_j}}, \quad j = 0, 1.$$

Since for each $j = 0, 1$, E is dense in L_{p_j} in the $\|\cdot\|_j$ norm, we have

$$E_j' = (E, \|\cdot\|_j)' = L_{p_j}'(\mu) = L_{p_j'}(\mu).$$

Therefore, $E^* = L_{p_0'}(\mu) \cap L_{p_1'}(\mu)$. The norm $\|\cdot\|_j$ ($j = 0, 1$) on E^* coincides with the norm $\|\cdot\|_{L_{p_j'}}$ on E^*.

It is well known that in this example the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are consistent on E. Moreover, given any $0 < \alpha < 1$ and $x \in E$, $x' \in E^*$, the interpolating norms of x and x' are

$$\|x\|_\alpha = \|x\|_{L_{p_\alpha}} \text{ and } \|x'\|_\alpha = \|x'\|_{L_{p_\alpha'}}, \quad \text{where } p_\alpha = \frac{p_0 p_1}{\alpha p_0 + (1-\alpha)p_1}.$$

Of course, equality (**) is also satisfied since $(E^*)_\alpha$ is dense in $L_{p_\alpha'}$, so that we can say E^* is a consistent norming space for E.

In the following proposition we require E^* to separate the points of E_0, this is clearly so for the above example.

PROPOSITION 2.1. *Let E be a linear space equipped with two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ such that E^* is a consistent norming space for E, and E^* separates the points of E_0. Let Ω be a locally compact topological space, μ a positive Radon measure and let $1 \le p_0, p_1 \le \infty$ with $p_0 < \infty$. Let*

$$B = L_{p_0}(\Omega, \mu, E_0) \cap L_{p_1}(\Omega, \mu, E_1)$$

and denote by $|||\cdot|||_j$ the norm induced on B by Δ_{p_j}, $j = 0, 1$.

Then $|||\cdot|||_0$ and $|||\cdot|||_1$ are consistent norms on B and the interpolating norm $|||\cdot|||_\alpha$ $(0 < \alpha < 1)$ coincides on B with the norm induced by the space $L_p(\Omega, \mu, E_\alpha)$ where $1/p_\alpha = (1-\alpha)/p_0 + \alpha/p_1$.

Proof. Let us first prove $|||\cdot|||_0$ and $|||\cdot|||_1$ are consistent norms on B. If $0 \ne f \in B$, there is a set A such that $\int_A f \ne 0$, and then there is $x^* \in E^*$ such that $\int_A x^*f \ne 0$ and so $\langle f, x^* \otimes \chi_A\rangle \ne 0$. Since $x^* \otimes \chi_A$ is continuous as a functional on $L_{p_j}(E_j)$ for $j = 0$ and 1, the norms $|||\cdot|||_0$ and $|||\cdot|||_1$ are consistent.

The space $[B]$ of simple functions is dense in B, in the norm induced on B by $L_{p_\alpha}(E_\alpha)$; observe $B \subseteq L_{p_\alpha}(E_\alpha)$, by the fact that for any $x \in E$ and $0 < \alpha < 1$, $\|x\|_\alpha \le \|x\|_0^{1-\alpha}\|x\|_1^\alpha$, and so a simple application of Hölder's inequality shows any $f \in B$ must belong to $L_{p_\alpha}(E_\alpha)$.

Thus it suffices to prove the equality of the norms $|||\cdot|||_\alpha$ and Δ_{p_α} on $[B]$. Let $f = \sum_{r=1}^n x_r \otimes \chi_{A_r} \in [B]$, where $x_r \in E$ and A_r are μ-measurable mutually disjoint subsets of Ω. Regard f as an element in $L_{p_\alpha}(E_\alpha)$ and assume that its norm $\Delta_{p_\alpha}(f) \le 1$.

Given $\varepsilon > 0$, there exist $F_r \in \beta(E)$ such that

$$F_r(\alpha) = x_r \quad \text{and} \quad \|F_r\| \le (1+\varepsilon)\|x_r\|_\alpha.$$

Define

$$G(z) = \sum_{r=1}^n \|F_r\|^{a(z-\alpha)}F_r(z) \otimes \chi_{A_r}$$

$$\text{where } a = \frac{p_0 - p_1}{\alpha p_0 + (1-\alpha)p_1} \quad (= 1/(\alpha - 1) \text{ if } p_1 = \infty).$$

Then $G \in \beta(B)$ and $G(\alpha) = f$, and we get

$$\begin{aligned} |||G(iy)|||_0 &= [\textstyle\sum \|F_r\|^{-a\alpha p_0}\|F_r(iy)\|_0^{p_0}\mu(A_r)]^{1/p_0} \\ &\le [\textstyle\sum \|F_r\|^{p_0(1-a\alpha)}\mu(A_r)]^{1/p_0} \\ &\le [\textstyle\sum (1+\varepsilon)^{p_\alpha}\|x_r\|_\alpha^{p_\alpha}\mu(A_r)]^{1/p_0} \le (1+\varepsilon)^{p_\alpha/p_0}, \\ |||G(1+iy)|||_1 &= [\textstyle\sum \|F_r\|^{p_1a(1-\alpha)}\|F_r(1+iy)\|_1^{p_1}\mu(A_r)]^{1/p_0} \\ &\le [\textstyle\sum \|F_r\|^{p_\alpha}\mu(A_r)]^{1/p_1} \le (1+\varepsilon)^{p_\alpha/p_1}. \end{aligned}$$

We have thus proved

$$|||f|||_\alpha = \inf\{\|G\|;\, G \in \beta(B),\, G(\alpha) = f\} \le \Delta_{p_\alpha}(f).$$

To prove $\Delta_{p_\alpha}(f) \geq |||f|||_\alpha$, let f be defined as above and assume $\Delta_{p_\alpha}(f) > 1$. Then there exist $y'_r \in (E^*)_\alpha$, $\|y'_r\|_\alpha < 1 + \varepsilon$ and $\langle x_r, y'_r\rangle \geq \|x_r\|_\alpha$. Let

$$g = (\Delta_{p_\alpha}(f))^{1-p_\alpha} \sum \|x_r\|_\alpha^{p_\alpha - 1} y'_r \otimes \chi_{A_r}.$$

Clearly $\Delta_{p_\alpha'}(g) < 1 + \varepsilon$, $\langle f, g\rangle > 1$. Proceeding as in [11] it is seen that for any $F \in \beta(B)$ with $F(\alpha) = f$, $\|F\| > 1$, so $|||f|||_\alpha \geq \Delta_{p_\alpha}(f)$.

Remark. Proposition 2.1 was proved above for complex Banach spaces, the real case could be done by using the real interpolation method of Lions and Peetre [2]. Other forms of Proposition 2.1 were obtained by Calderon [3].

Let $1 \leq p \leq q \leq \infty$. An operator $u: E \to F$ is called (p, q)-absolutely summing if there exists C such that for any finite subset $\{x_i\}_1^n \subset E$ the following inequality holds:

$$\left(\sum \|u(x_i)\|^q\right)^{1/q} \leq C \sup \left\{\left(\sum |\langle x_i, x'\rangle|^p\right)^{1/p}; \|x'\| = 1\right\}.$$

Let $\pi_{p,q}(u) = \inf C$. $\pi_{p,q}(u)$ is called the (p, q)-absolutely summing norm of u (see [12] and [15]).

PROPOSITION 2.2. *Let $1 \leq p \leq p_0, p_1 \leq \infty$ with $p_0 < \infty$, and let E be a linear space equipped with two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ such that E^* is a consistent norming space for E, and E^* separates the points of E_0. Let F be a normed linear space and $u: F \to E$ be a linear map. Given any $0 \leq \alpha \leq 1$ denote by u_α the operator u mapping F to E_α. Then*

$$\pi_{p,p_\alpha}(u_\alpha) \leq \pi_{p,p_0}^{1-\alpha}(u_0)\pi_{p,p_1}^{\alpha}(u_1).$$

Proof. $u_j \otimes 1$ as a map from $F \otimes_\varepsilon^\wedge l_p$ to $E_j \otimes_{\Delta_{p_j}}^\wedge l_{p_j}$ $(= l_{p_j}(E_j))$ has norm equal to $\pi_{p,p_j}(u_j)$, $j = 0, 1$. As above, if $B = l_{p_0}(E_0) \cap l_{p_1}(E_1)$, $u \otimes 1$ maps $F \otimes_\varepsilon^\wedge l_p$ to $(B, |||\cdot|||_\alpha)$, hence its norm satisfies the inequality

$$\|u \otimes 1\| \leq \|u_0 \otimes 1\|^{1-\alpha}\|u_1 \otimes 1\|^\alpha.$$

By Proposition 2.1 $|||\cdot|||_\alpha$ coincides with the norm induced on B by $l_{p_\alpha}(E_\alpha)$, and the proposition is established.

The last result is a useful tool for obtaining estimates on the $\pi_{p,q}$ norm in many cases. To illustrate this we have:

COROLLARY 2.3. *Let $1 \leq s < p' \leq 2$, $1 \leq r < p'$, $1 \leq q \leq \infty$. There exists a constant $a = a(r, p)$ such that if $v: L_p(\mu) \to I_q(v)$ is a bounded operator which also belongs to the class $\Delta_s(L_p(\mu), L_s(v))$, then for any $0 \leq \alpha \leq 1$, v is $(r, r/(1-\alpha))$-absolutely summing as a map from $L_p(\mu)$ to $L_{s_\alpha}(v)$, where $1/s_\alpha = (1-\alpha)/s + \alpha/q$. Moreover*

$$\pi_{r,r/(1-\alpha)}(v: L_p(\mu) \to L_{s_\alpha}(v) \leq [a\Delta_s(v: L_p(\mu) \to L_s(v))]^{1-\alpha}\|v: L_p(\mu) \to L_q(v)\|^\alpha,$$

and both sides in the inequality are equivalent for $\alpha = 0$ or 1.

Proof. By Corollary 1.7(2) we know that on the class of operators from $L_p(\mu)$ to $L_s(\nu)$ the norm π_r is equivalent to $\pi_s = \Delta_s$. The equality is by (*) and Corollary 1.4; checking the proof we see that $\pi_r(v) \leq a\Delta_s(v)$, where a depends only on r and p.

$\pi_r(v: L_p(\mu) \to L_s(\nu))$ is the norm of $v \otimes 1$ considered as a map from $L_p(\mu) \otimes_\varepsilon^\wedge l_r$ to $L_s(\nu) \otimes_{\Delta_r}^\wedge l_r$, and $\|v: L_p(\mu) \to L_q(\nu)\|$ is the norm of $v \otimes 1$ considered as mapping $L_p(\mu) \otimes_\varepsilon^\wedge l_r$ to $L_q(\nu) \otimes_{\Delta_\infty}^\wedge l_\infty$. The inequality follows by interpolating the norms of $v \otimes 1$.

Remarks. (1) The inequality implies that if $\{e_i\}_{i \geq 1}$ is the unit basis of l_{s_α} and $x'_i \in l_{p'}$, then the map $v = \sum_{i=1}^\infty x'_i \otimes e_i$ mapping l_p to l_{s_α} satisfies the inequality

$$\pi_{r, r/(1-\alpha)}(v) \leq [a(\sum \|x'_i\|^s)^{1/s}]^{1-\alpha}(\varepsilon_q(\{x'_i\}))^\alpha.$$

(2) The results may also be extended to include similar inequalities between $\mathscr{L}_p$-spaces.

LEMMA 2.4 *Let $2 \leq q \leq p \leq \infty$ and T be an operator from l_p to l_q. Let $1/r = 1/q - 1/p$ and $J: l_{r'} \to l_p$ be the inclusion map. Then:*

(1) $\nu_r(TJ) \leq \|T\|$.

(2) *If $q = 2$, then for any $1 < s < \infty$ there is a constant c_s such that $\nu_s(TJ) \leq c_s\|T\|$.*

Proof. In the proof we shall assume $p < \infty$, the case $p = \infty$ is proved in a similar manner. Let $\{f_i\}$ be the canonical basis of l_q, and $\{e_j\}$ the canonical basis of l_p, and assume $T(e_j) = \sum_{i=1}^\infty a_{ij}f_i$ for any j. If $x = \sum x_je_j$ has norm 1, then

$$\|T(x)\| = \left(\sum_i \left|\sum_j a_{ij}x_j\right|^q\right)^{1/q} \leq \|T\|.$$

Therefore if $r_j(t)$ is the jth Rademacher function on $[0, 1]$,

$$\sum_i \left|\sum_j a_{ij}x_jr_j(t)\right|^q \leq \|T\|^q \quad \text{for all } t,$$

and integrating this inequality with respect to t we obtain by Khintchine's inequality

$$\sum_i \left(\sum_j |a_{ij}x_j|^2\right)^{q/2} \leq \|T\|^q,$$

but $q \geq 2$, so that $\sum_i \sum_j |a_{ij}|^q|x_j|^q \leq \|T\|^q$, and minimizing this expression over all $x \in l_p$ with norm 1 we get

$$(\Delta) \qquad \sum_j \left(\sum_i |a_{ij}|^q\right)^{r/q} \leq \|T\|^r,$$

that is, $\Delta_r(J'T') \leq \|T\|$. Since l_q is a quotient of L_r, it follows that $\Delta_r(J'T') = \pi_r(TJ) = \nu_r(TJ)$, which proves (1).

To prove (2) observe that inequality (Δ) can be written as $\sum \|T(e_j)\|^r \leq \|T\|^r$, where now $q = 2$ and $1/r = \frac{1}{2} - 1/p$. With the aid of the Rademacher functions there is a bounded operator P: $L_s[0, 1] \to l_2$ such that if I is the injections of $C[0, 1]$ to $L_s[0, 1]$, then $Q = PI$ maps $C[0, 1]$ onto l_2 [10]. Hence l_2 is isomorphic to $C[0, 1]/Q^{-1}(0)$, therefore there are $\{x_k\}_{k=1}^{\infty} \subset C[0, 1]$ such that $Q(x_k) = T(e_k)$ and $\|x_k\| \leq K\|T(e_k)\|$ (where K is independent of k).

Since $\sum \|x_k\|^r$ converges, the map $b_i \to x_i$ (b_i the canonical basis of $l_{r'}$) defines a bounded operator R: $l_{r'} \to C[0, 1]$. Then the diagram

$$\begin{tikzcd}
 & l_p \arrow[r, "T"] & l_2 & \\
l_{r'} \arrow[ur, "J"] \arrow[r, "R"'] & C[0,1] \arrow[r, "I"'] & L_s[0,1] \arrow[u, "P"'] &
\end{tikzcd}$$

is commutative, so TJ is s-integral and since $l_{r'}$ is reflexive TJ is s-nuclear [17].

COROLLARY 2.5. *Let* $1 < s < \infty$, $2 \leq p \leq \infty$, $1/r = \frac{1}{2} - 1/p$. *Then, for any* $0 < \alpha < 1$, $1 \leq t \leq \infty$, *if T is a linear operator such that* T: $l_p \to l_2$ *and* T: $l_{r'} \to l_t$ *are bounded, then the operator*

$$T\colon l_{r'} \to l_{t_\alpha} \quad (1/t_\alpha = (1 - \alpha)/2 + \alpha/t)$$

satisfies the inequality

$$\pi_{s,\, s/(1-\alpha)}(T\colon l_{r'} \to l_{t_\alpha}) \leq c_s^{1-\alpha}\|T\colon l_p \to l_2\|^{1-\alpha}\| \quad : \quad _{r'} \to \quad _t$$

where c_s depends only on s.

Proof. Apply Lemma 2.4 with the fact that $\pi_s(TJ) \leq \nu_s(TJ) \leq c_s\|T\|$, so that the map $T \otimes 1$ of $l_{r'} \otimes_\varepsilon^\wedge l_s$ to $l_2 \otimes_{\Delta_s}^\wedge l_s$ $(= l_s(l_2))$ has norm $\leq c_s\|T\colon l_p \to l_2\|$, and $T \otimes 1$ considered as mapping $l_{r'} \otimes_\varepsilon^\wedge l_s$ to $l_t \otimes_{\Delta_\infty}^\wedge l_\infty$ $(= l_\infty(l_t))$ has norm equal to $\|T\colon l_{r'} \to l_t\|$. Finally, the result is obtained by interpolation.

COROLLARY 2.6. *Let E be as in Proposition* 2.1 *and let* $1 \leq p_j \leq q_j \leq \infty$ ($j = 0, 1$) *be arbitrary numbers. Let K be a compact Hausdorff space and let* u: $C \to E$ *be a linear map* ($C = C(K)$). *Then for any* $0 \leq \alpha \leq 1$,

$$\pi_{p_\alpha, q_\alpha}(u\colon C \to E_\alpha) \leq \pi_{p_0, q_0}^{1-\alpha}(u\colon C \to E_0)\pi_{p_1, q_1}^{\alpha}(u\colon C \to E_1),$$

where $1/p_\alpha = (1 - \alpha)/p_0 + \alpha/p_1$, $1/q_\alpha = (1 - \alpha)/q_0 + \alpha/q_1$.

Proof. If $1 \leq p \leq q \leq \infty$, $\pi_{p,q}(u\colon C \to F)$, for any Banach space F, is seen to be equal to the norm of the operator $\tilde{u}$ mapping $C(K, l_p)$, the space of continuous l_p valued functions on K equipped with the sup norm, to $l_q(F)$ defined by

$$\tilde{u}\left(\sum_{i=1}^{\infty} f_i(\cdot)e_i\right) = \sum_{i=1}^{\infty} u(f_i) \otimes \tilde{e}_i,$$

where $f_i \in C$, $\{e_i\}$ and $\{\tilde{e}_i\}$ are the unit bases of l_p and l_q, respectively.

This observation on the interpolating technique described earlier yields the inequality.

Remark. The referee noted that Corollary 2.6 was obtained for $p_j = q_j$ in [4]. Note that Corollary 2.6 differs from the previous results in that it allows to "interpolate" both subscripts (p, q) in the norm $\pi_{p,q}$ when the domain space is $C(K)$.

Let $1 \leq p \leq q \leq \infty$, F and G be normed linear spaces. An operator $u: F \to L_p(\Omega, \mu, G)$ is called q-cylindrical of type p (see [16]) if there is a constant C such that for every finite sequence $\{x_i\}_1^n \subset E$ the following inequality holds:

$$\left(\int (\sum \|u(x_i)\|^q)^{p/q}\right)^{1/p} \leq C(\sum \|x_i\|^q)^{1/q}.$$

Let $c_{p,q}(u) = \inf C$. This is clearly the norm of

$$\tilde{u}: l_q(E) \to l_q(G) \otimes_{\Delta_p}^{\wedge} L_p(\Omega, \mu)$$

which is defined by: $\tilde{u}(\sum x_i \otimes e_i) = \sum u(x_i)(\cdot) \otimes e_i$, where $\{e_i\}$ is the unit basis of l_q for $q < \infty$ and $x_i \in E$, with the obvious convention for $q = \infty$.

If $1 \leq p \leq q < \infty$ and if G is reflexive, then $l_q(G)$ has the R.N.P., hence by [1],

$$(L_p(l_q(G)))' = L_{p'}(l_{q'}(G'))$$

(this equality is also true for any G if Ω is discrete). Therefore in this case $c_{p,q}(u)$ is the norm of $\tilde{u}': L_{p'}(l_{q'}(G')) \to l_{q'}(E')$. In the particular case $p = 1$, $1 \leq q < \infty$, it follows from the proof of Corollary 2.6 that for any $u: E \to L_1(\mu)$, $\|\tilde{u}'\| = \pi_{q'}(u')$, so $c_{1,q}(u) = \pi_{q'}(u')$, a result obtained differently in [19, Theorem 1].

It is interesting to recall the following characterization of $c_{p,q}(u)$ proved in [16, Theorem 8].

THEOREM 2.7. *Let $u: F \to L_p(\Omega, \mu, G)$ be a bounded operator, $1 \leq p \leq q \leq \infty$, $1/p = 1/r + 1/q$, and given any $g \in L_r(\Omega, \mu)$ let $T_g: L_q(\Omega, \mu, G) \to L_p(\Omega, \mu, G)$ be the map defined by $T_g(f) = gf$. Then $c_{p,q}(u)$ is equal to the least of all possible values $\|v\|\ \|T_g\|$ where $T_g v = u$ and $v: F \to L_q(G)$ is a bounded operator.*

As done for $\pi_{p,q}$ it is possible to interpolate the constants $c_{p,q}$ using the following result.

THEOREM 2.8. *Let F, G be linear spaces each equipped with two norms denoted by $\|\cdot\|_{F_j}$, $\|\cdot\|_{G_j}$, respectively, $(j = 0, 1)$, such that F^* and G^* are consistent norming spaces for F and G and separate the points of F_0 and G_0, respectively. Let Ω be a locally compact space, μ a positive Radon measure on Ω and let $1 \leq p_j \leq q_j \leq \infty$ with $q_0 < \infty$. Let $B = L_{p_0}(G_0) \cap L_{p_1}(G_1)$ and u be a linear map from F to B such that*

$$c_j = c_{p_j, q_j}(u: F_j \to L_{p_j}(G_j)) < \infty \quad (j = 0, 1).$$

Then for every $0 \le \alpha \le 1$,

$$c_{p_\alpha, q_\alpha}(u: F_\alpha \to L_{p_\alpha}(G_\alpha)) \le c_0^{1-\alpha} c_1^\alpha.$$

Proof. The space $H = l_{q_0}(G_0) \cap l_{q_1}(G_1)$ has two norms induced by $\Delta_{q_j} = |\cdot|$ ($j = 0, 1$), and by Proposition 2.1 the interpolating norm $|\cdot|_\alpha$ on H coincides with the norm induced by $l_{q_\alpha}(G_\alpha)$. Similarly for the two norms on $E = l_{q_0}(F_0) \cap l_{q_1}(F_1)$.

By Proposition 2.1, H^* is a consistent norming space for H, and since G^* separates points in G_0, H^* separates the points in H_0. Let $D = L_{p_0}(H_0) \cap L_{p_1}(H_1)$ and $|||\cdot|||_j$ be the norm induced by $L_{p_j}(H_j)$ on D. By Proposition 2.1 the interpolating norm $|||\cdot|||_\alpha$ is the norm induced on D by $L_{p_\alpha}(H_\alpha)$.

Let $\tilde{u}_j: l_{q_j}(F_j) \to L_{p_j}(H_j)$ ($j = 0, 1$) be the map $\tilde{u}$ defined by

$$\tilde{u}(\sum x_i \otimes e_i) = \sum u(x_i)(\cdot) \otimes e_i.$$

It is clear that $\|\tilde{u}_j\| = c_{p_j q_j}(u: F_j \to L_{p_j}(G_j))$. If $f = \sum x_i \otimes e_i \in E$, then by Proposition 2.1, $|f|_\alpha = (\sum \|x_i\|_{F_\alpha}^{q_\alpha})^{1/q_\alpha}$, and since $\tilde{u}(f) \in D$,

$$|||\tilde{u}(f)|||_\alpha = \left(\int (\sum \|u(x_i)\|_{G_\alpha}^{q_\alpha})^{p_\alpha/q_\alpha}\right)^{1/p_\alpha}.$$

The operator $\tilde{u}$ maps E to D and is bounded as

$$(E, |\cdot|_j) \xrightarrow{\tilde{u}} (D, |||\cdot|||_j)$$

with norm $\le \|\tilde{u}_j\|$. Therefore $\tilde{u}$ is bounded as a map from $(E, |\cdot|_\alpha)$ to $(D, |||\cdot|||_\alpha)$ and its norm satisfies $\|\tilde{u}_\alpha\| \le \|\tilde{u}_0\|^{1-\alpha}\|\tilde{u}_1\|^\alpha$. It follows then that

$$\begin{aligned} |||\tilde{u}(f)|||_\alpha &= \left(\int (\sum \|u(x_i)\|_{G_\alpha}^{q_\alpha})^{p_\alpha/q_\alpha}\right)^{1/p_\alpha} \\ &\le \|\tilde{u}_\alpha\| \, |f|_\alpha \\ &\le \|\tilde{u}_0\|^{1-\alpha}\|\tilde{u}_1\|^\alpha (\sum \|x_i\|_{F_\alpha}^{q_\alpha})^{1/q_\alpha}, \end{aligned}$$

and this completes the proof.

As a consequence to Theorem 2.8 we have the following example:

Example 2.9. Let $p_0 = 1 \le p_1, q_0, q_1, r_0, r_1 \le \infty$, $p_1 = q_1$. For any $0 \le \alpha \le 1$, let $1/r_\alpha = (1-\alpha)/r_0 + \alpha/r_1$ and let p_α, q_α be similarly defined. Assume $u: l_{r_1} \to l_{p_1}$ is a bounded operator such that $\pi_{q_0'}(u': l_\infty \to l_{r_0'}) < \infty$. Then

$$c_{p_\alpha, q_\alpha}(u: l_{r_\alpha} \to l_{p_\alpha}) \le \|u: l_{r_1} \to l_{p_1}\|^\alpha \pi_{q_0'}^{1-\alpha}(u': l_\infty \to l_{r_0'}).$$

Proof. By the remark following the definition of $c_{p,q}$,

$$\pi_{q_0'}(u': l_\infty \to l_{r_0'}) = c_{p_0, q_0}(u: l_{r_0} \to l_{p_0})$$

and clearly $\|u: l_{r_1} \to l_{p_1}\| = c_{p_1, q_1}(u: l_{r_1} \to l_{p_1})$; this observation and Theorem 2.8 yield the result.

3. Unconditional structures of ideals of operators from l_1 to l_2

In this section we investigate the unconditional structure of the spaces $L(l_1^n, l_2^n)$ normed by an arbitrary ideal norm α. We recall the following definition of [8]: Given a real Banach space E, let

$$l(E) = \sup_{F \in \mathscr{F}(E)} \inf_{\{P_i\}} \sup_{N, \pm} \left\| \sum_1^N \pm \sqrt{r(P_i)}\, P_i \right\|$$

where $\mathscr{F}(E)$ is the collection of all finite-dimensional subspaces of E, the infimum ranges over all possible sequences $\{P_i\}$ of operators from F to E satisfying the equality $\sum P_i(f) = f$ for all $f \in F$, and $r(P)$ denotes the rank of an operator P.

If E is finite-dimensional, then clearly

$$l(E) = \inf_{\{P_i\}} \sup_{N, \pm} \left\| \sum_1^N \pm \sqrt{r(P_i)}\, P_i \right\|$$

where the infimum is taken over all sequences $\{P_i\}$ of operators from E to E satisfying the equality $\sum P_i(f) = f$ for all $f \in E$.

It is clear that if E is isomorphic to a complemented subspace of a Banach space X, where X has an unconditional Schauder decomposition into spaces all having the same finite dimension p, then $l(E) < \infty$. Examples of spaces with $l(E) = \infty$ were given in [8]. A new example will be provided by Corollary 3.2. First, we introduce the following notation: R^n will denote the n-dimensional linear space, $\{e_i\}_{i=1}^n$ its usual unit basis. Given any vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$, h_ε will denote the operator defined by $h_\varepsilon(e_i) = \varepsilon_i e_i$, $i = 1, \dots, n$. For any permutation σ of $\{1, 2, \dots, n\}$, g_σ will denote the operator defined by $g_\sigma(e_i) = e_{\sigma(i)}$ for all i.

G will be the compact group of isometries of l_2^n, dg the unique normalized Haar measure on G. S will denote the sphere $\{x \in l_2^n;\ \|x\|_2 = 1\}$, dx will be the measure on S defined by

$$\int_S f(x)\, dx = \int_G f(g(e))\, dg, \quad f \in C(S),$$

where $e \in S$ is any fixed point. All spaces are taken here over the reals. If α is an ideal norm, $\alpha(E)$ will denote the value $\alpha(1_E)$ where 1_E is the identity operator on E.

THEOREM 3.1. *Let α be an ideal norm and $E = (L(l_1^n, l_2^n), \alpha)$. Then*

$$l(E) \geq \pi^{-1} \sup_{0 \neq B \in E} \alpha(B)/\|B\|.$$

Proof. The topological dual of E is $E' = (L(l_2^n, l_1^n), \alpha^\Delta)$. Let $u = \sum_{k=1}^m A_k \otimes B_k$ be any rank-m operator from E' to E', where $A_k \in E$ and $B_k \in E'$. Let $\{e_i\}_1^n$ and $\{f_i\}_1^n$ be the canonical bases of l_2^n and l_1^n, respectively. Let

$$B_k(e_i) = \sum_{j=1}^n b_{kij} f_j \quad \text{and} \quad A_k(f_i) = \sum_{j=1}^n a_{kij} e_j, \quad 1 \leq i \leq n, 1 \leq k \leq m.$$

Let $A \in E'$ be an arbitrary nonzero operator. Consider the probability measure μ on $K = K_{E'} \times K_E$ defined by

$$\mu(f) = \frac{2^{-2n}}{n!} \sum_{\varepsilon,\theta} \int_S \int_G \sum_{\pi} f\left(\left(\frac{h_\varepsilon g_\pi A g}{\alpha^\Delta(A)}\right) \times (\theta \otimes x)\right) dg\, dx,$$

$f \in C(K)$, where the first $\sum$ ranges over all 2^{2n} possible choices of vectors ε, θ of the form $(\pm 1, \pm 1, \ldots, \pm 1)$, the second $\sum$ ranges over all $n!$ possible permutations π of the set $\{1, 2, \ldots, n\}$.

The operator u defines a function of $C(K)$ which we denote by $\langle u, \cdot \rangle$ and define by

$$\langle u, a \times b \rangle = \langle u(a), b \rangle = \text{trace}\,(b(u(a))), \quad a \in K_{E'}, b \in K_E.$$

Then,

$$\begin{aligned} \alpha^\Delta(A) u(|\langle u, \cdot \rangle|) &= \frac{2^{-2n}}{n!} \int_S \int_G \sum_{\varepsilon,\theta} \sum_{\pi} |\langle u(h_\varepsilon g_\pi A g), \theta \otimes x \rangle|\, dg\, dx \\ &= \frac{2^{-2n}}{n!} \int_S \int_G \sum_{\pi} \sum_{\varepsilon,\theta} |\langle (u(h_\varepsilon g_\pi A g))(x), \theta \rangle|\, dg\, dx. \end{aligned}$$

Using Khintchine's inequality, if $y \in l_2^n$ then $2^{-n} \sum_\theta |\langle y, \theta \rangle| \geq 2^{-1/2} \|y\|_2$ (the constant $2^{-1/2}$ is due to [25]) and denoting by v the operator $u(h_\varepsilon g_\pi A g)$ mapping l_2^n to l_1^n, writing $v(e_j) = \sum_{i=1}^n v_{ji} f_i$, we obtain

$$\begin{aligned} \int_S 2^{-n} \sum_\theta |\langle v(x), \theta \rangle|\, dx &\geq 2^{-1/2} \int_S \left(\sum_{i=1}^n \left| \sum_{j=1}^n v_{ji} x_j \right|^2 \right)^{1/2} dx \\ &\geq 2^{-1/2} \left(\sum_{i=1}^n \left(\int_S \left| \sum_{j=1}^n v_{ji} x_j \right| dx \right)^2 \right)^{1/2} \\ &= \frac{2^{-1/2}}{\pi_1(l_2^n)} \left(\sum_{i,j=1}^n v_{ij}^2 \right)^{1/2}. \end{aligned}$$

We see that

$$\begin{aligned} v_{ij} &= (u(h_\varepsilon g_\pi A g))_{ij} \\ &= \sum_{k=1}^m b_{kij}\, \text{trace}\,(h_\varepsilon g_\pi A g A_k) \\ &= \text{trace} \left\{ h_\varepsilon g_\pi A g \left(\sum_{k=1}^m b_{kij} A_k \right) \right\}, \end{aligned}$$

and denoting by $\{f_r'\}$ the unit basis of l_∞^n, and setting $w = g_\pi A g(\sum_{k=1}^m b_{kij} A_k)$, we get again by Khintchine's inequality

$$\begin{aligned} 2^{-n} \sum_\varepsilon |\text{trace}\,(h_\varepsilon w)| &= 2^{-n} \sum_\varepsilon |\sum \varepsilon_r w_{rr}| \\ &\geq 2^{-1/2} (\sum w_{rr}^2)^{1/2} \\ &= 2^{-1/2} (\sum \langle w(f_r), f_r' \rangle^2)^{1/2}. \end{aligned}$$

Thus we obtain

$$2^{1/2}\pi_1(l_2^n)\alpha^\Delta(A)\mu(|\langle u,\cdot\rangle|)$$

$$\geq \frac{1}{n!}\sum_\pi \int_G 2^{-n}\sum_\varepsilon \left[\sum_{i,j=1}^n \left(\operatorname{trace}\left\{h_\varepsilon g_\pi A g\left(\sum_{k=1}^m b_{kij}A_k\right)\right\}\right)^2\right]^{1/2} dg$$

$$\geq \frac{1}{n!}\sum_\pi \int_G \left[\sum_{i,j=1}^n \left(2^{-n}\sum_\varepsilon \left|\operatorname{trace}\left\{h_\varepsilon g_\pi A g\left(\sum_{k=1}^m b_{kij}A_k\right)\right\}\right|\right)^2\right]^{1/2} dg$$

$$\geq \frac{2^{-1/2}}{n!}\sum_\pi \int_G \left[\sum_{i,j,r=1}^n \left\langle g_\pi A g\left(\sum_{k=1}^m b_{kij}A_k\right)(f_r), f_r'\right\rangle^2\right]^{1/2} dg$$

$$\geq \frac{2^{-1/2}}{n!}\sum_\pi \left[\sum_{i,j,r=1}^n \left(\int_G \left|\left\langle g_\pi A g\left(\sum_{k=1}^m b_{kij}A_k\right)(f_r), f_r'\right\rangle\right| dg\right)^2\right]^{1/2},$$

and since $\int_G |\langle g(a), b\rangle|\, dg = \pi_1^{-1}(l_2^n)\|a\|_2\|b\|_2$ (cf. [5]) for any two vectors $a, b \in l_2^n$, we obtain

$$2\pi_1^2(l_2^n)\alpha^\Delta(A)\mu(|\langle u,\cdot\rangle|)$$

$$\geq \frac{1}{n!}\sum_\pi \left[\sum_{i,j,r=1}^n \left\|\sum_{k=1}^m b_{kij}A_k(f_r)\right\|_2^2 \left(\frac{1}{n!}\sum_\pi \|A' g_\pi'(f_r')\|_2\right)^2\right]^{1/2}$$

$$\geq \left[\sum_{i,j,r=1}^n \left\|\sum_{k=1}^m b_{kij}A_k(f_r)\right\|_2^2 \left(\frac{1}{n!}\sum_\pi \|A' g_\pi'(f_r')\|_2\right)^2\right]^{1/2}$$

$$= \frac{1}{n}\left[\sum_{i,j,r,s=1}^n \left(\sum_{k=1}^m a_{krs}b_{kij}\right)^2\right]^{1/2}\left(\sum_{t=1}^n \|A'(f_t')\|_2\right).$$

Now observe that $\sum_{t=1}^n \|A'(f_t')\|_2 = \Delta_1(A) = \nu_1(A)$ since A maps l_2^n to l_1^n. We shall also use the following inequality proved in [8, Lemma 2]: Let x_k, y_k, $k = 1, 2, \ldots, n$, be vectors in R^m. Then

$$\left(\sum_{k=1}^n \langle x_k, y_k\rangle\right)^2 \leq m \sum_{j=1}^n \sum_{k=1}^n \langle x_k, y_j\rangle^2.$$

The last inequality implies that

$$\left[\sum_{i,j,r,s=1}^n \left(\sum_{k=1}^m a_{krs}b_{kij}\right)^2\right]^{1/2} \geq m^{-1/2}\left|\sum_{k=1}^m \sum_{i,j=1}^n a_{kij}b_{kji}\right| = m^{-1/2}|\operatorname{trace}(u)|.$$

Finally, we have proved that for every $A\colon l_2^n \to l_1^n$,

$$nm^{1/2}2\pi_1^2(l_2^n)\alpha^\Delta(A)\mu(|\langle u,\cdot\rangle|) \geq |\operatorname{trace}(u)|\nu_1(A).$$

By duality we have for every $B\colon l_1^n \to l_2^n$,

$$nm^{1/2}2\pi_1^2(l_2^n)\|B\|\mu(|\langle u,\cdot\rangle|) \geq |\operatorname{trace}(u)|\alpha(B).$$

Let now $P_i \in L(E', E')$, $i = 1, \ldots, N$. Then since μ is a probability measure we obtain the inequality

$$2n\pi_1^2(l_2^n)\|B\| \max_{\pm} \left\| \sum_{i=1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\| \geq 2n\pi_1^2(l_2^n)\|B\|\mu\left(\sum_{i=1}^{N} \sqrt{r(P_i)}\, |\langle P_i, \cdot \rangle|\right)$$
$$\geq \alpha(B) \sum_{i=1}^{N} \operatorname{trace}(P_i),$$

so if $\sum_{i=1}^{N} P_i$ is the identity operator on E', and as dim $(E') = n^2$ and $\pi_1(l_2^n) \leq \sqrt{\pi n/2}$ [5] we have

$$\pi\|B\| \max_{\pm} \left\| \sum_{i=1}^{N} \pm \sqrt{r(P_i)}\, P_i \right\| \geq \alpha(B),$$

which implies the required result on $l(E)$.

Denote by $(L_0(E, F), \alpha)$ the closure in the α norm of the finite-rank operators from E to F.

COROLLARY 3.2. *If α is a perfect ideal norm not equivalent to the operator norm $\|\cdot\|$ on the space of operators from l_1 to l_2, then*

$$l((L(l_1^n, l_2^n), \alpha)) \to \infty \quad \textit{and} \quad l((L_0(l_1, l_2), \alpha)) = \infty.$$

Proof. It follows from the definition that if X and Y are normed spaces and if

$$A: X \to Y \quad \text{and} \quad B: Y \to X$$

are operators such that BA is the identity on X, then $l(X) \leq l(Y)\|A\|\,\|B\|$.

Since $E_n = (L(l_1^n, l_2^n), \alpha)$ is isometric to a norm-one complemented subspace of $(L_0(l_1, l_2), \alpha)$, it is sufficient to prove that $l(E_n) \to \infty$. Assume that $l(E_n) \leq \lambda < \infty$ for every $n = 1, 2, \ldots$. Then $\|B\| \leq \alpha(B) \leq \lambda\pi\|B\|$ for every $B: l_1^n \to l_2^n$, and as α is perfect, also for every $B: l_1 \to l_2$, which is a contradiction.

Remarks. (1) If α is any ideal norm and $C: l_2^n \to l_\infty^n$, then from Theorem 3.1 we obtain the inequality

$$\|C\| \leq \alpha(C) = \alpha'(C') \leq \pi\|C\| l((L(l_1^n, l_2^n), \alpha')) = \pi\|C\| l((L(l_2^n, l_\infty^n), \alpha))$$

and the analogue of Corollary 3.2 is also obvious.

(2) If X is a finite-dimensional normed space then clearly $l(X) = l(X')$, hence for any ideal norm α,

$$l((L(l_1^n, l_2^n), \alpha^\Delta)) = l((L(l_2^n, l_1^n), \alpha)),$$

and so if $D: l_2^n \to l_1^n$ is an arbitrary map we get the inequality

$$\begin{aligned} i_1(D) &= \alpha(D) \\ &= \sup\{\operatorname{trace}(uD)/\alpha^\Delta(u);\ u \in L(l_1^n, l_2^n)\} \\ &\geq [\pi l((L(l_1^n, l_2^n), \alpha^\Delta))]^{-1} \sup\{\operatorname{trace}(uD)/\|u\|;\ u \in L(l_1^n, l_2^n)\} \\ &= [\pi l((L(l_2^n, l_1^n), \alpha))]^{-1} i_1(D). \end{aligned}$$

As in Remark (1) it follows that for any $P\colon l_\infty^n \to l_2^n$,

$$i_1(P) \geq \alpha(P) \geq i_1(P)/\pi l((L(l_\infty^n, l_2^n),\alpha)).$$

Results analogous to Corollary 3.2 are now easily derived. The following theorem provides information on the rate of growth of the dimensions in many unconditional Schauder decomposition of $(L_0(l_1, l_2), \alpha)$ into finite-dimensional spaces.

THEOREM 3.3. *Let* $\alpha_n = \max\{\alpha(B)/\|B\|;\ B \in L(l_1^n, l_2^n)\}$. *Assume* $(L_0(l_1, l_2),\alpha)$ *has an unconditional Schauder decomposition into finite-dimensional spaces* E_i *having the following property: For every n, there is a subset of integers* I_n *such that* $(L_0(l_1^n, l_2^n), \alpha)$, *considered as a natural subspace of* $(L_0(l_1, l_2), \alpha)$, *is a subspace of* $\sum_{i \in I_n} \oplus E_i$ *and* $\sup_{i \in I_n} \dim(E_i) = p_n < \infty$. *Then* $\{\alpha_n/\sqrt{p_n}\}$ *is a bounded sequence.*

Proof. Fix n and consider the factorization

$$(L(l_1^n, l_2^n), \alpha) \xrightarrow{J_n} (L_0(l_1, l_2), \alpha) \xrightarrow{P_i} E_i \xrightarrow{T_i} (L_0(l_1, l_2), \alpha) \xrightarrow{Q_n} (L(l_1^n, l_2^n), \alpha)$$

where $i \in I_n$, J_n and T_i are the inclusion operators, P_n and Q_n the natural projections. Let $R_i = Q_n T_i P_i J_n$; then $r(R_i) \leq \dim(E_i) \leq p_n$ and $\sum_{i \in I_n} R_i(x) = x$ for all $x \in L(l_1^n, l_2^n)$. Then

$$\begin{aligned}
\infty &> \sup_{\pm, N} \left\| \sum_{i=1}^{N} \pm P_i \right\| \\
&\geq \sup_{\pm} \left\| \sum_{i \in I_n} \pm P_i \right\| \\
&\geq \sup_{\pm} \left\| \sum_{i \in I_n} \pm R_i \right\| \\
&\geq p_n^{-1/2} \sup_{\pm} \left\| \sum_{i \in I_n} \pm \sqrt{r(R_i)}\, R_i \right\| \\
&\geq p_n^{-1/2} l((L(l_1^n, l_2^n), \alpha)) \\
&\geq p_n^{-1/2} \alpha_n \pi^{-1};
\end{aligned}$$

the assertion is established.

Remarks. (1) Similar results may be obtained for the spaces considered in Remarks (1), (2).

(2) If $B\colon l_1^n \to l_2^n$ then for any ideal norm α,

$$\alpha(B) \leq i_1(B) = i_1(B') = \pi_1(B') \leq \|B\| \pi_1(l_2^n) \leq \sqrt{\pi n/2}\, \|B\|,$$

and if $j\colon l_1^n \to l_2^n$ is the inclusion map

$$\max\{i_1(B)/\|B\|\colon B \in L(l_1^n, l_2^n)\} \geq i_1(j) \geq \operatorname{trace}(jj^{-1})/\|j^{-1}\| = n/\sqrt{n} = \sqrt{n}.$$

Therefore $\sqrt{n} \leq \sup\{\alpha_n;\ \alpha \text{ is an ideal norm}\} \leq \sqrt{\pi n/2}$.

The following natural question arises:

Problem 2. If $r, p \notin \{1, 2, \infty\}$ characterize all ideal norms α (if any exist) such that if $E_n = (L(l_r^n, l_p^n), \alpha)$ then $\sup_n l(E_n) < \infty$.

The cases $\{r, p\} = \{1, 2\}$ or $\{2, \infty\}$ have been solved here. The cases $\{r, p\} = \{1, \infty\}$ or $\{2, 2\}$ were established in [8].

If $r, p \notin \{1, 2, \infty\}$ and $l_{r'}$ is an SQL_p-space it follows by the results of Section 2, that ν_p is (the only ideal norm up to equivalence) equivalent to Δ_p on $l_{r'} \otimes l_p$, and clearly the unconditional basis constant of $l_{r'} \hat{\otimes}_{\Delta_p} l_p$ is equal to 1. Problem 1 is open even in this case. However, if $l_{r'}$ is not an SQL_p-space, even the existence of an ideal norm α such that $\sup_n l(E_n) < \infty$ is open.

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