RANGES OF HYPONORMAL OPERATORS

BY

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It is shown that if T is a hyponormal operator on a Hilbert space H, if δ is a closed subset of the plane, and if $g: \mathbf{C}\backslash\delta \to H$ is a bounded function such that $(T - \lambda)g(\lambda) \equiv x$ for some $x \in H$, then there exists a (unique) analytic function $f: \mathbf{C}\backslash\delta \to H$ such that $(T - \lambda)f(\lambda) \equiv x$ (see Theorem 1). In case T is normal (or subnormal), the result is due to Putnam [7]; and in case T is spectral (or subspectral), the result is due to Fong and Radjabalipour [5, Lemma 2]. Actually, Putnam assumes no boundedness on g, while Fong and Radjabalipour show that the boundedness condition is necessary. (As in the case of hyponormal operators the necessity of the boundedness of g is an open question.) As an application of the above result we will show that if T is a cohyponormal operator, if S is a hyponormal operator, if W is an operator with a finite-dimensional null space, and if $WT = SW$, then T is normal (see Theorem 3). This answers a question raised by Stampfli and Wadhwa in [12, Remark to Theorem 3]; it is also a generalization of some results due to Stampfli, Wadhwa [12], Fong and Radjabalipour [5]. As byproducts we will also improve some results due to Stampfli (see Propositions 1 and 2).

From now on by an operator we mean a bounded linear transformation defined on a fixed separable Hilbert space H. The separability restriction will result in no loss of generality. The range and the null space of an operator T will be denoted by $R(T)$ and $N(T)$ respectively.

Recall that if T is normal or if the interior of the point spectrum $\sigma_p(T)$ of T is empty, then T has the single-valued extension property, i.e., there exists no nonzero, analytic, H-valued function f such that $(T - \lambda)f(\lambda) \equiv 0$. In particular every hyponormal operator has the single-valued extension property. Moreover if T has the single-valued extension property and if the manifold

$$X_T(\delta) = \{x \in H\text{: there exists an analytic function } f_x\colon \mathbf{C}\backslash\delta \to H \text{ such that } (T - \lambda)f_x(\lambda) \equiv x\}$$

is closed for some closed set δ, then $\sigma(T \mid X_T(\delta)) \subseteq \delta \cap \sigma(T)$ [3, Proposition 3.8, p. 23].

We first prove the following modest generalization of Theorem 2 of [11]. The result is known in case T has no residual spectrum.

PROPOSITION 1. *If T is hyponormal, then $X_T(\delta)$ is closed for all closed sets δ.*

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Proof. As in [11] we may and shall assume without loss of generality that T has no eigenvalue. Fix $\lambda \in \sigma(T)$ and define

$$P_\lambda: R(T - \lambda) \to \overline{R(T - \lambda)}$$

such that $P_\lambda x$ is the unique element in $\overline{R(T - \lambda)}$ satisfying $(T^* - \bar{\lambda})P_\lambda x = x$. (Note that in view of [4, Theorem 1], $R(T - \lambda) \subseteq R(T^* - \bar{\lambda})$.) For $y \in R(T - \lambda)$ we have

$$\begin{aligned}|(P_\lambda x \mid y)| &= |(x \mid (T - \lambda)^{-1}y)| \\ &= |((T - \lambda)^{-1}x \mid (T^* - \bar{\lambda})(T - \lambda)^{-1}y)| \\ &\leq \|(T - \lambda)^{-1}x\| \; \|(T - \lambda)(T - \lambda)^{-1}y\| \\ &\leq \|(T - \lambda)^{-1}x\| \; \|y\|.\end{aligned}$$

Since y is arbitrarily chosen from a dense subset of $\overline{R(T - \lambda)}$, $\|P_\lambda x\| \leq \|(T - \lambda)^{-1}x\|$ (compare [11, Lemma 1]). Now if $x \in R([T - \lambda]^n)$ for all n and if $\|x\| = 1$, it follows from the latter inequality that

$$\|(T - \lambda)^{-1}x\|^2 = (P_\lambda(T - \lambda)^{-1}x \mid x) \leq \|(T - \lambda)^{-2}x\|,$$

and, by induction on n, that

$$\|(T - \lambda)^{-n}x\|^2 \leq \|(T - \lambda)^{-(n+1)}x\| \; \|(T - \lambda)^{-(n-1)}x\|.$$

Using Lemma 2 of [11] yields $\|(T - \lambda)^{-1}x\|^n \leq \|(T - \lambda)^{-n}x\|$. (Compare [11, Lemma 3].)

Next let δ be a closed set and let $x \in X_T(\delta)$. Let f: $\mathbf{C}\backslash\delta \to H$ be an analytic function such that $(T - \lambda)f(\lambda) \equiv x$. One can use induction and differentiation to show that $x \in R([T - \lambda]^n)$ and $(T - \lambda)^{n+1}f^{(n)}(\lambda) \equiv n!\, x$ for all n. (Compare [11, Lemma 1].)

The rest of the proof is the same as in the case $\sigma_R(T) = \emptyset$ given by Stampfli in [11, Theorem 2]. His proof is based on Lemmas 3 and 4 of the same paper. In the preceding two paragraphs we proved Lemma 3 of [11]; and it is easy to see that Lemma 4 of [11] is true as long as $\sigma_p(T) = \emptyset$. The proof of the proposition is complete.

THEOREM 1. *Let T be a hyponormal operator and let δ be a closed subset of the plane. Let g: $\mathbf{C}\backslash\delta \to H$ be a bounded function such that $(T - \lambda)g(\lambda) \equiv x$ for some $x \in H$. Then $x \in X_T(\delta)$.*

Note. Since T has the single-valued extension property, there exists at most one analytic function f such that $(T - \lambda)f(\lambda) \equiv x$.

Proof of Theorem 1. In view of [7, Theorem 1] we assume without loss of generality that T has no reducing normal part; therefore T will have no invariant subspace M with area $(\sigma(T \mid M)) = 0$. (Use Putnam's inequality [6] and the fact that a normal part of a hyponormal operator is necessarily reducing.)

Let D be a fixed Cauchy domain containing δ. Choose a strictly increasing sequence $\{D_n\}$ of Cauchy domains converging to D such that $\delta \subset D_1$ and $\{|\partial D_n|\}$ is a bounded sequence. (Here $|\partial G|$ denotes the arc length of the boundary of G.) Let Γ be an open disc containing $\sigma(T)$, and let (at least formally)

$$u = (2\pi i)^{-1} \oint_{\partial(\Gamma\ D)} g(\lambda)\, d\lambda,$$

$$v_n = (2\pi i)^{-1} \oint_{\partial(D\ D_n)} g(\lambda)\, d\lambda,$$

$$w_n = x - u - v_n \quad (n = 1, 2, \ldots).$$

Since T has no eigenvalue, $g(\lambda)$ is weakly continuous [8, proof of Theorem 1]. Thus in view of [11, Scholium] the above integrals are well-defined, $u \in X_T(\mathbf{C}\backslash D)$, $v_n \in X_T(\bar{D}\backslash D_n)$, and $w_n \in X_T(\bar{D}_n)$ $(n = 1, 2, \ldots)$. Moreover $\|v_n\| \leq KL$ and $\|w_n\| \leq KL$, where K is a bound for g and L is a number greater than $|\partial\Gamma|$ and all $|\partial D_n|$. Since $\{v_n\}$ and $\{w_n\}$ are bounded, we can assume with no loss of generality that $\{v_n\}$ and $\{w_n\}$ converge weakly to some vectors v and w respectively. Since $X_T(F)$ is closed for all closed sets F (Proposition 1), it follows that $w \in X_T(\bar{D})$ and $v \in \bigcap X_T(\bar{D}\backslash D_n) = X_T(\partial D)$ (invoke the single-valued extension property of T). Thus $v = 0$ and hence $x = u + w$.

Let G be an open neighborhood of $\mathbf{C}\backslash D$ and let $y \in \bigcap_{\lambda \in G} R(T^* - \bar{\lambda})$. We show that $(g(\lambda) \mid y)$ is analytic in G. For fixed $\lambda_0 \in G$, we have

$$\begin{aligned}
&\lim_{\lambda \to \lambda_0} ((\lambda - \lambda_0)^{-1}[g(\lambda) - g(\lambda_0)] \mid y) \\
&\qquad = \lim_{\lambda \to \lambda_0} ((\lambda - \lambda_0)^{-1}[(T - \lambda)^{-1}x - (T - \lambda_0)^{-1}x] \mid y) \\
&\qquad = \lim_{\lambda \to \lambda_0} ((T - \lambda_0)^{-1}(T - \lambda)^{-1}x \mid y) \\
&\qquad = \lim_{\lambda \to \lambda_0} ((T - \lambda)^{-1}x \mid z) \\
&\qquad = ((T - \lambda_0)^{-1}x \mid z),
\end{aligned}$$

where z is any vector satisfying $y = (T^* - \bar{\lambda}_0)z$. (Here again use [8, proof of Theorem 1].) Therefore $(g(\lambda) \mid y)$ is analytic in G and thus $(u \mid y) = 0$. In particular, since $x \in \bigcap_{\lambda \notin \delta} R(T^* - \bar{\lambda})$ and $w_n \in \bigcap_{\lambda \notin \bar{D}_n} R(T^* - \bar{\lambda})$ [4, Theorem 1], we have $(u \mid x) = 0$ and $(u \mid w) = \lim (u \mid w_n) = 0$. Hence $u = 0$ and $x = w \in X_T(\bar{D})$. Since D is an arbitrary Cauchy domain containing δ, $x \in X_T(\delta)$. The proof of the theorem is complete.

As a corollary of Theorem 1 we have the following proposition.

PROPOSITION 2. *Let T be a hyponormal operator and let δ be a closed subset of the plane. Assume there exists a bounded function $g\colon \mathbf{C}\backslash\delta \to H$ such that $(T - \lambda)g(\lambda) \equiv x$ for some nonzero $x \in H$. Then T has a nonzero hyperinvariant subspace M with $\sigma(T \mid M) \subseteq \delta$. In particular if δ is a proper subset of $\sigma(T)$, then M is a nontrivial invariant subspace of T.* (Compare [11, Theorem 3].)

Proof. In view of Theorem 1, $X_T(\delta) \neq \{0\}$. Therefore $X_T(\delta)$ is a nonzero hyperinvariant subspace of T and since T has the single-valued extension property, $\sigma(T \mid X_T(\delta)) \subseteq \delta$. The rest of the proposition is obvious.

The following theorem is a sharpening of Proposition 2.

THEOREM 2. *For a hyponormal operator T the manifold $\bigcap_{\lambda \in \mathbf{C}} R(T - \lambda)$ is not dense in H. Moreover $\bigcap_{\lambda \in \mathbf{C}} R(T - \lambda) \subseteq N(T^*T - TT^*)$.*

Proof. Assume without loss of generality that $\sigma_p(T) = \sigma_p(T^*) = \emptyset$ and that T is not normal. Let $x \in \bigcap_{\lambda \in \mathbf{C}} R(T - \lambda)$, and let y be a nonzero vector for which $(T^* - \bar{\lambda})^{-1}y$ is weakly continuous everywhere (see [8, Theorem 1]). For fixed $\lambda_0 \in \mathbf{C}$, we have

$$\begin{aligned}
\lim_{\lambda \to \lambda_0} ((\lambda - \lambda_0)^{-1}&[((T - \lambda)^{-1}x - (T - \lambda_0)^{-1}x] \mid y) \\
&= \lim_{\lambda \to \lambda_0} ((T - \lambda)^{-1}(T - \lambda_0)^{-1}x \mid y) \\
&= \lim_{\lambda \to \lambda_0} ((T - \lambda_0)^{-1}x \mid (T^* - \bar{\lambda})^{-1}y) \\
&= ((T - \lambda_0)^{-1}x \mid (T^* - \bar{\lambda}_0)^{-1}y).
\end{aligned}$$

Therefore $((T - \lambda)^{-1}x \mid y)$ is analytic everywhere and thus $(x \mid y) = 0$. Hence $y \perp \bigcap_{\lambda \in \mathbf{C}} R(T - \lambda)$. Since such vectors y are dense in $R(T^*T - TT^*)$,

$$\bigcap_{\lambda \in \mathbf{C}} R(T - \lambda) \subseteq N(T^*T - TT^*).$$

The theorem is proved.

In view of [7, Theorem 1] one may expect $\bigcap_{\lambda \in \mathbf{C}} R(T - \lambda)$ to be $\{0\}$. Actually more is expected: the following conjecture is the most desirable form of a generalization of Theorem 1 of [7].

Conjecture. Let T be a hyponormal operator and let δ be a closed set. Then $X_T(\delta) = \bigcap_{\lambda \notin \delta} R(T - \lambda)$.

Remark 1. For convenience we admit the following definition of a decomposable operator. An operator T is called decomposable if for every finite open covering $\{G_1, \ldots, G_n\}$ of $\sigma(T)$ the manifolds $X_T(\bar{G}_1), \ldots, X_T(\bar{G}_n)$ are closed and $H = X_T(\bar{G}_1) + \cdots + X_T(\bar{G}_n)$ (see [1, Proposition 1.4] and [9, Remark 2]). The class of decomposable operators is a natural generalization of the class of spectral operators. In [10] we showed that there exists a decomposable, nonnormal, cosubnormal operator. Let T be such an operator. There exists a nonzero vector $x \in H$ and a bounded function $g: \mathbf{C} \to H$ such that $(T - \lambda)g(\lambda) \equiv x$ [8, Theorem 1]. This shows that Theorem 1 of [7], Lemma 2 of [5], and Theorem 1 of the present paper cannot be generalized to the class of decomposable operators.

THEOREM 3. *Let T, S, W, and D be operators satisfying the following conditions:*

(i) $(T - \lambda)(T^* - \bar{\lambda}) \geq D \geq 0$ *for all* $\lambda \in \mathbf{C}$;
(ii) S *is hyponormal*;
(iii) $\dim (N(W)) < \infty$;
(iv) $WT = SW$.

Then $D = 0$. In particular, if T is cohyponormal, then T is normal. (Compare [12, Theorem 1].)

Proof. If $D \neq 0$, by [8, Theorem 1] there exist a nonzero vector x and a bounded function $g: \mathbf{C} \to H$ such that $(T - \lambda)g(\lambda) \equiv x$. It follows that that $Wg: \mathbf{C} \to H$ is a bounded function for which $(S - \lambda)Wg(\lambda) \equiv Wx$. In view of Theorem 1, $Wx = 0$ and $Wg(\lambda) = 0$ for $\lambda \notin \sigma_p(S)$. Let T_1 be the restriction of T to $N(W)$ which is obviously an invariant subspace of T. The operator T_1 is normal and $(T_1 - \lambda)g(\lambda) = x$ for $\lambda \notin \sigma_p(S)$. It is easy to see that $g(\lambda) = (T_1 - \lambda)^{-1}x$ for $\lambda \notin (\sigma(T_1) \cup \sigma_p(S))$. Hence $(T_1 - \lambda)^{-1}x$ is an analytic function with finitely many singularities which is bounded on a dense subset of the plane. Therefore $(T_1 - \lambda)^{-1}x$ has an analytic extension everywhere and so $x = 0$, a contradiction. The proof of the theorem is complete.

COROLLARY 1. *Let T be a cohyponormal operator, let S be a hyponormal operator, and let W be a one-to-one operator such that $WT = SW$ and $N(W^*) = \{0\}$. Then T and S are two unitarily equivalent normal operators.*

Remark 2. An operator T is said to be a quasiaffine transform of an operator S if there exists a one-to-one operator W such that $WT = SW$ and $N(W^*) = \{0\}$. Corollary 1 says that if a cohyponormal operator T is a quasiaffine transform of a hyponormal operator S, then both T and S are normal. A slightly different result is true if T is cosubspectral: by Theorem 2 of [5] T is spectral and thus by Theorem 3(a) of the same paper S is a normal operator similar to T. However the following argument due to Berger and Shaw [2, Theorem 2.1] shows that the converse is not true in general; more precisely, given any cyclic operator T on an infinite-dimensional Hilbert space H, there exists a nonnormal, subnormal operator S which is a quasiaffine transform of T. The operator S is the multiplication by z in $R^2(G, dx\, dy)$ for some open neighborhood G of $\sigma(T)$ and the operator W satisfying $TW = WS$ can be chosen to be a trace class operator with $N(W) = N(W^*) = \{0\}$. For a different type of example see [12, Example 1].

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