CONVERGENCE OF RADON-NIKODYM DERIVATIVES AND MARTINGALES GIVEN SIGMA LATTICES

BY

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1. Introduction

For background, including examples and applications, on measure and integration with respect to a sigma lattice, we refer the reader to the basic reference [2]. Applications of sigma lattices to operator theory can be found in [1]. The Lebesgue decomposition theorem for lattices appears in [7], and the bridge between the finitely additive theory and the countably additive theory for lattices is displayed in [8]. An entree to vector valued martingale results and Orlicz spaces in the sigma algebra setting is provided by [15] and its list of references; [9] presents a closed martingale theorem for sigma lattices. Although the basic theory of Orlicz spaces can be found in [11], some relevant properties are recounted below.

The function $\Phi: R \to R$, is convex, $\Phi(-x) = \Phi(x) > 0$ if $x \neq 0$, $\Phi(0) = 0$, and Φ satisfies the Δ_2-condition: there exists a positive number K such that $\Phi(2x) \leq K\Phi(x)$, $x \in R$. Thus, there exists a sequence $\{K_n\}$ of positive numbers such that

$$\begin{aligned}\Phi(x + y) &= \Phi(2(x + y)/2)\\ &\leq K\Phi((x + y)/2)\\ &\leq (K/2)(\Phi(x) + \Phi(y))\\ &= K_2(\Phi(x) + \Phi(y))\end{aligned}$$

and $\Phi(\sum_{j=1}^{n} x_j) \leq K_n \sum_{j=1}^{n} \Phi(x_j)$. The set of $\mathscr{A}$-measurable functions $f: \Omega \to R$, with $\int_\Omega \Phi(h)\, d\mu < \infty$ is denoted by $L^\Phi = L^\Phi(\Omega. \mathscr{A}, \mu)$. Section 9 of [11] describes norms which make L^Φ into a Banach space; we shall use the Orlicz norm. Thus [11, Theorem 9.4], if $\{g_n\}$ is a sequence in L^Φ, $\|g_n\|_\Phi \to 0$, if, and only if, $\int_\Omega \Phi(g_n)\, d\mu \to 0$. For example, when $1 < \alpha < \infty, \alpha^{-1} + \beta^{-1} = 1$ $\Phi(t) = t^\alpha/\alpha$ and $L^\Phi = L_\alpha(\Omega, \mathscr{A}, \mu)$. Then

$$\|g\|_\Phi = \beta^{(1/\beta)}\|g\|_\alpha, \quad \text{where } (\|g\|_\alpha)^\alpha = \int_\Omega |g|^\alpha\, d\mu,\ 1 \leq \alpha < \infty.$$

Thus for $\alpha > 1$, L_α-convergence is a special case of L^Φ-convergence. Although L_1 does not fit into the Orlicz space framework, convergence in L_1 is determined by $\int |\cdot|\, d\mu$; and Theorem 9.4 of [11] permits us to restrict our attention to $\int_\Omega \Phi(\cdot)\, d\mu$ when considering convergence in the Orlicz space L^Φ. By focusing on

Received September 23, 1975.

this integral, we can give proofs of the L^Φ results which carry over to the L_1 case. The next paragraph explains why the proofs also establish L_1-convergence.

The function Φ has two additional properties:

$$\lim_{t\to 0} \Phi(t)/t = 0 \quad \text{and} \quad \lim_{t\to\infty} \Phi(t)/t = \infty.$$

If $\Phi(x)$ were $|x|$, $x \in R$, then Φ would not satisfy these two additional properties, but Φ would satisfy all of the properties mentioned in the first two sentences of the preceding paragraph; those properties suffice for verifying the properties of the integral that establish our assertions for L^Φ. Hence the appropriate statements of our results remain valid if we replace L^Φ by L_1. Some useful consequences of those properties follow.

Since $\Phi(x) \geq x\Phi(1)$, $x \geq 1$, $L^\Phi \subset L_1$, and

$$\int_{(|h|>a)} |h| \, d\mu \leq \Phi(1)^{-1} \int_{(|h|>a)} \Phi(h) \, d\mu, \quad a \geq 1.$$

Thus a sequence $\{h_k\}$ of $\mathscr{A}$-measurable functions is uniformly integrable in L_1 (cf. [12, II, D17]), i.e.,

$$\lim_{a\to\infty} \sup_k \int_{(|h_k|>a)} |h_k| \, d\mu = 0,$$

if $\{h_k\}$ is uniformly integrable in L^Φ, i.e.,

$$\lim_{a\to\infty} \sup_k \int_{(|h_k|>a)} \Phi(h_k) \, d\mu = 0,$$

which implies

$$\lim_{\delta\to 0} \sup_k \sup \left\{ \int_E \Phi(h_k) \, d\mu; \, \mu(E) < \delta \right\} = 0.$$

An L^Φ-Cauchy sequence $\{h_k\}$ is uniformly integrable in L^Φ, i.e.,

$$\lim_{a\to\infty} \sup_k \int_{(|h_k|>a)} \Phi(h_k) \, d\mu = 0.$$

An L_α-Cauchy sequence $\{h_k\}$ is uniformly integrable in L_α, i.e.,

$$\lim_{a\to\infty} \sup_k \int_{(|h_k|>a)} |h_k|^\alpha = 0.$$

If $a > 0$ and $x \geq 1$, then $\Phi(xa) \geq x\Phi(a) = (\Phi(a)/a)(xa)$. Hence,

$$\begin{aligned}\int_\Omega |h| \, d\mu &\leq \int_{(|h|>a)} |h| \, d\mu + a\mu(|h| \leq a) \\ &\leq (a/\Phi(a)) \int_{(|h|>a)} \Phi(h) \, d\mu + a,\end{aligned}$$

and (choosing a small) $\int_\Omega |h_n|\,d\mu \to 0$ if $\int_\Omega \Phi(h_n)\,d\mu \to 0$. We defer further discussion of this property to Section III, where the results of Section II will be available.

If $\alpha = 2$, conditional expectations have especially nice interpretations, namely projections on subspaces in the sub σ-algebra case and on convex cones [4], [10] in the sub σ-lattice case: If $h \in L_2$, then the derivative f of h given $\mathcal{M}$ is the best L_2-approximation to h by functions in $L_2(\Omega, \mathcal{M}, \mu)$. However, in contrast to the σ-algebra setting, projection on a convex cone need not be a linear operation and introducing intermediate projections may change the final result. Nevertheless, Johansen's characterization [10] implies that the Radon-Nikodym derivative is positive homogeneous and monotone on nonnegative functions (i.e., if $U, V, W \in L^\Phi$, $c \geq 0$, $0 \leq V < W$, then $(cU)_{\mathcal{M}} = cu$ and $0 \leq v \leq w$, where u, v, and w are the conditional expectations of U, V, and W given $\mathcal{M}$). Note that since the $\mathcal{M}_k$'s are merely nondecreasing, $\mathcal{M}$ can be any sub σ-lattice of $\mathcal{A}$. Moreover [9, Theorem 2], the derivatives f_k of an L^Φ function h given $\mathcal{M}_k$ converge in L^Φ to the derivative f of h given $\mathcal{M}$, i.e., $\int_\Omega \Phi(f - f_k)\,d\mu \to 0$. In Section II, the Radon-Nikodym derivative is shown to be a continuous map of L^Φ onto the closed convex cone $L^\Phi(\Omega, \mathcal{M}, \mu)$.

II. The Radon-Nikodym derivative is a continuous map of L^Φ onto the closed convex cone $L^\Phi(\Omega, \mathcal{M}, \mu)$

To establish this result, it suffices to show that the derivative is a continuous map on L^Φ and then verify that $L^\Phi(\Omega, \mathcal{M}, \mu)$ is complete in L^Φ.

We begin by recalling Johansen's characterization of the Radon-Nikodym derivative and exposing some of its relevant properties.

Let $h \in L_1$, let λ be defined on $\mathcal{A}$ by $\lambda(E) = \int_E h\,d\mu$, and let f denote the derivative of h given $\mathcal{M}$. Then (cf. [5, Theorem 1.9]) f is characterized by

$$\lambda((f > a) \cap B^c) \geq a\mu((f > a) \cap B^c), \quad B \in \mathcal{M} \tag{1}$$

and

$$\lambda((f \leq b) \cap B) \leq b\mu((f \leq b) \cap B), \quad B \in \mathcal{M}. \tag{2}$$

Notice that the σ-additivity of μ permits $(f > a)$ and $(f \leq b)$ to be replaced by $(f \geq a)$ and $(f < b)$. These inequalities imply that

$$\begin{aligned}\mu(|f| \geq a) &= \mu(f \geq a) + \mu(f \leq -a) \leq a^{-1}\{\lambda(f \geq a) - \lambda(f \leq -a)\} \\ &= a^{-1}\left\{\int_{(f \geq a)} h\,d\mu + \int_{(f \leq -a)} -h\,d\mu\right\}, \quad a \geq 0,\end{aligned}$$

so

$$\mu(|f| \geq a) \leq a^{-1}|\lambda|(|f| \geq a) \quad \text{where } |\lambda|(E) = \int_E |h|\,d\mu. \tag{3}$$

Again, $(|f| \geq a)$ can be replaced by $(|f| > a)$ both above and in the inequality

$$(4) \qquad \int_{(|f| \geq a)} \Phi(f)\, d\mu \leq \int_{(|f| \geq a)} \Phi(h)\, d\mu, \quad a \geq 0,$$

which follows from [9, p. 548–549] and σ-additivity. These inequalities provide a base from which to establish continuity.

Let $h \in L^{\Phi}$ and $\varepsilon > 0$; we shall find $\delta > 0$ such that if $g \in L^{\Phi}$ and

$$\int_{\Omega} \Phi(g - h)\, d\mu < \delta, \quad \text{then} \quad \int_{\Omega} \Phi(e - f)\, d\mu < \varepsilon,$$

where e and f are the derivatives of g and h given $\mathscr{M}$. To this end, let

$$\rho(E) = \int_E g\, d\mu \quad \text{and} \quad \lambda(E) = \int_E h\, d\mu, \quad E \in \mathscr{A};$$

and denote $\int_{\Omega} |g - h|\, d\mu$ by α, so $|\lambda(E) - \rho(E)| \leq \alpha$, $E \in \mathscr{A}$. Combining this inequality, (1) and (2) and their corresponding versions for ρ gives

$$(5) \qquad \lambda((e > a) \cap B^c) \geq a\mu((e > a) \cap B^c) - \alpha$$

and

$$(6) \qquad \lambda((e \leq b) \cap B) \leq b\mu((e \leq b) \cap B) + \alpha, \quad B \in \mathscr{M}.$$

Combining (1) with (6) and (2) with (5) yields

$$(7) \qquad (a - b)\mu((f > a) \cap (e \leq b)) \leq \alpha$$

and

$$(8) \qquad (a - b)\mu((f \leq b) \cap (e > a)) \leq \alpha.$$

Now let β be a positive number and m be a positive integer; then set b and a consecutive terms of the sequence $-(m - 1)\beta, \ldots, -\beta, 0, \beta, \ldots, (m - 1)\beta$. Applying (7) and (8), we obtain

$$(9) \quad \mu(|e - f| > 2\beta) < \mu(|e| \geq m\beta) + \mu(|f| \geq m\beta) + 2(2m - 1)\alpha\beta^{-1}.$$

Next apply (3) to (9) and let $\gamma = 2\beta$ to obtain

$$(10) \quad \mu(|e - f| > \gamma) < (m\beta)^{-1} \left\{ \int_{\Omega} |g|\, d\mu + \int_{\Omega} |h|\, d\mu \right\} + 4(2m - 1)\alpha\gamma^{-1}.$$

If g is near h in L^{Φ} then g is near h in L_1, so the right side of (10) is small if m is large enough and g is sufficiently close to h to make α much smaller than $\gamma/(8m)$. Thus,

$$\begin{aligned} &\int_{\Omega} \Phi(e - f)\, d\mu \\ &\quad \leq \Phi(\gamma) + K_2 \int_{(|e-f| > \gamma)} \{\Phi(e) + \Phi(f)\}\, d\mu \\ &\quad \leq \Phi(\gamma) + K_2 \left\{ 2\Phi(c)\mu(|e - f| > \gamma) + \int_{(|e| > c)} \Phi(e)\, d\mu + \int_{(|f| > c)} \Phi(f)\, d\mu \right\}; \end{aligned}$$

however,

$$\int_{(|f|>c)} \Phi(f)\, d\mu \le \int_{(|f|>c)} \Phi(h)\, d\mu$$

and

$$\int_{(|e|>c)} \Phi(e)\, d\mu \le \int_{(|e|>c)} \Phi(g)\, d\mu$$
$$\le K_2 \left\{ \int_{(|e|>c)} \Phi(g - h)\, d\mu + \int_{(|e|>c)} \Phi(h)\, d\mu \right\}.$$

Finally, (3) implies that

$$\mu(|e| > c) \le c^{-1} \int_\Omega |g|\, d\mu;$$

so we recall that, since $\Phi(h) \in L_1$, $\int_{(|e|>c)} \Phi(e)\, d\mu$ is small if c is large and g is near h in L_1. Hence, $\int_\Omega \Phi(e - f)\, d\mu < \varepsilon$ if we choose c large, then choose γ small and, finally, choose δ wisely. Thus, the Radon-Nikodym derivative is a continuous operator on L^Φ. For the sake of completeness, notice that δ is independent of $\mathscr{M}$.

To finish this section by showing that $L^\Phi(\Omega, \mathscr{M}, \mu)$ is complete in L^Φ, it will be convenient to have the following notation for the truncates of a function available.

Whenever n is a positive integer and u is a (real valued) function defined on Ω, let $u^n(x) = u(x)$, where $|u(x)| \le n$, and $u^n(x) = nu(x)/|u(x)|$ otherwise.

LEMMA 1. *The set of $\mathscr{M}$-measurable functions in L^Φ is complete in L^Φ.*

Proof. Let $h \in L^\Phi$ and (cf. [8, Theorem 2]) let h_k be a sequence of $\mathscr{M}$-measurable functions converging to h in L^Φ. Remembering that h is $\mathscr{M}$-measurable if h^n is $\mathscr{M}$-measurable for all positive integers n, we fix n and let ϕ_n denote the Radon-Nikodym derivative of h^n given $\mathscr{M}$. Since ϕ_n is $\mathscr{M}$-measurable, it suffices to show that $\phi_n = h^n$. Thus we fix n and notice that several functions to be encountered have values in $[-n, n]$; for example (cf. [9]) $\phi_n = \phi_n^n$, so it suffices to show that $\phi_n = h^n$ in L_2. To this end, remember that $h_k^n \in L_2(\Omega, \mathscr{M}, \mu)$ and that taking derivatives does not increase L_2-distance, so $\|\phi_n - h_k^n\| \le \|h^n - h_k^n\|$. Thus,

$$\begin{aligned} \|\phi_n - h^n\| &\le \|\phi_n - h_k^n\| + \|h_k^n - h^n\| \\ &\le 2\|h^n - h_k^n\| \\ &\le 2(2n\|h^n - h_k^n\|_1)^{1/2}, \end{aligned}$$

which goes to zero as $k \to \infty$ if $\int_\Omega \Phi(h^n - h_k^n)\, d\mu \to 0$ as $k \to \infty$. However,

$$|h^n - h_k^n| \le |h - h_k|,$$

so $\int_\Omega \Phi(h^n - h_k^n)\, d\mu \le \int_\Omega \Phi(h - h_k)\, d\mu \to 0$ as $k \to \infty$ and we are done.

III. Two other properties of L^Φ-Cauchy sequences

Suppose that $\{h_k\}$ is a Cauchy sequence in L^Φ such that h_k is $\mathscr{M}_k$-measurable. Since L^Φ is complete, Lemma 1 shows that there exists a $\mathscr{M}$-measurable function h such that $h_k \to h$ in L^Φ. For $g \in L^\Phi$, denote the derivative of g given $\mathscr{M}_k$ by $(g)_k$; notice that $(h_k)_k = h_k$. Let $u_k = (h)_k$; then for $j > k$,

$$\int_\Omega \Phi((h_j)_k - h_k)\, d\mu \leq K_2 \left\{\int_\Omega \Phi((h_j)_k - u_k)\, d\mu + \int_\Omega \Phi(u_k - h_k)\, d\mu\right\},$$

which is small if k is large enough because of the continuity of the derivative at h. But, $\int_\Omega \Phi(g)\, d\mu$ small implies that $\int_\Omega |g|\, d\mu$ is small; and $\mu(|g| > \varepsilon) \leq \varepsilon^{-1} \int_\Omega |g|\, d\mu$. Thus, $\{h_k\}$ satisfies

$$(**) \qquad \lim_m \sup_{n>m} \mu(|(h_n)_m - h_m| > \varepsilon) = 0, \quad \varepsilon > 0.$$

Since $|h_j^n - h_k^n| \leq |h_j - h_k|$ for $n \geq 1$,

$$\int_\Omega \Phi(h_j^n - h_k^n)\, d\mu \leq \int_\Omega \Phi(h_j - h_k)\, d\mu;$$

so the sequence $\{h_j^n\}_{j=1}$ is Cauchy in L^Φ and h_k^n is $\mathscr{M}_k$-measurable. Thus, $\{h_j\}$ satisfies

$$(*) \qquad \lim_k \sup_{j>k} \mu(|(h_j^n)_k - h_k^n| > \varepsilon) = 0, \quad n = 1, 2, \ldots.$$

Thus, uniform integrability in L^Φ, $(*)$ and $(**)$ are necessary conditions in order that a sequence $\{h_k\}$ of $\mathscr{M}_k$-measurable functions converge in L^Φ to a $\mathscr{M}$-measurable function. In the next section we shall verify that uniform integrability in L^Φ and $(*)$ are sufficient conditions; hence, in the presence of uniform integrability, $(*) \Rightarrow (**)$. But first we conclude this section with two examples. The first example is a very simple example to motivate the second, which shows that $(*)$ and $(**)$ arise naturally in the σ-lattice setting. In each example, f_k denotes the derivative of g given $\mathscr{M}_k$.

Example 1. Let $\Omega = \{1, 2, 3\}$; let $\mathscr{M}_1$ be comprised of the empty set, $\emptyset$, and Ω, let $\mathscr{M}_2 = \mathscr{M}_1 \cup \{1\}$ and let $\mathscr{M}_3 = \mathscr{M}_2 \cup \{1, 2\}$. A function f on Ω is determined by three numbers, $c_j = f(j), j \in \Omega$. The $\mathscr{M}_1$-measurable functions are constant functions, f is $\mathscr{M}_2$-measurable if $c_1 \geq c_2 = c_3$, and f is $\mathscr{M}_3$-measurable if $c_1 \geq c_2 \geq c_3$. Let g be defined on Ω by $g(1) = 2$, $g(2) = 4$, $g(3) = 1$. and let μ be the additive function defined on the subsets of Ω by $\mu(1) = \mu(2) = 1/4$, $\mu(3) = 1/2$. Then $f_1 \equiv c$ is obtained by minimizing

$$\int_\Omega |g - f|^2\, d\mu = 4^{-1}(2 - c)^2 + 4^{-1}(4 - c)^2 + 2^{-1}(1 - c)^2,$$

i.e., finding c so that $4^{-1}(2 - c) + 4^{-1}(4 - c) + 2^{-1}(1 - c) = 0$, $c = 2$. To find f_2, first minimize

$$4^{-1}(4 - c)^2 + 2^{-1}(1 - c)^2$$

and find $c_2 = c_3 = 2$, then let $c_1 = 2$. (Notice that $f_2 = f_1$ in this example.) To determine f_3, solve $4^{-1}(2 - c) + 4^{-1}(4 - c) = 0$ and find $c_1 = c_2 = 3$, then let $c_3 = 1$. Finally, to compute $(f_3)_2$, minimize $4^{-1}(3 - c)^2 + 2^{-1}(1 - c)^2$ to find $c_2 = c_3 = 5/3$ and then notice that $c_1 = 3$.

Example 2. Let Ω be the set of positive integers and, for each $k \in \Omega$, let $\mathscr{M}_k$ be comprised of $\emptyset$, Ω, and the sets $\{1, 2, \ldots, n\}$, $n < k$. Let μ be the sigma additive function defined on the subsets of Ω by $\mu(2n - 1) = \mu(2n) = 2^{-(n+1)}$. Let g be defined on Ω by $g(2n - 1) = 1/4^n$ and $g(2n) = 2/4^n$. We shall show that for each positive integer k, f_{2k} and f_{2k+1} behave like the functions in Example 1. To this end, first determine f_{2k+1} as follows. Look at the first $2k$ integers in pairs and find that the best possible choices are $c_{2n-1} = c_{2n} = (3/2)4^{-n}$, $n \leq k$. The tail equation to be solved for a minimum is

$$T_{2k+1}(c) = \sum_{n>k} 2^{-(n+1)}(4^{-n} - c) + 2^{-(n+1)}((2/4^n) - c) = 0.$$

Thus, $c_{2k+j} = c = (3/14)4^{-k}, j \in \Omega$. Turning now to f_{2k}, solve

$$T_{2k}(c) = 2^{-(k+1)}((2/4^k) - c) + T_{2k+1}(c) = 0$$

to find $c = (17/21)4^{-k}$; hence, $c_{2n-1} = c_{2n} = (3/2)4^{-n}$, $n < k$, $c_{2k-1} = 4^{-k}$, $c_{2k} = c_{2k+j} = (17/21)4^{-k}, j \in \Omega$. Finally, to compute $(f_{2k+1})_{2k}$, solve

$$2^{-(k+1)}(\{(3/2)4^{-n}\} - c) + 2^{-k}(\{(3/14)4^{-n}\} - c) = 0$$

to find $c = (9/14)4^{-n}$. Thus, $c_{2n-1} = c_{2n} = (3/2)4^{-n}, n < k, c_{2k-1} = (3/2)4^{-k}$, $c_{2k} = c_{2k+j} = (9/14)4^{-n}, j \in \Omega$.

Thus, while $\{f_k\}$ is a sequence of nonnegative functions bounded by 1, for each positive integer N there exist $n > m > N$ with $\mu(|f_n)_m - f_m| > 0) > 0$. This latter phenomenon does not occur in the σ-algebra setting. Nevertheless, the sequence $\{f_k\}$ is called a martingale in the σ-lattice literature. For the sake of completeness, notice that $\{f_j\}$ is Cauchy in L^Φ and in L_α, $1 \leq \alpha < \infty$; and $f_j = f_j^1, j \geq 1$.

IV. Uniform integrability in L^Φ and condition (*) are sufficient for a sequence $\{h_k\}$ of $\mathscr{M}_k$-measurable functions to converge in L^Φ to a $\mathscr{M}$-measurable function

To verify this assertion, notice that

$$\int_\Omega \Phi(h_j - h_k)\,d\mu = \int_\Omega \Phi(\{h_j - h_j^n\} + \{h_j^n - h_k^n\} + \{h_k^n - h_k\})\,d\mu$$
$$\leq K_3(A_{j,n} + B_{j,k,n} + A_{k,n}),$$

where

$$A_{j,n} = \int_{[|h_j| \geq n]} \Phi(h_j)\,d\mu, \quad j = 1, 2, \ldots,$$

and

$$B_{j,k,n} = \int_\Omega \Phi(h_j^n - h_k^n)\, d\mu$$
$$\leq \Phi(2n)\mu([|h_j^n - h_k^n| > \delta]) + \Phi(\delta)\mu(\Omega), \quad \delta > 0.$$

Uniform integrability of the sequence $\{\Phi(h_k)\}$ implies that $\lim_n \sup_j A_{j,n} = 0$, $\lim_{\delta\to 0} \Phi(\delta) = 0$ and L^Φ is complete [11]; so we conclude that there is an $\mathscr{A}$-measurable function h such that $h_k \to h$ in L^Φ if

$$\lim_m \sup_{j,k>m} \mu([|h_j^n - h_k^n| > \delta]) = 0, \quad \delta > 0, n = 1, 2, \ldots.$$

Since Lemma 1 implies that h is $\mathscr{M}$-measurable, L^Φ-convergence is established by showing convergence in probability as follows. Notice that Johansen's construction for the Radon-Nikodym derivative in [10] implies that $|(h_j^n)_k| \leq n$, $j \geq k$, $n = 1, 2, \ldots$. Then fix n and denote h_j^n by g_j, $j = 1, 2, \ldots$. Thus $|g_j| \leq n$ and it suffices to show that $\{g_j\}$ converges in probability by showing that it converges in L_2.

To this latter end, suppose on the contrary that $\{g_k\}$ does not converge in L_2. Then there exists $\beta > 0$ such that whenever $g \in L_2$ and p is a positive integer $\|g_k - g\|^2 = \int_\Omega |g_k - g|^2\, d\mu > 3\beta$ for some $k > p$. Thus, there exists a subsequence $\{g_{k_j}\}$ such that upon relabeling g_{k_j} as g_j, we have $\|g_{j+1} - g_j\|^2 > 3\beta$ and $\mu(|(g_j)_k - g_k|^2 > \varepsilon) < \delta, j > k = 1, 2, \ldots$, where ε and δ remain to be chosen. To this end, we set $j = k + 1$ and refer to [4, Corollary 2.1] and [10, Theorem 5] to obtain

$$\begin{aligned} \|g_{k+1} - g_1\|^2 = \|g_j - g_1\|^2 &\geq \|g_j - (g_j)_k\|^2 + \|(g_j)_k - g_1\|^2 \\ \text{(i)} \qquad &= \|(g_j - g_k) + (g_k - (g_j)_k)\|^2 \\ &\quad + \|((g_j)_k - g_k) + (g_k - g_1)\|^2. \end{aligned}$$

Next we remember that if u, v, and $w \in L_2$ with $u = v + w$, then

$$\text{(ii)} \qquad \|u\|^2 \geq (\|v\| - \|w\|)^2 \geq \|v\|^2 - 2\|v\|\,\|w\|.$$

Moreover, since $|g_i| \leq n$ and $|(g_j)_k| \leq n$,

$$\|g_j - g_k\|^2 \leq 4n^2 \quad \text{and} \quad \|g_k - (g_j)_k\|^2 \leq \varepsilon + 4n^2\delta = (\beta/4n)^2,$$

since we now let $\varepsilon = \beta^2 2^{-5} n^{-2}$ and $\delta = \beta^2 2^{-7} n^{-4}$. Thus,

$$\begin{aligned} 4n^2 \geq \|g_{k+1} - g_1\|^2 &\geq \|g_{k+1} - g_k\|^2 - \beta + \|g_k - g_1\|^2 - \beta \\ &> \beta + \|g_k - g_1\|^2 \\ &> (k + 2)\beta \to \infty \end{aligned}$$

which is a contradiction.

A similar result for L_2 where (*) is replaced by (**) is established in the next section.

V. Uniform integrability in L_2 and condition (**) are sufficient for a sequence $\{h_k\}$ of $\mathscr{M}_k$-measurable functions to converge in L_2 to a $\mathscr{M}$-measurable function

To establish convergence, suppose on the contrary that $\{h_k\}$ does not converge in L_2. Then there exists $\beta > 0$ such that whenever $g \in L_2$ and p is a positive integer $\|h_k - g\|^2 = \int_\Omega |h_k - g|^2 \, d\mu > 3\beta$ for some $k > p$. Thus, there exists a subsequence $\{h_{k_j}\}$ such that upon relabeling h_{k_j} as g_j, we have

$$\|g_{j+1} - g_j\|^2 > 3\beta \quad \text{and} \quad \mu(|(g_j)_k - g_k|^2 > \varepsilon) < \delta, \quad j > k = 1, 2, \ldots,$$

where ε and δ remain to be chosen. Since a uniformly integrable sequence is bounded in L_2, we can assume that $\|g_i\| \leq C$, $i \geq 1$. For the moment, also suppose that $4C\|g_k - (g_j)_k\| < \beta$. Then we can apply inequalities (i) and (ii) in the preceding section and thereby obtain the contradiction

$$\begin{aligned} 4C^2 \geq \|g_{k+1} - g_1\|^2 &\geq \|g_{k+1} - g_k\|^2 - \beta + \|g_k - g_1\|^2 - \beta \\ &> \beta + \|g_k - g_1\|^2 \\ &> (k+2)\beta \to \infty. \end{aligned}$$

Since L_2 is complete and $L_2(\Omega, \mathscr{M}, \mu)$ is closed in L_2, the theorem is established by verifying that $\lim_k \sup_{j>k} \|g_k - (g_j)_k\| = 0$ in the sequel.

We begin the aforementioned verification by recalling the list of ten properties of the Radon-Nikodym derivative exposed in Section II; then we observe

$$\begin{aligned} \sup_j \|h_j - h_j^n\|_1 &\leq \sup_j \|h_j - h_j^n\| \\ &\leq \sup_j \left\{ \int_{(|h_j| > n)} |h_j|^2 \, d\mu \right\}^{1/2} = \alpha_n, \end{aligned} \tag{11}$$

where $\alpha_n \to 0$ as $n \to \infty$.

Let $\rho > 0$, let $\beta = 1/2$, and let m satisfy the inequality $(4C)/m < \rho/2$. Next let n satisfy the inequality $4(2m - 1)\alpha_n < \rho/2$. Then recall that the derivative $(h^n)_{\mathscr{M}}$ of h^n given $\mathscr{M}$ satisfies $\|(h^n)_{\mathscr{M}}\|_\infty \leq n$, which implies

$$(|f| > n + 1) \subset (|(h^n)_{\mathscr{M}} - f| > 1). \tag{12}$$

Thus, we apply (10), (11), and (12) with $h = g_j$, $g = h^n$, and $\mathscr{M} = \mathscr{M}_k$ to obtain

$$\sup_{j,k} \mu(|(g_j)_k| > n + 1) < \rho. \tag{13}$$

Let $\nu > 0$. Then choose ρ so small that applying (4) with $h = g_j$ and $\mathscr{M} = \mathscr{M}_k$ yields

$$\int_{(|(g_j)_k| > n+1)} |(g_j)_k|^2 \, d\mu \leq \int_{(|(g_j)_k| > n+1)} |g_j|^2 \, d\mu < \nu, \quad j, k \geq 1, \tag{14}$$

because of uniform integrability.

Returning to our basic task, let $G(j, k, \varepsilon) = (|g_k - (g_j)_k| > \varepsilon)$ and notice that

$$\begin{aligned} \|g_k - (g_j)_k\|^2 &\le \varepsilon^2 + \int_{G(j,k,\varepsilon)} |g_k - (g_j)_k|^2 \, d\mu \\ &\le \varepsilon^2 + 4 \int_{G(j,k,\varepsilon)} (|g_k|^2 + |(g_j)_k|^2) \, d\mu \\ &< \varepsilon^2 + 4\{2\nu + 2(n+1)^2 \mu(G(j, k, \varepsilon))\} \quad \text{by (14).} \end{aligned} \tag{15}$$

Condition (**) permits us to choose k_0 such that if $j \ge k \ge k_0$, then $\mu(G(j, k, \varepsilon)) < \delta$, where δ satisfies the equation $(n + 1)^2\delta < \nu$. Hence $\|g_k - (g_j)_k\|^2 < \varepsilon^2 + 16\nu$ if $j \ge k \ge k_0$. Since ε and ν were arbitrary, we are done.

Perhaps a further remark about (*) and (**) is in order. If $\{\mathcal{M}_k\}$ is a non-decreasing sequence of σ-algebras and $\{f_k\}$ is a sequence of $\mathcal{M}_k$-measurable functions which is uniformly integrable in L_1, then [3], [13], and [14] imply that (*) and (**) are equivalent. We have shown that (*) implies (**) when the $\mathcal{M}_k$'s are σ-lattices and $\{f_k\}$ is uniformly integrable in L^Φ. When $L_2 \subset L^\Phi$ one can easily continue the argument in Section V to assert that (**) implies (*), so they are equivalent conditions for uniformly L^Φ-integrable sequences in this case. However, a verification of this latter implication in the general case (e.g., if $L^\Phi = L_1$) seems to involve a very tedious computation.

In conclusion we remark that [6] and [8] permit extensions of these results to sub-lattices of $\mathcal{A}$. For example, the approximation properties established in [8] imply that if the $\mathcal{M}_k$'s are merely sub-lattices of $\mathcal{A}$ with $\mathcal{M}_i \subset \mathcal{M}_{i+1}$, $i \ge 1$, then a uniformly integrable sequence $\{h_k\}$ of $\mathcal{M}_k$-measurable functions is Cauchy in the pre-Hilbert space L_2 if, and only if, it satisfies condition (**).

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