SPECTRAL INVARIANTS OF THE SECOND VARIATION OPERATOR

BY

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Introduction

We study the asymptotic expansion of the heat kernel for the second variation operator $\square$ which arises in the theory of minimal submanifolds. Specifically the first two terms in the expansion are calculated. If a manifold M is isometrically immersed in a manifold of constant curvature then there is a spectral condition determining whether or not M is totally geodesic. A similar result holds for complex submanifolds in a manifold of constant holomorphic sectional curvature.

1. Asymptotic expansion for the heat equation

In this section we summarize some known results concerning the asymptotic expansion of the heat kernel for Riemannian manifolds. The reader is referred to [1] for more details.

Let $V \to M$ be a smooth real r-dimensional vector bundle over the compact Riemannian manifold M of dimension m. For $D\colon \Gamma(V) \to \Gamma(V)$ a second order differential operator with leading order symbol given by the metric tensor, $\exp(-tD)$ is well defined when $t > 0$. Furthermore

$$\exp(-tD)(f)(x) = \int_M K(t, x, y, D)(f)\, dvol(y),$$

where $K(t, x, y, D)$ is an endomorphism from V_y, the fiber of V over y, to V_x.

When $K(t, x, y, D)$ is restricted to the diagonal $y = x$ it has an asymptotic expansion as $t \downarrow 0$, of the form

$$K(t, x, x, D) \sim \sum_{n=0}^{\infty} E_n(x, D) t^{(n-m)/2}.$$

The endomorphisms $E_n(x, D)$ are local invariants determined in any coordinate patch by the derivatives of the coefficients of D. Let $B_n(x, D)$ denote the trace of $E_n(x, D)$.

The asymptotic expansion is particularly interesting when V has an inner product and D is self-adjoint with respect to this inner product. Let $\{\lambda_i, \phi_i\}$ be a spectral resolution into smooth orthonormal eigensections ϕ_i. Then

$$K(t, x, y, D) = \sum_{i=1}^{\infty} \exp(-t\lambda_i)\phi_i(x) \otimes \phi_i(y)$$

Received October 28, 1975.

[1] This research was partially supported by a National Science Foundation grant.

and consequently

$$\sum_{i=1}^{\infty} \exp(-t\lambda_i) \sim \sum_{n=0}^{\infty} \left(\int_M B_n(x, D)\, dvol \right) t^{(n-m)/2}.$$

This formula shows that the integrals $\int_M B_n(x, D)\, dvol$ are determined by the spectrum. The calculation of these invariants for the second order operators D arising in Riemannian geometry is a topic of current research activity. McKean and Singer [2] studied the heat equation for the Laplace operator Δ acting on functions and obtained in particular the following:

THEOREM 1.1 (McKean-Singer). *Let Δ denote the Laplace operator of M acting on functions. Then*

$$B_0(x, \Delta) = (4\pi)^{-m/2}, \quad B_2(x, \Delta) = (4\pi)^{-m/2}(\tau/6),$$

where $\tau = \sum_{i,j} R_{ijij}$ is the scalar curvature of M and m is the dimension of M.

Gilkey [1] developed a systematic method for calculating local spectral invariants. Suppose we are given a connection ∇ on V; $\nabla: \Gamma(V) \to \Gamma(V \otimes T^*M)$. Since M is a Riemannian manifold there is a natural connection ∇_g^* on T^*M. Denote by D_∇ the second order operator defined by the composition:

$$\Gamma(V) \xrightarrow{\nabla} \Gamma(V \otimes T^*M) \xrightarrow{\nabla \otimes 1 + 1 \otimes \nabla_g^*} \Gamma(V \otimes T^*M \otimes T^*M) \xrightarrow{-1 \otimes g} \Gamma(V),$$

where g: $T^*M \otimes T^*M \to R$ is contraction via the Riemannian metric of M. Now suppose that $D = D_\nabla - E$, where E: $\Gamma(V) \to \Gamma(V)$ is an endomorphism. Then one has:

THEOREM 1.2 (Gilkey), *Let D: $\Gamma(V) \to \Gamma(V)$ be of the form $D = D_\nabla - E$ for some connection ∇ on V. Then*

$$B_0(x, D) = (4\pi)^{-m/2} r, \quad B_2(x, D) = (4\pi)^{-m/2}(r\tau/6 + \mathrm{Tr}(E)),$$

where $r = \dim V$ and $\mathrm{Tr}(E)$ denotes the trace of the endomorphism E.

2. Second variation operator

This section is devoted to some preliminaries involving Riemannian immersions. A fuller account may be found in [3].

Suppose M is a Riemannian manifold of dimension m isometrically immersed in the Riemannian manifold $\overline{M}$ of dimension $\overline{m}$. The normal bundle NM is then a real $r = \overline{m} - m$ dimensional vector bundle with inner product induced by the metric on $\overline{M}$.

If $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ then $\bar{\nabla}$ induces a connection ∇ on NM via

$$\bar{\nabla}_X Y = (\nabla_X Y)^N, \quad X \in TM, \; Y \in \Gamma(NM)$$

where $(V)^N$ is the normal component of V. Let $D_\nabla \colon \Gamma(NM) \to \Gamma(NM)$ denote the second order differential operator associated to ∇ via the construction in Section 1.

The second fundamental form is a map from $TM \otimes TM \to N(M)$ defined by

$$B(X, Y) = (\bar{\nabla}_X Y)^N, \quad X \in TM, \; Y \in \Gamma(TM).$$

B is a symmetric tensor on M with values in the normal bundle. Now define a map $A \colon N(M) \to T(M) \otimes (T(M))^*$ by the equation

$$\langle A^W(X), Y \rangle = \langle B(X, Y), W \rangle, \quad X, Y \in T(M), \; W \in N(M).$$

Here $\langle \ , \ \rangle$ denotes the inner product on $T\bar{M}$. Let $S(M)$ denote the fiber bundle over M whose fiber at p is the symmetric linear transformations $(TM)_P \to (TM)_P$. Then, since $B(X, Y) = B(Y, X)$, we may regard A as a linear map $A \colon N(M) \to S(M)$. Let ${}^tA \colon S(M) \to N(M)$ denote the transpose of A.

The following lemma [3, p. 70] is well known:

LEMMA 2.1. *Let R and $\bar{R}$ be the curvature tensors in M and $\bar{M}$ respectively. Then for X, Y, Z, $W \in TM$,*

$$\langle R_{X,Y}Z, W \rangle = \langle \bar{R}_{X,Y}Z, W \rangle + \langle B(X, W), B(Y, Z) \rangle - \langle B(X, Z), B(Y, W) \rangle.$$

There is a second order differential operator $\Box$ called the second variation operator which is important in the study of minimal submanifolds. It is defined by

$$\Box V = D_\nabla V + \bar{R}(V) - {}^tAA(V), \quad V \in \Gamma(NM),$$

where $\bar{R} \colon \Gamma(NM) \to \Gamma(NM)$ is the partial Ricci transformation given by

$$\bar{R}(V) = \sum_{i=1}^{m} (\bar{R}_{e_i, V} e_i)^N, \quad V \in \Gamma(NM),$$

for $e_1, \ldots, e_m$ an orthonormal basis of TM.

3. Spectral invariants and the second variation operator

The second variation operator $\Box$ is an elliptic second order differential operator with leading symbol given by the metric tensor. Since $\Box$ is self-adjoint with respect to the inner product on NM, it has real pure point spectrum. In particular the heat kernel theory of Section 1 is applicable.

THEOREM 3.1.

$$B_0(x, \square) = (4\pi)^{-m/2} r,$$
$$B_2(x, \square) = (4\pi)^{-m/2}(r\tau/6 - \mathrm{Tr}(\bar{R}) + \|B\|^2),$$

where r is the codimension of M in $\bar{M}$.

Proof. From Theorem 1.2,

$$B_0(x, \square) = (4\pi)^{-m/2} r,$$
$$B_2(x, \square) = (4\pi)^{-m/2}(r\tau/6 - \mathrm{Tr}(\bar{R}) + \mathrm{Tr}({}^tAA)).$$

Letting $e_1, \ldots, e_r$ be an orthonormal basis of $(NM)_p$ for any $p \in M$, we have

$$\mathrm{Tr}({}^tAA) = \sum_{i=1}^{r} \langle {}^tAAe_i, e_i\rangle = \sum_{i=1}^{r} \langle Ae_i, Ae_i\rangle = \|A\|^2 = \|B\|^2$$

at p. Thus $B_2(x, \square) = (4\pi)^{-m/2}(r\tau/6 - \mathrm{Tr}(\bar{R}) + \|B\|^2)$.

There are some interesting applications if the ambient manifold $\bar{M}$ has constant curvature or if $\bar{M}$ is complex and has constant holomorphic sectional curvature.

THEOREM 3.2. *Let $\bar{M}$ have constant curvature c. Then*

$$B_0(x, \square) = (4\pi)^{-m/2} r,$$
$$\begin{aligned} B_2(x, \square) &= (4\pi)^{-m/2}(+\, mrc + r\tau/6 + \|B\|^2) \\ &= (4\pi)^{-m/2}(+\, mrc - m(m-1)r(c/6) \\ &\quad + (r+6)\|B\|^2/6 - r\|K\|^2/6), \end{aligned}$$

where $K \in \Gamma(NM)$ denotes the mean curvature vector.

Proof. Since $\bar{M}$ has constant sectional curvature c we have $\mathrm{Tr}(\bar{R}) = -mrc$. This gives the first formula for $B_2(x, \square)$. Now Lemma 2.1 implies

$$\tau = -m(m-1)c + \|B\|^2 - \|K\|^2.$$

This yields the final formula for $B_2(x, \square)$.

COROLLARY 3.3. (i) *Let M, M′ be immersed in some $\bar{M}$ with constant curvature c and suppose M, M′ are isospectral with respect to the Laplacian Δ on functions and the second variation operator □. Then M, M′ have the same codimension. If M is totally geodesic then so is M′.*

(ii) *Let M, M′ be minimally immersed in some $\bar{M}$ of constant curvature c. If M, M′ are isospectral with respect to □ and M is totally geodesic then M′ is totally geodesic.*

Proof. (i) Since M, M' are isospectral with respect to Δ we have $m = m'$, $\int_M 1 = \int_{M'} 1$, $\int_M \tau = \int_{M'} \tau'$. Then, because M, M' are isospectral with respect to $\Box$, we have $r = r'$ from $B_0(x, \Box)$. Finally using these results and $B_2(x, \Box)$ we have $\int_M \|B\|^2 = \int_{M'} \|B'\|^2$. Since M, M' is totally geodesic if and only if $\|B\|$, $\|B'\| = 0$, this completes the proof of (i).

(ii) $m = m'$ from the leading term in the asymptotic expansion. From $B_0(x, \Box)$, $\int_M r = \int_{M'} r'$. Using $B_2(x, \Box)$,

$$(r + 6)\int_M \|B\|^2 = (r' + 6)\int_{M'} \|B'\|^2.$$

This completes the proof of (ii).

THEOREM 3.4. *Let $\overline{M}$ have constant holomorphic curvature c and suppose M is a complex submanifold of $\overline{M}$. Then*

$$B_0(x, \Box) = (4\pi)^{-m/2}r,$$
$$B_2(x, \Box) = (4\pi)^{-m/2}(+ mr(c/4) + r(\tau/6) + \|B\|^2)$$
$$= (4\pi)^{-m/2}(+ mr(c/4) + -rm(m + 2)(c/4) + (r + 6)\|B\|^2/6).$$

Proof. Recall that on a manifold of constant holomorphic curvature c we have

$$-R(X, \cdot)X = \begin{cases} 0 & \text{on } R\cdot X \\ c \times \mathrm{Id} & \text{on } R\cdot JX \\ c/4 \times \mathrm{Id} & \text{on the orthogonal complement of } R\cdot X \oplus R\cdot JX, \end{cases}$$

where J is the almost complex structure and $R\cdot V$ denotes real multiples of V.

Since M is a complex submanifold we have $\mathrm{Tr}(\overline{R}) = -mr(c/4)$. This gives the first formula for $B_2(x, \Box)$. Now applying Lemma 2.1 we find

$$\tau = -m(m + 2)(c/4) + \|B\|^2,$$

using the well known fact that $K = 0$ for a complex submanifold of a Kaehler manifold [3, p. 72]. This gives the second formula for $B_2(x, \Box)$.

COROLLARY 3.5. *Let M, M' be complex submanifolds of some $\overline{M}$ with constant holomorphic curvature c. If M, M' are isospectral with respect to $\Box$ and M is totally geodesic, then so is M'.*

The proof of Corollary 3.5 is similar to that of Corollary 3.3.

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