MEASURE ALGEBRAS ON INFINITE DIMENSIONAL SPACES

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Hewitt and Zuckerman [3] studied the measure algebra of a compact interval and Ross [7] extended these results to locally compact intervals. Baartz [1] then considered finite Cartesian products of intervals; however, he demonstrated the impossibility of extending his results to infinite Cartesian products. The purpose of this article is to extend the above works to infinite dimensional spaces by using weak products, rather than Cartesian products. The maximal ideal space of the measure algebra will be identified, the Gelfand topology described, and many Banach algebra type results, such as semisimplicity, regularity, Choquet boundary, etc., will be investigated. Finally, a Herglotz-Bochner theorem will be obtained.

1. Preliminaries

Let S be a totally ordered set, with a least element 0. We make S into a topological semigroup by putting on the interval topology and defining multiplication by $xy = \max(x, y)$. We shall assume that S is compact, and write $S = [0, 1]$; however, the extension of the results of this article to the locally compact case can be accomplished in a manner precisely the same as in Ross [7]. If $\{S_\gamma \mid \gamma \in \Gamma\}$ is a collection of these *order intervals*, we define the weak product,

$$\begin{aligned} S &= \prod^{w} \{S_\gamma \mid \gamma \in \Gamma\} \\ &= \{(x_\gamma) \in \prod \{S_\gamma \mid \gamma \in \Gamma\} \mid x_\gamma = 0_\gamma \text{ for all but finitely many coordinates}\}. \end{aligned}$$

A basis for the topology on S is given by

$$\left\{\prod^{w} \{U_\gamma \mid \gamma \in \Gamma\} \mid U_\gamma \text{ open in } S_\gamma, 0_\gamma \in U_\gamma \text{ for all but finitely many coordinates}\right\}.$$

Multiplication and order is coordinatewise. Finally

$$L_x = \{y \in S \mid y \leq x\} \quad \text{and} \quad M_x = \{y \in S \mid y \geq x\}.$$

We let $M(S)$ denote the space of all bounded, regular Borel measures on S, equipped with the total variation norm. We make $M(S)$ into a Banach algebra by defining

$$\mu * \nu(E) = \int_S \int_S \chi_E(xy)\, d\mu(x)\, d\nu(y), \quad \mu, \nu \in M(S),$$

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where χ_E is the characteristic function of E. Let δ_x denote the point mass at x.

It should be pointed out here that since S is no longer locally compact, we cannot employ the Riesz representation theorem. In other words, our measures can no longer be realized as the space of continuous linear functionals on the space of continuous functions on S that vanish at infinity.

The following lemma, the proof of which is straightforward and follows precisely as in Baartz [1], will provide a basis for much of what is to follow.

LEMMA 1. *Suppose* μ, $\nu \in M(S)$ *and* x, $y \in S$. *Then:*

(a) $\mu * \nu(E) = \int_S \mu(Ex^{-1})\, d\nu(x)$, *where* $Ex^{-1} = \{y \in S \mid xy \in E\}$.
(b) $\mu * \delta_x(E) = \mu(Ex^{-1})$.
(c) $\delta_x * \delta_y = \delta_{xy}$.
(d) $\mu * \delta_x = \mu$ *if and only if* support $\mu = S(\mu) \subseteq M_x$.

Notation.

$$\prod^{w} (A_\gamma;\, \gamma \in V) \prod^{w} (B_\gamma;\, \gamma \in T) = \left\{(x_\gamma) \in \prod^{w} \{S_\gamma \mid \gamma \in \Gamma\} \mid x_\gamma \in A_\gamma \text{ for } \gamma \in V \text{ and } x_\gamma \in B_\gamma \text{ for } \gamma \in T\right\}.$$

The following proposition, stating that compact subsets of S are finite dimensional, is crucial to the following results.

PROPOSITION 1. *Suppose K is closed in S. Then K is compact if and only if for some finite subset* $\{\gamma_1, \ldots, \gamma_N\}$ *of* Γ,

$$K \subseteq \prod^{w} (S_\gamma;\, \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma;\, \gamma \neq \gamma_1, \ldots, \gamma_N).$$

Proof. Since each S_{γ_i} is compact, it is clear that

$$\prod^{w} (S_\gamma;\, \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma;\, \gamma \neq \gamma_1, \ldots, \gamma_N)$$

is compact (Tychonov's theorem). Hence, K closed implies K compact. To see the opposite implication, suppose K were not contained in any "finite product." Then for infinitely many points $\{x^i\} \in K$, and infinitely many coordinates $\{\gamma_i\} \in \Gamma$, $x^i_{\gamma_i} \neq 0_{\gamma_i}$. Let V_{γ_i} be open in S_{γ_i} with $0_{\gamma_i} \in V_{\gamma_i}$ but $x^i_{\gamma_i} \notin V_{\gamma_i}$, and define the set $\mathcal{O}_N$ as follows:

$$\mathcal{O}_N = \prod^{w} (S_\gamma;\, \gamma \neq \gamma_{N+1}, \ldots) \prod^{w} (V_\gamma;\, \gamma = \gamma_{N+1}, \ldots).$$

Thus, $\{\mathcal{O}_N\}$ covers K (in fact, it covers S), but no finite subcover covers K—contradiction.

DEFINITION. A *prime subsemigroup* (pssg) of S is a subsemigroup whose complement is an ideal.

The following two propositions will be stated without proof. The first one follows very similarly to the corresponding proof of Baartz. The second is a general fact in the theory of idempotent semigroups.

PROPOSITION 2. *A is a pssg of S if and only if $A = \prod^{w} A_\gamma$, where $A_\gamma = [0, x_\gamma)$ or $[0, x_\gamma]$ (we write $A_\gamma = [0, x_\gamma\})$.*

As a consequence of this proposition, we see that all pssg's are Borel sets, something which is false in the case of an infinite Cartesian product.

By a semicharacter on S, we mean a multiplicative map from S into the unit disc. Denote the set of all nonzero measurable semicharacters on S by $\hat{S}$.

PROPOSITION 3. *The function $\chi \in \hat{S}$ if and only if χ is the characteristic function of a pssg of S.*

Although convolution of measures is quite unlike pointwise multiplication, the concepts turn out to be the same when applied to pssg's.

PROPOSITION 4. *If $\mu, \nu \in M(S)$, A a pssg of S, then $\mu * \nu(A) = \mu(A)\nu(A)$.*

Proof. First notice that $xy \in A$ if and only if $x \in A$ and $y \in A$. Hence,

$$\begin{aligned} \mu * \nu(A) &= \int_S \int_S \chi_A(xy)\, d\mu(x)\, d\nu(y) \\ &= \int_S \int_S \chi_A(x)\chi_A(y)\, d\mu(x)\, d\mu(y) \\ &= \left\{\int_S \chi_A(x)\, d\mu(x)\right\}\left\{\int_S \chi_A(y)\, d\nu(y)\right\} \\ &= \mu(A)\nu(A) \end{aligned}$$

by Fubini's theorem.

2. The main theorem

The main theorem in this section deals with the relationship between the maximal ideal space of $M(S)$ and the space of semicharacters on S. Since the proof is fairly long, it is broken up into a sequence of lemmas. Let π_β stand for the projection onto the βth coordinate of S.

LEMMA 2. *Let A be a pssg of S, $\mu \in M(S)$, and $\varepsilon > 0$ be given. Then there exists an $x_0 \in A$ such that $|\mu|(A \sim L_{x_0}) < \varepsilon$.*

Proof. Since μ is regular, there exists a compact $K \subseteq A$ satisfying $|\mu|(A \sim K) < \varepsilon$, and by Proposition 1,

$$K \subseteq \prod^{w} (S_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N).$$

Suppose $A = \prod^w A_\gamma$ and $t_{\gamma_i} = \max(\pi_{\gamma_i}(K))$ for $i = 1, 2, \dots, N$. Then $t_{\gamma_i} \in A_{\gamma_i}$ for $i = 1, \dots, N$, since A_{γ_i} is a pssg of S_{γ_i}. If $x_0 = (t_\gamma)$, where

$$t_\gamma = \begin{cases} t_{\gamma_i} & \text{for } \gamma = \gamma_i \\ 0_\gamma & \text{otherwise,} \end{cases}$$

then $x_0 \in A$, and $L_{x_0} = \prod^w ([0_\gamma, t_\gamma]; \gamma = \gamma_1, \dots, \gamma_N) \prod^w (0_\gamma; \gamma \neq \gamma_1, \dots, \gamma_N)$. From this, $K \subseteq L_{x_0} \subseteq A$, and hence, $|\mu|(A \sim L_{x_0}) < \varepsilon$.

The maximal ideal space of $M(S)$ consists of all nonzero algebra homomorphisms of $M(S)$ into the complex numbers, and this is denoted by $\mathscr{M}(M(S))$.

DEFINITION. If $\tau \in \mathscr{M}(M(S))$, then τ *determines* the set $A \subseteq S$ if

$$A = \{x \in S \mid \tau(\delta_x) = 1\}.$$

LEMMA 3. *If τ determines A, then A is a pssg of S.*

Proof. From Lemma 1(c), we know that $\delta_x * \delta_y = \delta_{xy}$ for all $x, y \in S$. Consequently, the function χ on S defined by $\chi(x) = \tau(\delta_x)$ is multiplicative. Therefore, either χ is a semicharacter, in which case A is a pssg, or χ is identically 0. If we can eliminate the latter possibility, we are done. Suppose χ were identically 0; i.e., $\tau(\delta_x) = 0$ for all $x \in S$. Defining $0 \in S$ by $\pi_\gamma(0) = 0_\gamma$, for all $\gamma \in \Gamma$, we have $M_0 = S$. If $\mu \in M(S)$, then obviously $S(\mu) \subseteq M_0$, and Lemma 1(d) implies that $\tau(\mu) = \tau(\mu * \delta_0) = \tau(\mu)\tau(\delta_0) = 0$. Thus, τ is identically zero, which is a contradiction.

DEFINITION. Given $\mu \in M(S)$, define $\mu_P \in M(S)$ by $\mu_P(E) = \mu(E \cap P)$.

LEMMA 4. *Suppose $\mu \in M^+(S)$ (the positive measures on S), $A = \prod^w A_\gamma$ is a pssg, and $\varepsilon > 0$. Then there exists an open set U of the form*

$$U = \prod^w ([0_\gamma, d_\gamma); \gamma = \gamma_1, \dots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \dots, \gamma_N)$$

with the property that $A \subseteq U$ and $\mu(U \sim A) < \varepsilon$.

Proof. Being that the proof of this lemma is rather long, it is broken down into three steps.

Step I. There exists countably many $\{\gamma_j\}$ such that

$$\mu\left(\prod^w (A_\gamma; \gamma = \gamma_1, \gamma_2, \dots) \prod^w (S_\gamma; \gamma \neq \gamma_1, \gamma_2, \dots)\right) = \mu(A).$$

For each finite subset $L = \{\gamma_1, \dots, \gamma_N\}$ of Γ, define

$$I_L = \prod^w (P_\gamma; \gamma \in L) \prod^w (A_\gamma; \gamma \notin L) \quad \text{where } P_{\gamma_i} = S_{\gamma_i} \sim A_{\gamma_i}.$$

Note that for two different finite subsets L and L', $I_L \cap I_{L'} = \emptyset$. Hence $\mu(I_L) > 0$ for at most countably many L, for if not, we could choose countably many L, say $\{L_K\}$, with $\sum \mu(I_{L_K}) = \infty$, and this would contradict the boundedness of the measure μ. Let $\{\gamma_j\}$ be those γ_j associated with the countably many L such that $\mu(I_L) > 0$. The assertion is that if

$$B = \prod^w (A_\gamma; \gamma = \gamma_1, \gamma_2, \ldots) \prod^w (S_\gamma; \gamma \neq \gamma_1, \gamma_2, \ldots),$$

then $\mu(B) = \mu(A)$. If this were not the case, there would exist a compact $K \subseteq B \sim A$ with $\mu(K) > 0$. By the characterization of compact sets, it is known that for some $\beta_1, \ldots, \beta_m \in \Gamma$,

$$K \subseteq \prod^w (S_\gamma; \gamma = \beta_1, \ldots, \beta_m) \prod^w (0_\gamma, \gamma \neq \beta_1, \ldots, \beta_m).$$

For those $\beta_\iota \in \{\gamma_j\}$, we must have $\pi_{\beta_\iota}(K) \subseteq A_{\beta_\iota}$. Hence, the only coordinates which need not have $\pi_{\beta_i}(K) \subseteq A_{\beta_i}$ are those $\beta_i \notin \{\gamma_j\}$. Without loss of generality, suppose that $\beta_i \notin \{\gamma_j\}$ for $i = 1, \ldots, s$; and $\beta_i \in \{\gamma_j\}$ for $i = s + 1, \ldots, m$. Then the claim is that

$$(*) \qquad K \subseteq \bigcup_T \left(\prod^w (P_{\beta_j}; j \in T) \prod^w (A_\gamma; \gamma \neq \beta_j, j \in T) \right),$$

where the union is taken over all nonempty subsets T of $\{1, 2, \ldots, s\}$. To see this, suppose that $t = (t_\gamma) \in K$. Then $t_\gamma = 0_\gamma$ for $\gamma \neq \beta_1, \ldots, \beta_m$, $t_\gamma \in A_\gamma$ for $\gamma = \beta_{s+1}, \ldots, \beta_m$ (since $t \in B$), and $t_\gamma \notin A_\gamma$ for some $\gamma \in \{\beta_1, \ldots, \beta_s\}$ (since $t \notin A$). We define the set T_0 to be exactly those $i \in \{1, \ldots, s\}$ with the property that $t_{\beta_i} \in A_{\beta_i}$, and consequently,

$$t \in \prod^w (P_{\beta_j}; j \in T_0) \prod^w (A_\gamma; \gamma \neq \beta_j, j \in T_0).$$

However, all of the sets in the union in $(*)$ have measure zero, and since there are only finitely many of them, their union has measure zero which implies $\mu(K) = 0$ —a contradiction.

Step II. There exist finitely many $\{\gamma_j\}_1^N$ satisfying

$$\mu\left(\prod^w (A_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N) < \mu(A) + \varepsilon/2\right).$$

Suppose $B = \prod^w (A_\gamma; \gamma = \gamma_1, \gamma_2, \ldots) \prod^w (S_\gamma; \gamma \neq \gamma_1, \gamma_2, \ldots)$ is as in Step I, that is, $\mu(B) = \mu(A)$. Let

$$B_n = \prod^w (A_\gamma; \gamma = \gamma_1, \gamma_2, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \gamma_2, \ldots, \gamma_N).$$

Notice $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$, and $\bigcap B_n = B$. Consequently, by a standard fact in measure theory, there is an N such that $\mu(B_N) < \mu(B) + \varepsilon/2$.

Step III. There exists an open set U of the form

$$U = \prod^w ([0_\gamma, d_\gamma); \gamma = \gamma_1, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$$

with $A \subseteq U$ and $\mu(U \sim A) < \varepsilon$.

If $C = \prod^w (A_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$ as in Step II, we need only show that $\mu(U \sim C) < \varepsilon/2$. Without loss of generality, suppose that $A_{\gamma_i} = [0_{\gamma_i}, c_{\gamma_i}]$ for $i = 1, \ldots, M$ and that $A_{\gamma_i} = [0_{\gamma_i}, c_{\gamma_i})$ for $i = M + 1, \ldots, N$. Thus

$$C = \prod^w ([0_\gamma, c_\gamma]; \gamma = \gamma_1, \ldots, \gamma_M) \prod^w ([0_\gamma, c_\gamma); \gamma = \gamma_{M+1}, \ldots, \gamma_N) \times \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N).$$

Define the set

$$L_j = \begin{cases} (c_{\gamma_j}, 1_{\gamma_j}] \times \prod^w (S_\gamma; \gamma \neq \gamma_j), & j = 1, \ldots, M \\ [c_{\gamma_j}, 1_{\gamma_j}] \times \prod^w (S_\gamma; \gamma \neq \gamma_j); & j = M + 1, \ldots, N. \end{cases}$$

We can then write $C^c = \bigcup_1^N L_j$. Again, the regularity of μ implies there exist compact $\{K_j\}_1^N$, $K_j \subset L_j$, with the property that $\mu(C^c) < \mu(\bigcup_1^N K_j) + \varepsilon/2$. If $\pi_{\gamma_j}(K_j) = D_j$, then D_j is compact in S_{γ_j}, and

$$D_j \subseteq \begin{cases} (c_{\gamma_j}, 1_{\gamma_j}], & j = 1, \ldots, M \\ [c_{\gamma_j}, 1_{\gamma_j}], & j = M + 1, \ldots, N. \end{cases}$$

For $j = 1, \ldots, M$, define $d_{\gamma_j} = \min \{D_j\}$. We see that $d_{\gamma_j} > c_{\gamma_j}$. If

$$U = \prod^w ([0_{\gamma_j}, d_{\gamma_j}); j = 1, \ldots, M) \prod^w ([0_{\gamma_j}, c_{\gamma_j}); j = M + 1, \ldots, N) \times \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N),$$

we have $A \subseteq C \subseteq U$ and $U \cap (\bigcup_1^N K_j) = \emptyset$, since

$$K_j \subseteq \begin{cases} [d_{\gamma_j}, 1_{\gamma_j}] \times \prod^w (S_\gamma; \gamma \neq \gamma_j), & j = 1, \ldots, M \\ [c_{\gamma_j}, 1_{\gamma_j}] \times \prod^w (S_\gamma; \gamma \neq \gamma_j), & j = M + 1, \ldots, N. \end{cases}$$

Therefore $U \sim C \subseteq (S \sim C) \sim (\bigcup_1^N K_j)$, and $\mu(U \sim C) < \varepsilon/2$.

Remark. The above lemma is strictly stronger than just the regularity of the measure, even for finite products. Its significance is suggested by the following counterexample. Suppose $S = [0, 1] \times [0, 1]$, $A = \{0\} \times [0, 1)$ is a pssg in S, and $U = \bigcup \{[0, 1/n) \times [0, n - 1/n)\}$ is an open set in S containing A. Then there does not exist $d_1, d_2 \in [0, 1]$ such that $A \subseteq [0, d_1) \times [0, d_2) \subseteq U$.

With the use of the previous lemma, the following lemma can be proven.

LEMMA 5. *Suppose $\tau \in \mathscr{M}(M(S))$, τ determines $A = \prod^w A_\gamma$, and $P = S \sim A$. Then if $\mu \in M^+(S)$, $\tau(\mu_P) = 0$.*

Proof. Given $\varepsilon > 0$, choose U as in Lemma 4, with

$$U = \prod^w ([0_{\gamma_i}, d_{\gamma_i}); \gamma = \gamma_1, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N),$$

$A \subseteq U$, and $\mu(U \sim A) < \varepsilon$. Since τ is a multiplicative linear functional, τ has norm one. If $T = S \sim U$, then $\mu_P = \mu_T + \mu_{U \sim A}$ where $\mu(U \sim A) < \varepsilon$. Consequently,

$$|\tau(\mu_P)| \leq |\tau(\mu_T)| + |\tau(\mu_{U \sim A})|$$

and since $\|\mu_{U \sim A}\| < \varepsilon$, we have $|\tau(\mu_{U \sim A})| < \varepsilon$. Because $\varepsilon > 0$ is arbitrary, it suffices to show that $\tau(\mu_T) = 0$. Let $y^j \in S$ be defined by

$$(y^j)_\gamma = \begin{cases} d_{\gamma_j} & \text{for } \gamma = \gamma_j \\ 0_\gamma & \text{otherwise,} \end{cases}$$

and notice that $T = \bigcup_1^N M_{y^j}$ and $y^j \in S \sim A$ for $j = 1, \ldots, N$. If $T_1 = M_{y^1}$ and $T_k = M_{y^k} \sim \bigcup_1^{k-1} M_{y^j}$, then $T = \bigcup_{k=1}^N T_k$ is a disjoint union. Thus we have $\mu_T = \mu_{\bigcup_1^N T_k} = \sum_1^N \mu_{T_k}$. Noticing that $T_k \subset M_{y^k}$, or equivalently, that $S(\mu_{T_k}) \subseteq M_{y^k}$, and applying Lemma 1(d) yields the fact that $\mu_{T_k} = \mu_{T_k} * \delta_{y^k}$, and therefore, $\mu_T = \sum_1^N \mu_{T_k} * \delta_{y^k}$. Then,

$$\tau(\mu_T) = \tau\left(\sum_1^N \mu_{T_k} * \delta_{y^k}\right) = \sum_1^N \tau(\mu_{T_k} * \delta_{y^k}) = \sum_1^N \tau(\mu_{T_k})\tau(\delta_{y^k}) = 0,$$

since $y^k \in S \sim A$ for $k = 1, \ldots, N$.

With all of the preliminaries out of the way, we are now ready to state and prove the major theorem of this section.

THEOREM 1. *Suppose S is as above, A a pssg of S, and the homomorphism τ_A of $M(S)$ defined by $\tau_A(\mu) = \mu(A)$. Then the mapping $\phi\colon \hat{S} \to \mathscr{M}(M(S))$ defined by $\phi(\chi_A) = \tau_A$ is a one-one correspondence.*

Proof. The fact that τ_A is a multiplicative linear functional is straightforward, just using Proposition 4. To see ϕ is one-one, let A and B be pssg's of S with $A \neq B$. Without loss of generality, there exists an $x \in S$ with $x \in A$ but $x \notin B$. Then $1 = \tau_A(\delta_x) \neq \tau_B(\delta_x) = 0$. Finally, we must show that ϕ is onto, so suppose that $\tau \in \mathscr{M}(M(S))$ and τ determines A. It is known by Lemma 3 that A is a pssg; consequently, it suffices to show that $\tau = \tau_A$, i.e., $\tau(\mu) = \mu(A)$. By Lemma 2, given $\varepsilon > 0$, choose $x \in A$ such that $\mu(A \sim L_x) < \varepsilon$. Without loss of generality, $\mu \in M^+(S)$, since if not, we can decompose μ by $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ and use the linearity of τ. By Lemma 1(b), we know that

$$\begin{aligned} \mu_A * \delta_x(E) &= \mu_A(Ex^{-1}) \\ &= \mu_{L_x}(Ex^{-1}) + \mu_{A \sim L_x}(Ex^{-1}) \\ &= \mu_{L_x}(Ex^{-1}) + \mu_{A \sim L_x} * \delta_x(E) \\ &= \mu_{L_x}(Ex^{-1}) + \nu(E) \end{aligned}$$

where $\nu(E) = \mu_{A \sim L_x} * \delta_x(E)$ and $\|\nu\| < \varepsilon$ since $\nu(S) \leq \mu(A \sim L_x) < \varepsilon$. Notice that if $x \in E$, then $Ex^{-1} = \{y \in S \mid xy \in E\} \supseteq L_x$ and if $x \notin E$, then

$Ex^{-1} \cap L_x = \emptyset$. Thus $\mu_{L_x}(Ex^{-1}) = \mu_A(L_x)\, \delta_x(E)$. Therefore, $\mu_A * \delta_x = \mu_A(L_x)\, \delta_x + \nu$. Lemma 5 and the fact that $\tau(\delta_x) = 1$ (since $x \in A$) allows us to write

$$\tau(\mu) = \tau(\mu_{A^c} + \mu_A) = \tau(\mu_{A^c}) + \tau(\mu_A) = \tau(\mu_A) = \tau(\mu_A)\tau(\delta_x)$$
$$= \tau(\mu_A * \delta_x) = \tau(\mu_A(L_x)\delta_x + \nu) = \mu_A(L_x) + \tau(\nu).$$

Hence, $|\tau(\mu) - \mu(A)| \leq |\mu_A(L_x) - \mu(A)| + |\tau(\nu)| \leq \varepsilon + \varepsilon = 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get that $\tau(\mu) = \mu(A)$.

If $S = [0, 1]$, Ross showed in [7] that $\hat{S}$ can be identified with the set $\{x] \mid 0 \leq x \leq 1\} \cup \{x) \mid 0 < x \leq 1\}$. Again using the symbol $x\}$ to stand for either $x)$ or $x]$, the order on $\hat{S}$ is given by $x\} < y\}$ if either $x < y$ or $x = y$ and $x\} = x)$, $y\} = y]$. Furthermore, he demonstrated that the maximal ideal space of $M[0, 1]$ is homeomorphic and semigroup isomorphic to $\hat{S}$ with the order topology, where the multiplication is given by $x\}y\} = \min(x\}, y\})$. Following this idea, the next item on the agenda is identifying the Gelfand topology and obtaining a similar type of semigroup isomorphism. The reader should note that in the next theorem, the Gelfand topology is realized in the form of a full Cartesian product, which is exactly what happens when considering the dual of a weak product of groups as shown by Kaplan in [4].

THEOREM 2. *If we make $\prod \hat{S}_\gamma$ into a semigroup by defining the multiplication on each $\hat{S}_\gamma$ by $x\}y\} = \min(x\}, y\})$, we have $\hat{S} \cong \prod \hat{S}_\gamma$, where $\hat{S}$ has the Gelfand topology and $\cong$ denotes homeomorphic and semigroup isomorphic.*

Proof. The semigroup isomorphism is straightforward. For the other part, we let $\mu \in M(S)$ and we want to show $\hat{\mu}$ is continuous on $\prod \hat{S}_\gamma$. Without loss of generality, suppose $\mu \in M^+(S)$, for

$$(\mu_1 - \mu_2 + i\mu_3 - i\mu_4)^\wedge = \hat{\mu}_1 - \hat{\mu}_2 + i\hat{\mu}_3 - i\hat{\mu}_4.$$

Let $\varepsilon > 0$ be given and $A = \prod^w [0_\gamma, c_\gamma\} \subseteq S$. We will show that $\hat{\mu}$ is continuous at $(c_\gamma\}) \in \hat{S}$. By Lemma 4, there exists a

$$U = \prod^w ([0_\gamma, d_\gamma); \gamma = \gamma_1, \ldots, \gamma_N) \prod^w (S_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$$

with $A \subseteq U$ and $\mu(U \sim A) < \varepsilon$. Choose a compact $K \subseteq A$ with $\mu(A \sim K) < \varepsilon$. The fact that K is compact implies that

$$K \subseteq \prod^w (S_\gamma; \gamma = \gamma_1, \ldots, \gamma_M) \prod^w (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_M).$$

If $K_i = \pi_{\gamma_i}(K)$, $i = 1, \ldots, M$, then A a pssg implies that

$$K' = \prod^w (K_i; i = 1, \ldots, M) \prod^w (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_M) \subseteq A,$$

and $\mu(A \sim K') < \varepsilon$. If $t_{\gamma_i} = \max(K_i)$, then again because A is a pssg, we have

$$K'' = \prod^w ([0_\gamma, t_\gamma]; \gamma = \gamma_1, \ldots, \gamma_M) \prod^w (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_M) \subseteq A,$$

and $\mu(A \sim K'') < \varepsilon$. Let

$$s_{\gamma_j} = \begin{cases} c_{\gamma_j}) \text{ if } t_{\gamma_j} = c_{\gamma_j} & \text{for } j = 1, \ldots, M \\ t_{\gamma_j}] \text{ if } t_{\gamma_j} < c_{\gamma_j} & \text{for } j = 1, \ldots, M \\ 0_{\gamma_j}] \text{ if } N > M & \text{for } j = M+1, \ldots, N \end{cases}$$

and

$$p_{\gamma_j} = \begin{cases} d_{\gamma_j}) & \text{for } j = 1, \ldots, N \\ 1_{\gamma_j}] \text{ if } M > N & \text{for } j = N+1, \ldots, M; \end{cases}$$

wherefore $s_{\gamma_j}, p_{\gamma_j} \in \hat{S}_{\gamma_j}$.

Define V by

$$V = \begin{cases} \prod ((s_\gamma, p_\gamma); \gamma = \gamma_1, \ldots, \gamma_M) \prod ([s_\gamma, p_\gamma); \gamma = \gamma_{M+1}, \ldots, \gamma_N) \\ \qquad \prod (\hat{S}_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N) \quad \text{if } N \geq M \\ \prod ((s_\gamma, p_\gamma); \gamma = \gamma_1, \ldots, \gamma_N) \prod ((s_\gamma, p_\gamma]; \gamma = \gamma_{N+1}, \ldots, \gamma_M) \\ \qquad \prod (\hat{S}_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_M) \quad \text{if } M > N. \end{cases}$$

In either case, V is open in $\prod \hat{S}_\gamma$ and, for any $(x_\gamma\}) \in V$,

$$K'' \subseteq \prod^{w} [0_\gamma, x_\gamma\} \subseteq U$$

and hence, $\mu(A) - \varepsilon < \mu\,(\prod^w [0_\gamma, x_\gamma\}) < \mu(A) + \varepsilon$, or in other words,

$$\hat{\mu}((c_\gamma\})) - \varepsilon < \hat{\mu}((x_\gamma\})) < \hat{\mu}((c_\gamma\})) + \varepsilon.$$

Therefore, $\hat{\mu}$ is continuous on $\prod \hat{S}_\gamma$ with the Cartesian product topology; consequently, the Gelfand topology is weaker than or equal to the Cartesian product topology on $\prod \hat{S}_\gamma$. But both topologies are compact and Hausdorff, since each $\hat{S}_\gamma$ was shown to be compact and Hausdorff in [3]. Thus, by a standard theorem in topology, the two topologies coincide.

3. More on $M(S)$

In Section 2, the maximal ideal space of $M(S)$ was completely described. In this section, many of the other aspects of $M(S)$ that one usually looks at when studying any Banach algebra are investigated. The Gelfand theory yields a map $^\wedge : M(S) \to C(\hat{S})$ and one can ask several questions about this map: Is the map one-one and what does the range look like? If the map is one-one, then $M(S)$ is called *semisimple* and this is shown in the next proposition. A description of the range is given in the Herglotz-Bochner theorem in the next section.

PROPOSITION 5. *$M(S)$ is semisimple.*

Proof. By Theorem 1, $M(S)$ is semisimple if and only if for all $\lambda \in M(S)$, $\lambda(A) = 0$ for all pssg's A implies that $\lambda \equiv 0$. Suppose $\lambda(A) = 0$ for all pssg's $A \subseteq S$ but $\lambda \not\equiv 0$. Then there would exist a compact set

$$K \subseteq \prod^{w} (S_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$$

in S with $|\lambda|(K) > 0$. Define a measure $\tilde{\lambda}$ on the finite Cartesian product $T = \prod_1^N S_{\gamma_i}$ as follows: if $\tilde{E} \subseteq T$ is measurable, and

$$E = \left(\tilde{E} \times \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)\right) \subseteq S,$$

then $\tilde{\lambda}(\tilde{E}) = \lambda(E)$. It is an easy matter to check that $\tilde{\lambda} \in M(T)$, and it is also clear that $\tilde{\lambda}(\tilde{A}) = 0$ for all pssg's $\tilde{A}$ in T. Hence, Baartz has shown in [1; 3.6] that $\tilde{\lambda} = 0$, and thus $|\tilde{\lambda}| = 0$. But if

$$\tilde{K} = \{(k_{\gamma_i}) \in T \mid \text{for some } k \in K, \pi_{\gamma_i}(k) = k_{\gamma_i} \text{ for } i = 1, \ldots, N\},$$

then $0 = |\tilde{\lambda}|(\tilde{K}) = |\lambda|(K)$, which is a contradiction.

In the finite dimensional case, $\hat{S}$ is known to be a totally disconnected space. From this fact and Theorem 2, we can easily deduce that $\hat{S}$ is totally disconnected where S is a weak product.

COROLLARY 1. *The space $\hat{S}$ is Hausdorff and totally disconnected.*

DEFINITION. If X is a compact Hausdorff space, a subalgebra B of $C(X)$ is called *regular on X* if for each closed subset F of X and each point $t \notin F$, there exists an $f \in B$ such that $f(t) \neq 0$ but $f(x) = 0$ for all $x \in F$. A commutative Banach algebra A is called *regular* if $\hat{A}$ is regular on $\mathscr{M}(A)$.

The next project will be to show that $M(S)$ is a regular Banach algebra.

THEOREM 3. *The algebra $M(S)$ is regular.*

Proof. Let F be closed in $\hat{S}$ and $x \in \hat{S} \sim F$. We want to show that there exists a measure $\mu \in M(S)$ with the property that $\hat{\mu}(x) = 1$, but $\hat{\mu}(y) = 0$ for all $y \in F$. Since $\hat{S} = \prod \hat{S}_\gamma$ is compact Hausdorff and hence a regular topological space, there exist neighborhoods U of x and V of F such that $U \cap V = \emptyset$. Without loss of generality, we can choose U a basic open set of the form

$$U = \prod (U_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod (\hat{S}_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N),$$

where each U_{γ_i} is an open interval.

For each U_{γ_i}, determine the points $d_{\gamma_i}, f_{\gamma_i}$ from one of the following four cases:

Case I. $x_{\gamma_i}\} = x_{\gamma_i}]$.

Subcase α. x_{γ_i} has no immediate successor in S_{γ_i}. Therefore, U_{γ_i} must contain certain points of the form $y_{\gamma_i}]$, where $y_{\gamma_i} > x_{\gamma_i}$. Choose one such point and let $d_{\gamma_i} = x_{\gamma_i}, f_{\gamma_i} = y_{\gamma_i}$.

Subcase β. x_{γ_i} has an immediate successor y_{γ_i} in S_{γ_i}. In this case, $x_{\gamma_i}] = y_{\gamma_i})$ and let $d_{\gamma_i} = x_{\gamma_i}, f_{\gamma_i} = y_{\gamma_i}$.

Case II. $x_{\gamma_i}\} = x_{\gamma_i})$.

Subcase α. x_{γ_i} has no immediate predecessor in S_{γ_i}. Thus, U_{γ_i} must contain points of the form $y_{\gamma_i}]$, where $y_{\gamma_i} < x_{\gamma_i}$. Choose one such point and let $d_{\gamma_i} = y_{\gamma_i}, f_{\gamma_i} = x_{\gamma_i}$.

Subcase β. x_{γ_i} has an immediate predecessor y_{γ_i} in S_{γ_i}. In this case, $y_{\gamma_i}] = x_{\gamma_i})$ and let $d_{\gamma_i} = y_{\gamma_i}, f_{\gamma_i} = x_{\gamma_i}$. Now define d and $f^j \in S$ by

$$d_\gamma = \begin{cases} d_{\gamma_i} & \text{for } \gamma = \gamma_i,\ i = 1, \dots, N \\ 0_\gamma & \text{for } \gamma \neq \gamma_1, \dots, \gamma_N,\ \gamma \in \Gamma; \end{cases}$$

$$f_\gamma^j = \begin{cases} f_{\gamma_j} & \text{for } \gamma = \gamma_j \\ d_{\gamma_i} & \text{for } \gamma = \gamma_i,\ i = 1, \dots, N,\ i \neq j \\ 0_\gamma & \text{for } \gamma \neq \gamma_1, \dots, \gamma_N,\ \gamma \in \Gamma, \end{cases}$$

and let $\mu_j = \delta_d - \delta_{f^j} \in M(S)$. The first claim is that $\hat{\mu}_j(x) = 1$ for all $j = 1, 2, \dots, N$. By construction, $(d_\gamma]) \leq (x_\gamma\})$—for certainly $0_\gamma] \leq x_\gamma\}$, $\gamma \in \Gamma$, $\gamma \neq \gamma_1, \dots, \gamma_N$ and $d_{\gamma_i}] \leq x_{\gamma_i}\}$, $i = 1, \dots, N$. As a result, $\hat{\delta}_d(x) = 1$. But $f^j_{\gamma_j}] > x_{\gamma_j}\}$, implying that $(f^j_\gamma]) \not\leq (x_\gamma\})$ and thus $\hat{\delta}_{f^j}(x) = 0$. Therefore, $\hat{\mu}_j(x) = \hat{\delta}_d(x) - \hat{\delta}_{f^j}(x) = 1 - 0 = 1$. We now assert that for $t = (t_\gamma\}) \in F$, there exists an $s \in \{1, 2, \dots, N\}$ with $\hat{\mu}_s(t) = 0$. Since $t \notin U$, there exists an s such that $t_{\gamma_s} \notin U_{\gamma_s}$. Since U_{γ_s} is an open interval, there are two cases involved in showing that $\hat{\mu}_s(t) = 0$.

Case (1). $t_{\gamma_s}\} < y\}$, for all $y\} \in U_{\gamma_s}$. Consequently, $t \not\geq d$ and $t \not\geq f^s$; therefore, $\hat{\mu}_s(t) = \hat{\delta}_d(t) - \hat{\delta}_{f^s}(t) = 0 - 0 = 0$.

Case (2). $t_{\gamma_s}\} > y\}$, for all $y\} \in U_{\gamma_s}$.

Subcase (a). $t_{\gamma_i}\} \geq d_{\gamma_i}]$ for all $i = 1, \dots, N$, $i \neq s$. Then $t \geq d$ and $t \geq f^s$, and thus, $\hat{\mu}_s(t) = \hat{\delta}_d(t) - \hat{\delta}_{f^s}(t) = 1 - 1 = 0$.

Subcase (b). For some $p \in \{1, 2, \dots, N\}$, $p \neq s$, $t_{\gamma_p}\} < d_{\gamma_s}]$. As a result, $t \not\geq d$ and $t \not\geq f^s$, and $\hat{\mu}_s(t) = \hat{\delta}_d(t) - \hat{\delta}_{f^s}(t) = 0 - 0 = 0$. Define the measure $\mu = \mu_1 * \mu_2 * \cdots * \mu_N$. Then $\hat{\mu}(x) = 1$, since each of the $\hat{\mu}_j(x) = 1$, and $\hat{\mu} = \hat{\mu}_1 \hat{\mu}_2 \cdots \hat{\mu}_N$. Also, $\hat{\mu}(y) = 0$ for all $y \in F$, since for some s, $\hat{\mu}_s(y) = 0$. This shows that $M(S)$ is regular.

Our next goal will be to identify the Choquet boundary of $M(S)$, which recall is the set of all elements of $\mathscr{M}(M(S))$ having unique representing measures. First of all, notice that $M(S)^\wedge$ is dense in $C(\hat{S})$. This is because $M(S)$ is symmetric ($\bar{\mu}^\wedge = \hat{\mu}^-$) and we can employ the Stone-Weierstrass theorem. Hence, we get the following result.

PROPOSITION 6. *The Choquet boundary of $M(S)$ equals $\hat{S}$.*

Proof. Let $x \in \hat{S}$ and suppose there were two representing measures λ_1 and λ_2 for x. That is, $\int f\, d\lambda_1 = f(x) = \int f\, d\lambda_2$ for all $f \in M(S)^\wedge$, or equivalently, $\int f\, d(\lambda_1 - \lambda_2) = 0$ for all $f \in M(S)^\wedge$. But $M(S)^\wedge$ is dense in $C(\hat{S})$, and consequently, $\lambda_1 - \lambda_2 = 0$, or $\lambda_1 = \lambda_2$.

4. Idempotents and Herglotz-Bochner

We come now to the final two major theorems of this paper. In the first of these, the idempotents of $M(S)$ are characterized in precisely the same way as in the case of finite products. The second theorem deals with the Gelfand image as a subalgebra of $C(\hat{S})$ and relies heavily upon the notion of a function of bounded variation on a commutative idempotent semigroup with identity as described by Newman in [6].

Recall that an idempotent measure μ is one that satisfies $\mu * \mu = \mu$. We omit the proof of Theorem 4 due to the fact that it is pretty much an application of Baartz's [1] result on idempotent measures on finite dimensional products. The main idea of the proof is to find a compact set K which approximates the support of the measure μ to within 1/4 and choose a compact pssg L containing the compact set. Then realizing that the measure μ_L is also idempotent, use Baartz's result to characterize μ_L, from which, it can be easily deduced that $\mu_L = \mu$.

THEOREM 4. *A measure $\mu \in M(S)$ is idempotent if and only if μ is a discrete measure of the form $\mu = \sum_1^M \alpha_k \delta_{t_k}$, with the property that if $q = |B|$, where*

$$B = \{j \mid \pi_{\gamma_j}(t_k) \neq 0_{\gamma_j} \text{ for some } t_k,\ k = 1, \ldots, M\},$$

then the coefficients α_k are nonzero integers between -2^{q-1} and 2^{q-1} satisfying for each $x \in S$, $\sum (\alpha_k; t_k \leq x) = 0$ or 1.

We now proceed to prove a Herglotz-Bochner theorem which characterizes the Gelfand image of $M(S)$. All of the ideas below about functions of bounded variation were taken from Newman [6]. The idea in the proof of the main theorem is exactly as in the proof of the previous theorem, that is, to reduce the case of a weak product to that of a finite product of intervals and use the known results.

For $\chi \in \hat{S}$, define A_χ to be the pssg associated with $\chi = \chi^{-1}(1)$. Let $J_\chi = \chi^{-1}(0)$ so that $A_\chi \cup J_\chi = S$, $A_\chi \cap J_\chi = \emptyset$. Let $\mathscr{A}$ be the Boolean algebra of subsets of S generated by the $J_\chi(\chi \in \hat{S})$. Suppose

$$X = \{\chi_1, \chi_2, \ldots, \chi_n\} \subseteq \hat{S},$$

and $\sigma \in T_n$, the Boolean algebra of all n-tuples of 0's and 1's. We can then let

$$(**) \qquad B(X, \sigma) = \{\textstyle\bigcap (A_{\chi_i}; \sigma(i) = 1)\} \cap \{\textstyle\bigcap J_{\chi_i}; \sigma(i) = 0\}.$$

Therefore, $\mathscr{A}$ consists of all finite unions of sets of the form (**).

If F is a function on $\hat{S}$, define an operator L by

$$L(X, \sigma)(F) = \sum_{\substack{\tau \in T_n \\ \tau \geq \sigma}} \mu(\sigma, \tau) F\left(\prod_i \chi_i^{\tau(i)}\right)$$

where

$$\mu(\sigma, \tau) = \begin{cases} (-1)^{|\tau| - |\sigma|} & \text{if } \tau \geq \sigma \\ 0 & \text{otherwise} \end{cases}$$

and $|\sigma|$ denotes the number of ones in the n-tuple σ.

We are now ready to define what is meant by a function of bounded variation on $\hat{S}$.

DEFINITION. F is a function of *bounded variation* on $\hat{S}$ if

$$\sup_X \sum_{\sigma \in T_n} |L(X, \sigma)(F)| < \infty,$$

where the supremum is taken over all finite subsets X of $\hat{S}$.

The following theorem of Newman's gives us the relationship between the functions of bounded variation on $\hat{S}$ and the finitely additive set functions on $\mathscr{A}$.

THEOREM 5. [6; 3.2] *The algebra $ba(\mathscr{A})$ of all bounded, finitely additive measures on $\mathscr{A}$, with convolution multiplication is a Banach algebra. The algebra $BV(\hat{S})$ of all functions of bounded variation on $\hat{S}$, with pointwise multiplication and bounded variation norm, is also a Banach algebra. The map $\mu \to \hat{\mu}$ (where $\hat{\mu}(\chi) = \mu(A_\chi)$) maps $ba(\mathscr{A})$ isomorphically and isometrically onto $BV(\hat{S})$.*

Suppose that Σ is the σ-algebra generated by $\mathscr{A}$. In other words, Σ is generated by the pssg's of S, all of which are Borel sets, and hence the Borel algebra $\mathscr{B}(S) \supseteq \Sigma$. The following lemma is a parallel to Caratheodory's extension theorem for measures.

LEMMA 6. *Each $\mu \in ba(\mathscr{A})$ can be extended to an element $\tilde{\mu} \in ba(\Sigma)$, i.e., a finitely additive measure on the σ-algebra Σ. Moreover, this can be done in such a way that $\|\hat{\mu}\| = \|\mu\|$.*

Proof. Let

$$B(\mathscr{A}) = \left\{\text{uniform limits of functions of the form } f = \sum_1^n \alpha_i \chi_{E_i} \mid E_i \in \mathscr{A}\right\},$$

and similarly

$$B(\Sigma) = \left\{\text{uniform limits of functions of the form } f = \sum_1^n \alpha_i \chi_{E_i} \mid E_i \in \Sigma\right\}.$$

Clearly, both $B(\mathscr{A})$ and $B(\Sigma)$ are normed linear spaces with the sup norm. Let $\mu \in ba(\mathscr{A})$ and define $l \in B(\mathscr{A})^*$ by $l(f) = \int_S f\, d\mu$—it is easily seen that $\|l\| = \|\mu\|$. Hence, by the Hahn-Banach theorem, we can extend l to $\tilde{l} \in B(\Sigma)^*$; i.e.,

$l(f) = \tilde{l}(f)$ for all $f \in B(\mathscr{A})$, and $\|l\| = \|\tilde{l}\|$. Define $\tilde{\mu} \in ba(\Sigma)$ by $\tilde{\mu}(E) = \tilde{l}(\chi_E)$, thus getting an extension of μ with the property that $\|\mu\| = \|\tilde{\mu}\|$.

Let us examine the following diagram:

$$\begin{array}{lllll}
(1) & ba(\mathscr{A}) & \xrightarrow{\wedge} & BV(\hat{S}) & \text{(as in Theorem 5)} \\
 & \cup| & & & \\
(2) & ba(\Sigma) & \xrightarrow{\wedge} & BV(\hat{S}) & \text{(obtained by factoring through } ba(\mathscr{A})\text{;} \\
 & \cup| & & & \text{Lemma 6 assures us that this is onto)} \\
(3) & M(S) & \xrightarrow{\text{Gelfand}} & C(\hat{S}) &
\end{array}$$

We see from this diagram that

$$M(S) \xrightarrow{\text{Gelfand}} CBV(\hat{S})$$

where $CBV(\hat{S})$ is the set of continuous functions of bounded variation on $\hat{S}$.

We want to show that the Gelfand map is onto, and this result is called a Herglotz-Bochner theorem. The fact that the map is onto for the case of $S = \prod_1^N S_n$ will be used. Since map (2) above is onto, it suffices to show that $ba(\Sigma) \sim M(S) \xrightarrow{\wedge} NBV(\hat{S})$, the set of noncontinuous functions of bounded variation on $\hat{S}$. Again, for $S = \prod_1^N S_n$, this is known to be true.

THEOREM 6. *The Gelfand map* $\wedge : M(S) \to CBV(\hat{S})$ *is onto.*

Proof. Suppose $\mu \in ba(\Sigma) \sim M(S)$. We break this up into several cases.

Case I. μ is not countably additive on $\mathscr{B}(S)$. Consequently, there exist disjoint sets $\{E_n\}_1^\infty$ such that

$$|\mu(\bigcup E_n) - \sum \mu(E_n)| > \varepsilon > 0.$$

Choose a compact

$$K \subseteq \prod^{w} (S_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N) = L$$

such that $\|\mu_K - \mu\| < \varepsilon/3$, and hence, $\|\mu_L - \mu\| < \varepsilon/3$. Consider μ_L as a finitely additive measure on $\prod_1^N S_{\gamma_j} = T$, and call it $\tilde{\mu}_L$, that is,

$$\tilde{\mu}_L(E) = \mu_L\left(E \times \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)\right).$$

For a set $E \subseteq S$, let $E_T = \{(x_{\gamma_1}, \ldots, x_{\gamma_N}) \in T \mid$ there exists a $y \in E$ with $\pi_{\gamma_i}(y) = x_{\gamma_i}$ for all $i = 1, 2, \ldots, N\}$. We then have

$$|\tilde{\mu}_L(\bigcup (E_n)_T) - \sum \tilde{\mu}_L((E_n)_T)| > \varepsilon - \varepsilon/3 - \varepsilon/3 = \varepsilon/3 > 0.$$

Thus, $\tilde{\mu}_L$ is not countably additive and by the remark just before the statement of the theorem, $(\tilde{\mu}_L)^\wedge$ is not continuous. This implies that there exists a sequence

$$\{(x_{n,\gamma_1}\}, x_{n,\gamma_2}\}, \ldots, x_{n,\gamma_N}\})\} \to (x_{\gamma_1}\}, x_{\gamma_2}\}, \ldots, x_{\gamma_N}\})$$

with the property that

$$(\tilde{\mu}_L)^\wedge(x_{n,\gamma_1}\}, x_{n,\gamma_2}\}, \ldots, x_{n,\gamma_N}\}) \nrightarrow (\tilde{\mu}_L)(x_{\gamma_1}\}, x_{\gamma_2}\}, \ldots, x_{\gamma_N}\}).$$

Let y^n, $y \in \hat{S}$ be such that $\pi_{\gamma_j}(y^n) = x_{n,\gamma_j}\}$ and $\pi_{\gamma_j}(y) = x_{\gamma_j}\}$, for $j = 1, \ldots, N$ and $\pi_\gamma(y^n) = \pi_\gamma(y) = 0_\gamma]$, $\gamma \neq \gamma_j$. Hence, $y^n \to y$ in $\hat{S}$ but

$$\begin{aligned}\hat{\mu}(y^n) &= \mu\left(\prod^{w} [0_\gamma, y^n_\gamma\}\right)\\ &= \tilde{\mu}_L\left(\prod_1^N [0_{\gamma_j}, x_{n,\gamma_j}\}\right)\\ &= (\tilde{\mu}_L)^\wedge((x_{n,\gamma_1}\}, \ldots, x_{n,\gamma_N}\}))\\ &\not\to (\tilde{\mu}_L)^\wedge(x_{\gamma_1}\}, \ldots, x_{\gamma_N}\})\\ &= \tilde{\mu}_L\left(\prod_1^N [0_{\gamma_j}, x_{\gamma_j}\}\right)\\ &= \mu\left(\prod^{w} [0_\gamma, y_\gamma\}\right)\\ &= \hat{\mu}(y),\end{aligned}$$

wherefore, $\hat{\mu}$ is not continuous.

Case II. $\hat{\mu}$ is countably additive, but not regular.

Subcase α. μ is not inner regular. Consequently, there exists a set $E \subseteq S$ and an $\varepsilon > 0$ such that for all compact $K \subseteq E$, $|\mu|(E \sim K) > \varepsilon$. As before, find

$$L = \prod^{w} (S_\gamma; \gamma = \gamma_1, \ldots, \gamma_N) \prod^{w} (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$$

with $\|\mu_L - \mu\| < \varepsilon$, and consider $\tilde{\mu}_L$ defined on $T = \prod_1^N S_{\gamma_j}$. Let $E_T \subseteq T$ be defined as before. If $\tilde{\mu}_L$ were inner regular, there would exist a compact $K' \subseteq E_T \subseteq T$ satisfying $|\tilde{\mu}_L|(E_T \sim K') < \varepsilon/3$. Let $K = K' \times \prod (0_\gamma; \gamma \neq \gamma_1, \ldots, \gamma_N)$. Then $|\mu|(E \sim K) \leq \varepsilon/3 + |\mu_L|(E_T \sim K') = 2\varepsilon/3 < \varepsilon$, which is a contradiction. Hence $\tilde{\mu}_L$ is not regular, from which we can conclude that μ is not continuous by the same argument as in Case I.

Subcase β. μ is not outer regular. Claim that this case follows from subcase α, since μ not outer regular implies that μ is not inner regular, or equivalently, μ inner regular implies μ is outer regular. For suppose that μ is inner regular and $E \subseteq \mathscr{B}(S)$. Given $\varepsilon > 0$, find a compact set $K \subseteq S \sim E$ such that $|\mu|((S \sim E) \sim K) < \varepsilon$. Then let $U = S \sim K$. Clearly, $E \subseteq U$, U is open, and $|\mu|(U \sim E) = |\mu|((S \sim E) \sim K) < \varepsilon$.

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