ON THE WEYL SPECTRUM II

BY

KIRTI K. OBERAI[1]

Abstract

In this paper we show that if T is an isoloid operator for which Weyl's theorem holds and if $p(t)$ is a polynomial then Weyl's theorem holds for $p(T)$ if and only if $p(\omega(T)) = \omega(p(T))$ where $\omega(T)$ is the Weyl spectrum of T. We also prove that if Weyl's theorem holds for T and if N is a nilpotent operator commuting with T then Weyl's theorem holds for $T + N$.

1. Preliminaries

Let X be a complex Banach space and let $\mathscr{L}(X)$ be the space of continuous linear operators on X considered with the norm topology. For $T \in \mathscr{L}(X)$ let $\sigma(T)$, $\mathscr{P}(T)$, and $\pi_{00}(T)$ be respectively the spectrum, the resolvent set, and the isolated points of $\sigma(T)$ which are eigenvalues of finite (geometric) multiplicity. Let $\mathscr{N}(T)$ and $\mathscr{R}(T)$ respectively denote the null space and the range space of T. Let $\mathfrak{F}$ be the class of Fredholm operators on X ($T \in \mathfrak{F}$ if and only if $\mathscr{R}(T)$ is closed and the dimension of $\mathscr{N}(T)$ and the codimension of $\mathscr{R}(T)$ are both finite) and let $\mathfrak{F}_0$ be the class of Fredholm operators of index 0, i.e., those operators in $\mathfrak{F}$ for which $\dim \mathscr{N}(T) = \operatorname{codim} \mathscr{R}(T)$. If $\mathscr{K}(X)$ is the ideal of compact operators on X then $\hat{T}$ will denote the image of T under the canonical mapping of $\mathscr{L}(X)$ into the quotient algebra $\mathscr{L}(X)/\mathscr{K}(X)$. Finally, let $\mathscr{C}$ be the set of complex numbers.

DEFINITION 1. The *Weyl spectrum* $\omega(T)$ of $T \in \mathscr{L}(X)$ is defined by $\omega(T) = \{\lambda \in \mathscr{C} : \lambda I - T \notin \mathfrak{F}_0\}$.

Remark. If X is finite dimensional then $\omega(T) = \emptyset$. However, if X is infinite dimensional (and from now on we shall assume X to be so) then $\omega(T)$ is a nonempty compact subset of $\sigma(T)$ and it always contains $\sigma(\hat{T})$. Also, if $\pi_0(T)$ is the set of eigenvalues of finite multiplicity of T then $\sigma(T) \sim \pi_0(T) \subset \omega(T)$.

We say that *Weyl's theorem* holds for T if $\omega(T) = \sigma(T) \sim \pi_{00}(T)$.

From the above remark it follows immediately that if $\pi_0(T) = \emptyset$ then Weyl's theorem holds for T.

Received September 15, 1975.

[1] This research was partially supported by an N.R.C. grant.

2. Spectral mapping theorem for the Weyl spectrum

In this section we give conditions under which $f(\omega(T)) = \omega(f(T))$ for a holomorphic function $f(t)$ defined in a neighborhood of spectrum of T. We may remark (see [1, Example 3.3]) that in general even for a polynomial $p(t)$, $p(\omega(T)) \neq \omega(p(T))$.

To avoid trivialities, in the sequel, whenever we consider a polynomial we shall assume that it is not a constant polynomial.

LEMMA 1. *Let $T \in \mathscr{L}(X)$. Then for any polynomial $p(t)$ we have $\sigma(p(T)) \sim \pi_{00}(p(T)) \subset p(\sigma(T) \sim \pi_{00}(T))$.*

Proof. Let $\lambda \in \sigma(p(T)) \sim \pi_{00}(p(T)) = p(\sigma(T)) \sim \pi_{00}(p(T))$.

Case I. λ is not an isolated point of $p(\sigma(T))$. In this case there exists a sequence (λ_n) contained in $p(\sigma(T))$ such that $\lambda_n \to \lambda$. There exists a sequence (μ_n) in $\sigma(T)$ such that $p(\mu_n) = \lambda_n \to \lambda$. This implies that (μ_n) contains a convergent subsequence and we may assume that $\lim \mu_n = \mu_0$. Hence $\lambda = \lim p(\mu_n) = p(\mu_0)$. Since $\mu_0 \in \sigma(T) \sim \pi_{00}(T)$ then $\lambda \in p(\sigma(T) \sim \pi_{00}(T))$.

Case II. λ is an isolated point of $\sigma(p(T))$ so that either λ is not an eigenvalue of $p(T)$ or it is an eigenvalue of infinite multiplicity. Let $p(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I)$.

If λ is not an eigenvalue of $p(T)$ then none of $\mu_1, \ldots, \mu_n$ can be an eigenvalue of T and of course, at least one of $\mu_1, \ldots, \mu_n$ is in $\sigma(T)$. Therefore,

$$\lambda \in p(\sigma(T) \sim \pi_{00}(T)).$$

If λ is an eigenvalue of $p(T)$ of infinite multiplicity then at least one of $\mu_1, \ldots, \mu_n$, say μ_1, is an eigenvalue of T of infinite multiplicity. Then $\mu_1 \in \sigma(T) \sim \pi_{00}(T)$ and $p(\mu_1) = \lambda$ so that $\lambda \in p(\sigma(T) \sim \pi_{00}(T))$.

DEFINITION 2. An operator T is called *isoloid* if isolated points of $\sigma(T)$ are eigenvalues of T.

PROPOSITION 1. *Let $T \in \mathscr{L}(X)$ be isoloid. Then for any polynomial $p(t)$ we have $p(\sigma(T) \sim \pi_{00}(T)) = \sigma(p(T)) \sim \pi_{00}(p(T))$.*

Proof. In the presence of Lemma 1 we need only to show that $p(\sigma(T) \sim \pi_{00}(T)) \subset \sigma(p(T)) \sim \pi_{00}(p(T))$.

Let $\lambda \in p(\sigma(T) \sim \pi_{00}(T))$. Since $p(\sigma(T)) = \sigma(p(T))$ then $\lambda \in \sigma(p(T))$. If possible let $\lambda \in \pi_{00}(p(T))$ so that in particular, λ is an isolated point of $\sigma(p(T))$. Let

$$(1) \qquad p(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I).$$

The relation (1) shows that if any of $\mu_1, \ldots, \mu_n$ is in $\sigma(T)$ then it must be an isolated point of $\sigma(T)$ and hence an eigenvalue (since T is isoloid). Since λ is an eigenvalue of finite multiplicity any such μ must also be an eigenvalue of finite multiplicity and hence belongs to $\pi_{00}(T)$. This contradicts the fact that $\lambda \in p(\sigma(T) \sim \pi_{00}(T))$. Therefore, $\lambda \notin \pi_{00}(p(T))$ and

$$p(\sigma(T) \sim \pi_{00}(T)) \subset \sigma(p(T)) \sim \pi_{00}(p(T)).$$

THEOREM 1. *Let T be an isoloid operator and let Weyl's theorem hold for T. Then for any polynomial $p(t)$ Weyl's theorem holds for $p(T)$ if and only if $p(\omega(T)) = \omega(p(T))$.*

Proof. From Proposition 1 $p(\sigma(T) \sim \pi_{00}(T)) = \sigma(p(T)) \sim \pi_{00}(p(T))$. If Weyl's theorem holds for T then $\omega(T) = \sigma(T) \sim \pi_{00}(T)$ so that

$$p(\omega(T)) = p(\sigma(T) \sim \pi_{00}(T)) = \sigma(p(T)) \sim \pi_{00}(p(T)).$$

The theorem follows immediately from this relationship.

Example 1. We give an example to show that both Proposition 1 and Theorem 1 may fail if T is not assumed to be isoloid.

Define T_1 and T_2 on l_2 by

$$T_1(x_1, x_2, \ldots) = (x_1, 0, x_2/2, x_3/2, \ldots)$$

and

$$T_2(x_1, x_2, \ldots) = (0, x_1/2, x_2/3, x_3/4, \ldots).$$

Let T be defined on $X = l_2 \oplus l_2$ by $T = T_1 \oplus (T_2 - I)$. Then

$$\sigma(T) = \{1\} \cup \{z: |z| \leq 1/2\} \cup \{-1\}, \qquad \pi_{00}(T) = \{1\}$$

and

$$\omega(T) = \{z: |z| \leq 1/2\} \cup \{-1\}.$$

Thus Weyl's theorem holds for T.

Let $p(t) = t^2$. It is easy to verify that

$$\sigma(p(T)) = \{z: |z| \leq 1/4\} \cup \{1\}, \qquad \pi_{00}(p(T)) = \{1\}$$

and

$$\omega(p(T)) = \{z: |z| \leq 1/4\} \cup \{1\}.$$

Thus $1 \in p(\sigma(T) \sim \pi_{00}(T))$ but $1 \notin \sigma(p(T)) \sim \pi_{00}(p(T))$. Also, $\omega(p(T)) = p(\omega(T))$ but Weyl's theorem does not hold for $p(T)$.

For the proof of the next theorem we need the concept of limit of a sequence of compact subsets of the complex plane. For this we refer to [7].

THEOREM 2. *Let $T \in \mathscr{L}(X)$ be such that for any polynomial $p(t)$ then $p(\omega(T)) = \omega(p(T))$. Then if $f(t)$ is a holomorphic function defined in a neighborhood of $\sigma(T)$ then $f(\omega(T)) = \omega(f(T))$.*

Proof. Let $(p_n(t))$ be a sequence of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T) \to f(T)$. Since $f(T)$ commutes with each $p_n(T)$ by [7, Theorem 2] we have

$$\omega(f(T)) = \lim \omega(p_n(T)) = \lim p_n(\omega(T)) = f(\omega(T)).$$

For the definitions of spectral operators (in the sense of Dunford) and the related concepts we refer to [2, Chapter XV].

COROLLARY 1. *Let T be a spectral operator of finite type, in particular let T be a normal operator on a Hilbert space. Then for any holomorphic function $f(t)$ defined on a neighborhood of $\sigma(T)$ we have $\omega(f(T)) = f(\omega(T))$.*

Proof. For any polynomial $p(t)$, $p(T)$ is a spectral operator of finite type. Hence, $p(T)$ is isoloid and Weyl's theorem holds for $p(T)$ [7, Theorem 4]. By Theorem 1, $p(\omega(T)) = \omega(p(T))$. The result now follows from Theorem 2.

3. Two perturbations theorems

In this section we prove the conjecture made in [7] and give one more result on the same lines.

LEMMA 2. *Let $T \in \mathscr{L}(X)$ and let N be a quasinilpotent operator commuting with T. Then $\omega(T + N) = \omega(T)$.*

Proof. It is enough to show that if $0 \notin \omega(T)$ then $0 \notin \omega(T + N)$.

Let $0 \notin \omega(T)$ so that $0 \notin \sigma(\hat{T})$. For all $\lambda \in \mathscr{C}$ we have $\sigma((T + \lambda N)^\wedge) = \sigma(\hat{T})$. Hence $0 \notin \sigma((T + \lambda N)^\wedge)$ for all $\lambda \in \mathscr{C}$.

Thus for all $\lambda \in \mathscr{C}$, $T + \lambda N$ is a Fredholm operator and in particular has closed range and has an index. By [4, Theorem V.1.8], $T + \lambda N$ has the same index for all $\lambda \in \mathscr{C}$. (This is not explicitly stated in the theorem quoted. However it follows immediately from the theorem and the fact that the index stays stable in a neighborhood of a Fredholm operator.) Since T is a Fredholm operator of index 0 then $T + N \in \mathfrak{F}_0$ so that $0 \notin \omega(T + N)$.

COROLLARY 2. *Let T be a spectral operator and let S be its scalar part. Then $\omega(T) = \omega(S)$. Also, if $\sigma(T)$ does not have isolated points then Weyl's theorem holds for T.*

Proof. $T = S + N$ where N is a quasinilpotent operator commuting with T. Hence $\omega(T) = \omega(S)$.

If $\sigma(T)$ does not have isolated points then $\sigma(S)$ $(= \sigma(T))$ does not have isolated points. Since Weyl's theorem holds for S [7, Theorem 4],

$$\omega(T) = \omega(S) = \sigma(S) = \sigma(T)\ (= \sigma(T) \sim \pi_{00}(T)).$$

Hence, Weyl's theorem holds for T.

The next theorem proves the conjecture made in [7].

THEOREM 3. *Let $T \in \mathscr{L}(X)$ and let N be a nilpotent operator commuting with T. If Weyl's theorem holds for T then it also holds for $T + N$.*

Proof. We show that $\pi_{00}(T + N) = \pi_{00}(T)$.

Let $0 \in \pi_{00}(T)$ so that $\mathscr{N}(T)$ is finite dimensional. Let $(T + N)x = 0$ for some $x \neq 0$. Then $Tx = -Nx$. Since N commutes with T it follows that for every positive integer

$$T^m x = (-1)^m N^m x. \tag{2}$$

Let n be the smallest positive integer such that $N^n = 0$. The relation (2) shows that for some r with $1 \leq r \leq n$, $T^r x = 0$ and then $T^{r-1}x \in \mathscr{N}(T)$. Thus

$$\mathscr{N}(T + N) \subset \mathscr{N}(T^{n-1}).$$

Therefore, $\mathscr{N}(T + N)$ is finite dimensional. Also if for some x ($\neq 0$) $Tx = 0$ then $(T + N)^n x = 0$ so that 0 is an eigenvalue of $T + N$. Again since $\sigma(T + N) = \sigma(T)$ it follows that $0 \in \pi_{00}(T + N)$.

By symmetry $0 \in \pi_{00}(T + N)$ implies $0 \in \pi_{00}(T)$. Thus we have

$$\begin{aligned}\omega(T + N) &= \omega(T) \quad \text{(by Lemma 2)}\\ &= \sigma(T) \sim \pi_{00}(T) \quad \text{(since Weyl's theorem holds for } T)\\ &= \sigma(T + N) \sim \pi_{00}(T + N).\end{aligned}$$

Therefore, Weyl's theorem holds for $T + N$.

Example 2. Let $X = l_2$ and let T and N in $\mathscr{L}(X)$ be defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1/2, x_2/3, \ldots)$$

and

$$N(x_1, x_2, x_3, \ldots) = (0, -x_1/2, 0, 0, \ldots).$$

Since the point spectrum of T is empty then Weyl's theorem holds for T. Also N is a nilpotent operator. Since

$$0 \in \pi_{00}(T + N) \cap \omega(T + N)$$

then Weyl's theorem does not hold for $T + N$.

This example shows that Theorem 3 may fail if N is not assumed to commute with T. It also shows that if Weyl's theorem holds for T and F is a finite rank operator (i.e., $\mathscr{R}(T)$ is finite dimensional) then Weyl's theorem may not hold for $T + F$. The next theorem gives some conditions under which Weyl's theorem would hold for $T + F$ when it holds for T.

Recall that if λ is an isolated point of $\sigma(T)$ and P is the projection associated with λ then the dimension of P is called the *algebraic multiplicity* of λ. If dimension of P is finite and not zero then λ must be an eigenvalue of T. By $\pi_{0A}(T)$

we denote the set of isolated eigenvalues of T of finite algebraic multiplicity. It is well known that $\pi_{0A}(T) \subset \pi_{00}(T)$. For the details we refer to [6, III.6.5].

THEOREM 4. *Let Weyl's theorem hold for T and let F be a finite rank operator. Let $\pi_{00}(T) = \pi_{0A}(T)$ and let $\pi_{00}(T + F) = \pi_{0A}(T + F)$. Then Weyl's theorem holds for $T + F$.*

Remark. By [3, Theorem 4.2] the hypothesis $\pi_{00}(T) = \pi_{0A}(T)$ is satisfied if $\lambda \in \pi_{00}(T)$ implies $\lambda I - T$ is normally solvable.

Proof. As in [6, IV.6.2] we define the multiplicity function $\tilde{\nu}(\lambda, T)$ for T by

$$\tilde{\nu}(\lambda, T) = \begin{cases} 0 & \text{if } \lambda \in \mathscr{P}(T) \\ \dim P & \text{if } \lambda \text{ is an isolated point of } \sigma(T) \\ \infty & \text{in all other cases.} \end{cases}$$

Let $\Delta = \mathscr{P}(T) \cup \pi_{0A}(T)$.

The first Weinstein-Aronszajn formula [6, Theorem IV.6.2] gives

$$(*) \qquad \tilde{\nu}(\lambda, T + F) = \tilde{\nu}(\lambda, T) + \nu(\lambda, \omega), \quad \lambda \in \Delta,$$

where $\nu(\lambda, \omega)$ is a finite integer valued function. (For the details of the definition of $\nu(\lambda, \omega)$ refer to [6, IV.6.2]. The only property of $\nu(\lambda, \omega)$ that we shall use is that it is finite integer valued function and so we do not include details of its definition.)

Let $\lambda \in \pi_{00}(T) \cup \mathscr{P}(T) = \pi_{0A}(T) \cup \mathscr{P}(T)$. Then (*) shows that $\tilde{\nu}(\lambda, T + F)$ is finite and hence

$$\lambda \in \pi_{0A}(T + F) \cup \mathscr{P}(T + F) = \pi_{00}(T + F) \cup \mathscr{P}(T + F).$$

Hence

$$\pi_{00}(T) \cup \mathscr{P}(T) \subset \pi_{00}(T + F) \cup \mathscr{P}(T + F).$$

Similarly

$$\pi_{00}(T + F) \cup \mathscr{P}(T + F) \subset \pi_{00}(T) \cup \mathscr{P}(T).$$

Thus

$$\pi_{00}(T) \cup \mathscr{P}(T) = \pi_{00}(T + F) \cup \mathscr{P}(T + F)$$

so that

$$\sigma(T) \sim \pi_{00}(T) = \sigma(T + F) \sim \pi_{00}(T + F).$$

The theorem now follows from the fact that $\omega(T + F) = \omega(T)$.

We conclude this paper by mentioning a few questions that we have not been able to answer.

1. Does there exist a Toeplitz operator T such that Weyl's theorem does not hold for T^2? We know (see, e.g. [5, Problem 195]) that T^2 is not Toeplitz unless T is analytic or coanalytic.

We may add that Example 3.3 in [1] along with Theorem 1 may be used to show that there exists a Toeplitz operator T and a polynomial $p(t)$ such that Weyl's theorem does not hold for $p(T)$. Note that a Toeplitz operator is isoloid.

2. Does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2? Note that T^2 may not be hyponormal if T is hyponormal [5, Problem 164].

3. If Weyl's theorem holds for T and F is a finite rank operator commuting with T then does Weyl's theorem hold for $T + F$?

We may remark that if F is required to be a compact operator then Weyl's theorem may not hold for $T + F$ if it holds for T. A simple example is to take $T = 0$ and F to be adjoint of the operator T_2 given in Example 1.

References

1. S. K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. J., vol. 20 (1970), pp. 529–544.
2. N. Dunford and J. T. Schwartz, *Linear operators, part III*, Wiley, New York, 1971.
3. I. C. Gohberg and M. G. Krein, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Trans. Amer. Math. Soc. (2), vol. 13 (1964), pp. 185–264.
4. S. Goldberg, *Unbounded linear operators*, McGraw Hill, New York, 1966.
5. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.
6. T. Kato, *Perturbation theory for linear operators*, Springer Verlag, Berlin, 1966.
7. K. K. Oberai, *On the Weyl spectrum*, Illinois J. Math., vol. 18 (1974), pp. 208–212.

Queen's University
Kingston, Ontario, Canada