A CHARACTERIZATION OF $PSL(4, q)$, q EVEN, $q > 4$

BY

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1. Introduction

We prove the following result:

THEOREM. *Let G be a group with the same character table as $PSL(4, q)$, q even, $q > 4$. The $G \simeq PSL(4, q)$.*

The argument hinges on properties of G derived from the class algebra in Section 3 which, taken in conjunction with Suzuki's work on (C)-groups in [8] and [9], enable us to obtain three subgroups of G isomorphic to $SL(2, q)$ satisfying the conditions of K. W. Phan's characterization of special linear groups in [7].

The theorem has already been proved by different methods when $q = 2$ ($PSL(4, 2) \simeq A_8$; see [4]) and when $q = 4$ (see [6]). In the present proof, all results up to and including (6.1) can be obtained for the case $q = 4$ with a little extra difficulty. But Phan's theorem in [7], which excludes the case $q = 4$, is not so easily adaptable.

The reader is referred to Section 2 of [4] where techniques of obtaining group-theoretical information from a character table are discussed: results 2.1–2.8 in [4] will be used frequently in the present paper and will be referred to henceforth as A2.1–A2.8. Most of the notation is standard; if $g \in G$ then $o(g)$ denotes the order of g and if n is a positive integer then $\pi(n)$ denotes the set of primes dividing n.

2. Products of transvections in $SL(4, q)$

A *transvection* in $SL(4, q)$ is a conjugate of

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}.$$

The $q - 1$ nonidentity elements in the center of the Sylow 2-subgroup

$$\left\{\begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{pmatrix}\right\}$$

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of $SL(4, q)$ are all transvections. One can verify the following statement concerning the class algebra of $SL(4, q)$ by vector space calculations of the kind exhibited in Section 2 of [5].

(2.1) *There are $q + 2$ nonidentity classes of $SL(4, q)$ containing products of two transvections, represented by the elements*

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & & \\ \alpha & \alpha + 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$0 \neq \alpha \in GF(q).$$

The number of ways in which the element

$$\begin{pmatrix} 1 & 1 & & \\ \alpha & \alpha + 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

can be expressed as a product of two transvections is $q + 1 - N_\alpha$ where N_α is the number of solutions in $GF(q)$ of the quadratic equation $x^2 + \alpha x + \alpha = 0$.

3. The class algebra of G

By A2.8, G and $PSL(4, q) = SL(4, q)$ have isomorphic class algebras. In particular, if $\mathscr{C}$ is the class of G corresponding in the character table to the class of transvections in $SL(4, q)$, and if $\mathscr{D}$, $\mathscr{L}$, $\mathscr{M}_\alpha (\alpha \in GF(q)^*)$ correspond to the respective classes of the other elements listed in (2.1), then

$$\mathscr{C}^2 = \{1\} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{L} \cup \{\mathscr{M}_\alpha \mid \alpha \in GF(q)^*\}$$

and if $z_\alpha \in \mathscr{M}_\alpha$ and $t \in \mathscr{C}$ then

$$\#(z_\alpha = t^\circ t^\circ)_G = q + 1 - N_\alpha.$$

The following table is compiled by computing the orders of the elements of $SL(4, q)$ listed in (2.1), and their centralizers, and using A2.3 and A2.5:

Class	Centralizer order	Primes dividing order of an element
$\mathscr{C}$	$q^6(q^2 - 1)(q - 1)$	2
$\mathscr{D}$	$q^5(q^2 - 1)$	2
$\mathscr{L}$	$q^4(q - 1)$	2
$\mathscr{M}_\alpha$	$q(q^2 - 1)(q - 1)(q + 1 - N_\alpha)$	odd

It is evident from centralizer orders and a remark in Section 2 that $\mathscr{C}$ is the only class of G containing 2-elements central in Sylow 2-subgroups of G; therefore $\mathscr{C}$ contains involutions.

3.1) *For each $\alpha \in GF(q)^*$ the element $z_\alpha \in \mathscr{M}_\alpha$ has the same order as*

$$\begin{pmatrix} 1 & 1 & & \\ \alpha & \alpha + 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and is conjugate to its inverse but to no other power of itself.

Proof. In $SL(4, q)$ the classes of elements of odd order which are products of two transvections are in 1-1 correspondence with the distinct classes of elements of odd order in the subgroup

$$\left\{ \left(\begin{array}{c|cc} X & & \\ \hline & 1 & \\ & & 1 \end{array} \right) : \det X = 1 \right\} \approx SL(2, q).$$

Each such element lies inside a cycle of order $q \pm 1$ and is conjugate to its inverse but to no other power. Further, elements of a given odd order generate conjugate cyclic subgroups.

Consequently, once we establish the orders of the elements $z_\alpha \mid \alpha \in GF(q)^*$ it will follow from A2.6 that each is conjugate to its inverse but to no other power.

Let $\varepsilon = \pm 1$ and suppose $q + \varepsilon = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. It is enough to identify the orders of the p_i-elements in $\mathscr{C}^2$ for each i. The orders of the elements of composite order can then progressively be determined using A2.7.

Suppose the element

$$\begin{pmatrix} 1 & 1 & & \\ \beta & \beta + 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

of $SL(4, q)$ has order $q + \varepsilon$ and let $o(z_\beta) = n$. If $z_\beta = t_1 t_2$ is a solution of the equation $(z_\beta = t^\circ t^\circ)_G$ then $\langle t_1, t_2 \rangle \approx D_{2n}$ contains n solutions, whence $n \leq q + \varepsilon$. By A2.5, $\pi(n) \neq \pi(q + \varepsilon)$. Now $z_\beta^{n+1} = z_\beta$ and by A2.6,

$$\begin{pmatrix} 1 & 1 & & \\ \beta & \beta + 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

is conjugate to its $(n + 1)$st power, whence $n \equiv 0$ or $-2 \bmod q + \varepsilon$. Therefore $n = q + \varepsilon$. Thus we identify the order of the p_i-part of z_β for each i. It follows from A2.6 and previous remarks that the classes of p_i-elements in $\mathscr{C}^2$ form a_i orbits under the action of field automorphisms on the character table, and that the elements in each orbit have the same order as they generate conjugate cycles. If we number the orbits so that the jth orbit contains elements which in the $SL(4, q)$ case have order p_i^j then we have

(i) the a_ith orbit contains elements of order $p_i^{a_i}$, and
(ii) an element in the jth orbit is not conjugate to its $(p_i^k + 1)$st power if $k < j$ (A2.6).

In view of this there is only one way to assign orders to the orbits and the result is proved.

(3.2) *Let N be a subgroup of G all of whose 2-elements lie in $\mathscr{C}$. Then a Sylow 2-subgroup of N is either normal or a T.I. set. In the latter case, N has a single class of involutions.*

Proof. We may assume that N has two distinct Sylow 2-subgroups V and W. Suppose $V \cap W \cap \mathscr{C}$ is not empty. If $t \in V \cap W \cap \mathscr{C}$ and $v \in V \cap \mathscr{C}$, $w \in W \cap \mathscr{C}$ and $o(vw) = k$ then $\langle t, v, w\rangle \approx D_{2k}$ or $D_{2k} \times Z_2$, the former only if $k = 4$ and $(vw)^2 = t$. If k is odd, $t(vw) = (tv)w \in \mathscr{C}^2$ has twice odd order, a contradiction. If k is a power of 2 then by hypothesis $vw \in \mathscr{C}$ and V, W commute, a contradiction. Therefore V and W intersect trivially. Let $x, y \in N \cap \mathscr{C}$. If $o(xy)$ is odd, then x is conjugate to y in $\langle x, y\rangle$. If not, then $\langle x, y\rangle$ is contained in a Sylow 2-subgroup T of N. Let $z \in N \cap \mathscr{C}\backslash T$. Then x is conjugate to z in $\langle x, z\rangle$ and z is conjugate to y in $\langle z, y\rangle$. It follows that all 2-elements in N are conjugate in N.

4. A subgroup of G isomorphic to $SL(2, q) \times SL(2, q) \times Z_{q-1}$

There is a self-centralizing element A in $GL(2, q)$ of order $q^2 - 1$ which is conjugate to A^q but to no other power of A. Let g belong to the class in G corresponding in the character table to the class in $SL(4, q)$ of

$$\left(\begin{array}{cc|c} \lambda & & \\ & \lambda & \\ \hline & & A \end{array}\right)$$

where $\lambda^2 = (\det A)^{-1}$. By considering the centralizer in $SL(4, q)$ and applying A2.3 we have $|C_G(g)| = q(q^2 - 1)^2$. Now

$$\left(\begin{array}{cc|c} \lambda & & \\ & \lambda & \\ \hline & & A \end{array}\right)$$

commutes with the transvection

$$\left(\begin{array}{cc|cc} 1 & & & \\ 1 & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

Hence by A2.7 there exists $t \in \mathscr{C}$ commuting with g and the class of gt corresponds to the class of

$$\left(\begin{array}{cc|c} \lambda & & \\ \lambda & \lambda & \\ \hline & & A \end{array}\right).$$

It follows that $|C_G(gt)| = q(q^2 - 1)$.

(4.1) *The order of g is $q^2 - 1$ and $C_G(g) = L_1 \times \langle g \rangle$ where $L_1 \approx SL(2, q)$.*

Proof. If $o(g) = n$ then, by A2.5, $\pi(n) = \pi(q^2 - 1)$. As $g \in C_G(gt)$ we also have $n \mid q^2 - 1$. Since $g^{n+1} = g$,

$$\left(\begin{array}{cc|c} \lambda & & \\ & \lambda & \\ \hline & & A \end{array}\right)$$

is conjugate to its $(n + 1)$st power (A2.6). Therefore $n + 1 = q$ or q^2. It follows that $n = q^2 - 1$.

Since

$$\left(\begin{array}{cc|c} \lambda & & \\ & \lambda & \\ \hline & & A \end{array}\right)$$

commutes with transvections but no other 2-elements in $SL(4, q)$, the 2-elements of $C_G(g)$ lie in $\mathscr{C}$ (A2.7). By (3.2), a Sylow 2-subgroup of $C_G(g)$ is normal or a T.I. set. Since

$$|C_G(g): C_G(gt)| = q^2 - 1$$

the latter is true and $C_G(g)$ has $q^2 - 1$ involutions. It follows that in $\bar{C} = C_G(g)/\langle g \rangle$ (of order $q(q^2 - 1)$) there are $q + 1$ trivially intersecting Sylow 2-subgroups. By a result of Suzuki in [8], $\bar{C} \approx SL(2, q)$. The multiplier of $SL(2, q)$ is trivial when q is even, $q > 4$ (see [2]) whence $C_G(g) \approx SL(2, q) \times \langle g \rangle$.

It follows from the structure of $SL(2, q)$ that the elements of odd order in L_1 lie in $(\mathscr{C} \cap L_1)^2$. The $q - 1$ classes of elements of odd order in L_1 are, by (3.1), in 1-1 correspondence with $\{\mathscr{M}_\alpha \mid \alpha \in GF(q)^*\}$; there is no fusion in G of distinct classes in L_1. Let b be an element of L_1 of order $q + 1$, and let $u = g^{q+1}$.

(4.2) $C_G(b) = \langle b \rangle \times L_2 \times \langle u \rangle$ *where* $L_2 \approx SL(2, q)$.

Proof. The class of b is one of the $\mathscr{M}_\alpha$ and the class of u is known since u is the $(q - 1)$-part of g; therefore the class of bu is known (A2.7). By considering what happens in $SL(4, q)$ we can deduce that:

(i) $|C_G(b)| = q(q^2 - 1)^2$.
(ii) $|C_G(bu)| = |C_G(b)|$, i.e., $C_G(bu) = C_G(b)$.
(iii) $C_G(b)$ meets $\mathscr{C}$ but no other class of 2-elements of G.
(iv) If $s \in C_G(b) \cap \mathscr{C}$ then $|C_G(bs)| = q(q^2 - 1)$.

Arguing as in (4.1) we deduce that $C_G(b)/\langle bu\rangle$ has order $q(q^2 - 1)$ and $q + 1$ T.I. Sylow 2-subgroups, and that $C_G(b) \approx SL(2, q) \times \langle bu\rangle$ as required.

The class of G containing u corresponds in the character table to the class of $SL(4, q)$ containing

$$\left(\begin{array}{cc|cc} \mu & & & \\ & \mu & & \\ \hline & & \mu^{-1} & \\ & & & \mu^{-1} \end{array}\right)$$

where $\langle \mu\rangle = GF(q)^*$. The centralizer in $SL(4, q)$ of this element is isomorphic to $SL(2, q) \times SL(2, q) \times Z_{q-1}$.

(4.3) $C_G(u) = L_1 \times L_2 \times \langle u\rangle$.

Proof. Since $L_1 \cap (L_2 \times \langle u\rangle) \subseteq L_1 \cap C_G(b)$, it is plain that L_1 and $L_2 \times \langle u\rangle$ intersect trivially. That their product equals $C_G(u)$ is immediate from order considerations (A2.3). It remains to be shown that $L_i \lhd C_G(u)$ $(i = 1, 2)$.

In $SL(4, q)$ there is only one class of elements whose 2-parts lie in $\mathscr{C}$ and whose 2′-parts are conjugate to

$$\left(\begin{array}{cc|cc} \mu & & & \\ & \mu & & \\ \hline & & \mu^{-1} & \\ & & & \mu^{-1} \end{array}\right).$$

The centralizer of such an element has order $q^2(q^2 - 1)(q - 1)$. It follows from A2.7 and A2.3 that if $t \in L_i \cap \mathscr{C}$ then

$$|C_G(u)\colon C_G(ut)| = q^2 - 1 = |L_i\colon C_{L_i}(t)|.$$

Thus the involutions in L_i form a conjugate class of $C_G(u)$ and since they generate L_i, L_i is normal as required.

(4.4) *The class $\mathscr{D}$ consists of involutions.*

Proof. It was shown in Section 3 that $\mathscr{D}$ is a class of 2-elements. Since a Sylow 2-subgroup of $C_G(u)$ is elementary abelian it suffices to prove that $\mathscr{D}$ meets $C_G(u)$. This is immediate from A2.7 because in $SL(4, q)$,

$$\left(\begin{array}{cc|cc} 1 & & & \\ 1 & 1 & & \\ \hline & & 1 & \\ & & 1 & 1 \end{array}\right) \text{ commutes with } \left(\begin{array}{cc|cc} \mu & & & \\ & \mu & & \\ \hline & & \mu^{-1} & \\ & & & \mu^{-1} \end{array}\right)$$

(4.5) $N_G(L_i) = L_1 \times L_2 \times \langle u\rangle$ $(i = 1, 2)$.

Proof. By (4.2), $C_G(L_1) = L_2 \times \langle u\rangle$. Therefore $g \in L_2 \times \langle u\rangle$. We may write $g = vu$, $v \in L_2$ of order $q + 1$. Since $v \in \mathscr{C}^2$ the centralizer order of v is given in Section 3. In fact, $C_G(v) = L_1 \times \langle v \rangle \times \langle u \rangle$. Hence $C_G(L_2) = L_1 \times \langle u \rangle$. By (3.1) the elements of odd order in L_i are conjugate to their inverses but to no other powers. The subgroup of the automorphism group of $SL(2, q)$ *not* inducing further conjugation in the $(q \pm 1)$-cycles is $SL(2, q)$ itself. Hence $N_G(L_i)/C_G(L_i) \approx L_i$. The result follows.

(4.6) *There is an involution $\tau \in \mathscr{D}$ such that*

$$N_G(\langle u \rangle) = (L_1 \times L_2 \times \langle u \rangle) \cdot \langle \tau \rangle,$$

where $u^\tau = u^{-1}$ and $L_1^\tau = L_2$.

Proof. Since

$$\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right) \text{ inverts } \left(\begin{array}{c|c} \mu I & \\ \hline & \mu^{-1} I \end{array}\right)$$

the latter element can be written as a product of two conjugates in $SL(4, q)$ of the former. Then by A2.8, $u \in \mathscr{D}^2$. In particular there is an involution $\tau \in \mathscr{D}$ inverting u. We can show by familiar arguments that u is conjugate to u^{-1} but to no other power. Thus $N_G(\langle u \rangle) = C_G(u) \cdot \langle \tau \rangle$. Now $L_1 \times L_2$ is a characteristic subgroup of $C_G(u)$ and so by the Krull-Schmidt theorem, τ interchanges or normalizes the L_i $(i = 1, 2)$. Since $\tau \notin N_G(L_i)$ (by (4.5)) the result is proved.

(4.7) (i) *An element of odd order in $\mathscr{C}^2$ lies in a unique conjugate of L_1.*
(ii) *If $c \in L_1$ has odd order and belongs to the cyclic subgroup $\langle c_0 \rangle$ of order $q \pm 1$ in L_1, the centralizer of c in G is $\langle c_0 \rangle \times L_2 \times \langle u \rangle$.*

Proof. Part (ii) follows directly from the order of $C_G(c)$ given in Section 3. From the equation $|L_1 : C_{L_1}(c)| \cdot |G : N_G(L_1)| = |G : C_G(c)|$ it follows that two distinct conjugates of L_1 cannot both contain a given conjugate of c.

5. A subgroup of G isomorphic to $GL(3, q)$

In $SL(4, q)$ every element of the subgroup

$$\left\{\left(\begin{array}{c|cc} X & & \\ \hline & 1 & \\ & & 1 \end{array}\right)\right\} \approx SL(2, q)$$

commutes with the element

$$\begin{pmatrix} \mu & & & \\ & \mu & & \\ & & \mu & \\ & & & \mu^{-3} \end{pmatrix}$$

which has order $q - 1$ and is not conjugate to any proper power of itself. It follows from A2.5, A2.7 and remarks in Sections 3–4 that there is an element a in the corresponding class of G which commutes with the element $b \in L_1$

defined in Section 4. The following facts also derive from equivalent statements about $SL(4, q)$. Let $H = C_G(a)$. Then

$$|H| = q^3(q^3 - 1)(q^2 - 1)(q - 1);$$

the 2-elements in H lie in $\mathscr{C} \cup \mathscr{L}$; if $x, y \in H \cap \mathscr{C}$ (respectively, $H \cap \mathscr{L}$) then ax and ay are conjugate in G and $|C_G(ax)| = q^3(q - 1)^2$ (respectively, $q^2(q - 1)$).

In particular, $H \cap \mathscr{C}$ and $H \cap \mathscr{L}$ are classes of H, and the order of a divides $q - 1$. It can be shown that the order of a is exactly $q - 1$ by the methods used in proving (3.1).

(5.1) *$L_1 \leq H$; there is an element $w \in L_2$ of order $q - 1$ such that $N_H(L_1) = L_1 \times \langle w \rangle \times \langle u \rangle$.*

Proof. By definition, $a \in C_G(b) = \langle b \rangle \times L_2 \times \langle u \rangle$. Therefore, there is an element w of order $q - 1$ in L_2 such that $a \in \langle w \rangle \times \langle u \rangle$. It follows that L_1 centralizes a and

$$N_H(L_1) = H \cap (L_1 \times L_2 \times \langle u \rangle) = C_{L_1 \times L_2 \times \langle u \rangle}(a) = L_1 \times \langle w \rangle \times \langle u \rangle.$$

(5.2) *H is a (C)-group.*

Proof. $H \cap \mathscr{C}$ and $H \cap \mathscr{L}$ are the only classes of 2-elements in H. The centralizer of an element in $H \cap \mathscr{L}$ has order $q^2(q - 1)$ and is thus 2-closed. Let $t \in L_1 \cap \mathscr{C}$. We show that $K = C_H(t)$ is 2-closed by proving that if p is an odd prime dividing $|K|$, a Sylow p-subgroup of K has a normal p-complement; the intersection of these complements for all such p is the (necessarily unique) Sylow 2-subgroup of K.

$|K| = q^3(q - 1)^2$. If $C_{L_1}(t) = V$ then $V \times \langle w \rangle \times \langle u \rangle$ is contained in K. Let P be the Sylow p-subgroup of K contained in $\langle w \rangle \times \langle u \rangle$. By the Burnside Transfer Theorem (cf. [3, Theorem 7.4.3] for instance) it is enough to show that $P \leq Z(N_K(P))$. Now if $x \in N_K(P)$ then $(L_1 \times P)^x = L_1^x \times P$, i.e., $L_1^x \leq C_G(P)$. By (4.7(ii)) (applied to L_2) $C_G(P) = L_1 \times \langle w \rangle \times \langle u \rangle$ whence $L_1^x = L_1$, i.e., $x \in N_H(L_1) = C_G(P)$ by (5.1). The result is proved.

(5.3) *If $q \equiv 1$ (3) then H does not contain a subgroup isomorphic to $PSL(3, q)$.*

Proof. Let ω be an element of $GF(q)$ of order 3. In $PSL(3, q)$, the element

$$\begin{bmatrix} \omega & & \\ & \omega^2 & \\ & & 1 \end{bmatrix}$$

has a nonabelian centralizer of order $(q - 1)^2$ $(q \ddagger 4)$. Suppose that H contains a subgroup isomorphic to $PSL(3, q)$. It follows easily that there must be an element of order 3 in $H \cap \mathscr{C}^2$ whose centralizer contains a nonabelian subgroup of order $(q - 1)^2$. But by (4.7) the centralizer in G of such an element is isomorphic to $SL(2, q) \times Z_{q-1} \times Z_{q-1}$ which has no nonabelian subgroup of the required order.

(5.4) $H \approx GL(3, q)$.

Proof. It is clear from the sizes of the classes $H \cap \mathscr{C}$ and $H \cap \mathscr{L}$ that H has no nontrivial normal 2-subgroup. By a theorem of Suzuki in [9], there are normal subgroups H_1, H_2 in H such that $H_2 \leq Z(H_1)$, H/H_1 and H_2 have odd order and H_1/H_2 is a simple (C)-group or the linear fractional group M_9 over the noncommutative near-field of 9 elements. Since H_1 contains all the 2-elements of H, $H_1 \geq L_1$ and $Z(H_1) \leq C_{H_1}(L_1) = \langle w \rangle \times \langle u \rangle$. Also if $y \in H \cap \mathscr{L}$ then $Z(H_1)$ is contained in $C_H(y)$ which is the product of $\langle a \rangle$ and a 2-group. Therefore $Z(H_1) \leq \langle a \rangle$, i.e., $H_2 \leq \langle a \rangle$. Since H_1/H_2 contains a full Sylow 2-subgroup of H and $q > 4$, $H_1/H_2 \not\approx M_9$; thus H_1/H_2 is simple and $H_2 = H_1 \cap \langle a \rangle$. The order of $H_1/H_2 \approx H_1\langle a \rangle/\langle a \rangle$ is divisible by $q^3(q^2 - 1)$ (because $L_1 \leq H_1$ and $|H/H_1|$ is odd) and divides $|H/\langle a \rangle| = q^3(q^3 - 1)(q^2 - 1)$. The only group in Suzuki's list of simple (C)-groups having this property is $PSL(3, q)$. The multiplier of $PSL(3, q)$ can be found in Feit's paper [2]. If $q \equiv 2\ (3)$, $PSL(3, q) = SL(3, q)$ has trivial multiplier, i.e., $H_1 \approx SL(3, q) \times H_2$ and it follows from order considerations that $H \approx SL(3, q) \times \langle a \rangle \approx GL(3, q)$. If $q \equiv 1\ (3)$ the multiplier of $PSL(3, q)$ has $2'$-part Z_3 and so

$$H_1 \approx PSL(3, q) \times H_2 \quad \text{or} \quad H_1 \approx SL(3, q) \times H_2' \quad \text{where } H_2 \approx H_2' \times Z_3.$$

By (5.3) the former cannot occur. It follows that H has a normal subgroup

$$J \approx SL(3, q) \circ \langle a \rangle \approx \{X \in GL(3, q) \mid \det X \text{ is a cube}\}$$

of index 3.

Clearly $L_1 < J$. If $t \in L_1 \cap \mathscr{C}$ then by (5.1), $C_H(t) > \langle w \rangle \times \langle u \rangle \approx Z_{q-1} \times Z_{q-1}$. But $|C_J(t)| = [q^3(q - 1)^2]/3$. Hence there exists $h \in \langle w \rangle \times \langle u \rangle \backslash J$ and $H = \langle J, h \rangle$. Now h acts on J (in particular on the $SL(3, q)$ contained in J) and centralizes L_1 (see (5.1)). By considering the automorphisms of $SL(3, q)$ (cf. [1] for example) it is easily checked that in these circumstances h induces a diagonal automorphism on the $SL(3, q)$. Also $[h, a] = 1$ (we can prove that $Z(H) = \langle a \rangle$ by the method used to prove $Z(H_1) \leq \langle a \rangle$ above). It follows that $H \approx GL(3, q)$, as required.

6. Identification of G

(6.1) *There is a conjugate L_3 of the subgroup L_1 and an element $v \in L_3$ of order $q - 1$ such that*

$$\langle L_1, L_3 \rangle \approx \langle L_2, L_3 \rangle \approx SL(3, q), \qquad \langle w, v \rangle \approx \langle w^\tau, v \rangle \approx Z_{q-1} \times Z_{q-1},$$

$$\langle L_1, v \rangle \approx \langle L_2, v \rangle \approx \langle L_3, w \rangle \approx \langle L_3, w^\tau \rangle \approx GL(2, q)$$

where τ is as defined in (4.6).

Proof. In the isomorphism $\phi: H \to GL(3, q)$ we may take $\phi(L_1) = L$ where

$$L = \left\{ \left(\begin{array}{c|c} X & \\ \hline & 1 \end{array} \right) \middle| \det X = 1 \right\}$$

by combining ϕ with a conjugation map if necessary. Now $N(L) = L \times C(L)$ where $C(L)$ is the subgroup

$$\left\{ \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \alpha \end{pmatrix} \middle| \lambda\alpha \neq 0 \right\}$$

contained in the diagonal subgroup D of $GL(3, q)$. It follows that

$$\phi(C_H(L_1)) = \phi(\langle w \rangle \times \langle u \rangle) \leq D.$$

Consider the subgroup $H \cap H^\tau$ of H. $H \cap H^\tau = C_H(a^\tau)$ and

$$|H \cap H^\tau| \geq |H|^2/|G| = \{(q^2 + q + 1)/(q^2 + 1)\}(q - 1)^3.$$

Now $a^\tau \in \langle w^\tau \rangle \times \langle u \rangle$ and $w^\tau \in L_2^\tau = L_1$. Replacing τ by an element $\eta\tau (\eta \in L_1)$ if necessary we may assume that

$$\phi(w^\tau) = \begin{pmatrix} \mu & & \\ & \mu^{-1} & \\ & & 1 \end{pmatrix} \in D.$$

Already $\phi(u) \in D$ so that now $\phi(a^\tau) \in D$. The centralizer of an element in $D^\#$ is either D or a subgroup $L' \times Z_{q-1} \times Z_{q-1}$ containing D, where L' equals either

$$\left\{ \left(\begin{array}{c|c} 1 & \\ \hline & X \end{array} \right) : \det X = 1 \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} \alpha & & \beta \\ & 1 & \\ \gamma & & \delta \end{pmatrix} : \alpha\delta + \beta\gamma = 1 \right\}$$

and is conjugate in $GL(3, q)$ to L. It follows from the inequality derived for $|H \cap H^\tau|$ that there is a conjugate L_3 of L_1 in G such that

$$H \cap H^\tau = L_3 \times Z_{q-1} \times Z_{q-1}.$$

In addition, $\langle L_1, L_3 \rangle \approx SL(3, q)$. Since τ normalizes $H \cap H^\tau$, $L_3^\tau = L_3$ and

$$\langle L_2, L_3 \rangle = \langle L_1, L_3 \rangle^\tau \approx SL(3, q).$$

Now $\langle w^\tau \rangle \times \langle w \rangle \times \langle u \rangle$ centralizes both a and a^τ and by order considerations, $L_3 \cap (\langle w^\tau \rangle \times \langle w \rangle \times \langle u \rangle)$ has order $q - 1$ and is thus cyclic. Say

$$L_3 \cap (\langle w^\tau \rangle \times \langle w \rangle \times \langle u \rangle) = \langle v \rangle.$$

By the nature of the construction, $\langle w^\tau \rangle \times \langle w \rangle \times \langle u \rangle$ normalizes L_1, L_2, and L_3, each of whose normalizer in H is isomorphic to $SL(2, q) \times Z_{q-1} \times Z_{q-1}$.

The relations

$$\langle w, v\rangle \approx \langle w^{\tau}, v\rangle \approx Z_{q-1} \times Z_{q-1},$$

$$\langle L_3, w\rangle \approx \langle L_3, w^{\tau}\rangle \approx GL(2, q), \qquad \langle L_1, v\rangle \approx \langle L_2, v\rangle \approx GL(2, q)$$

follow, since τ normalizes $\langle v\rangle$ and by (4.7(ii)) no power of w or w^{τ} lies in L_3 and no power of v lies in L_1 or L_2. (Q.E.D)

The conditions on the subgroups L_1, L_2, L_3 derived in (6.1) are precisely those in the hypothesis of K. W. Phan's Theorem 1 in [7]. The conclusion is that (since $q > 4$) G is a homomorphic image of $SL(4, q)$. But $SL(4, q)$ is simple. Hence, we get the final result, $G \simeq SL(4, q) = PSL(4, q)$.

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