HOMOTOPY GROUPS OF PRO-SPACES

BY

JERROLD W. GROSSMAN

1. Introduction

In this paper we continue the investigation [4], [5] of the homotopy theory of pro-spaces indexed over the positive integers. It is known that the homotopy type of a "nice" pro-space $\{X_s\}$ is dependent upon (among other things) its homotopy pro-groups $\{\pi_n X_s\}$. We show here that in fact, homotopy groups $\pi_n\{X_s\}$—defined as the set of homotopy classes of maps from a kind of pro-n-sphere $\{S_s^n\}$ into $\{X_s\}$—capture the same information as $\{\pi_n X_s\}$. More generally we show that pro-groups indexed over the positive integers contain no more information than groups, by exhibiting a functor P from such pro-groups to groups, such that a map f between pro-groups is an isomorphism if and only if Pf is an isomorphism.

In Section 2 we review pro-spaces and define the homotopy groups. The more general algebraic situation is discussed in Section 3. In Section 4 we show that $\pi_n\{X_s\} \cong P\{\pi_n X_s\}$ and comment on the connection with the proper homotopy groups of a complex.

2. Pro-spaces

For more details see [4]. Let $\mathscr{S}_0$ be the category of pointed, connected spaces, i.e., pointed, connected simplicial sets; $*$ is the basepoint or a one-point space. Then tow-$\mathscr{S}_0$ consists of towers in $\mathscr{S}_0$,

$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = *,$$

denoted $\{X_s\}$, and informally called a pro-space, with maps defined by

$$\mathrm{Hom}_{\text{tow-}\mathscr{S}_0}(\{X_s\}, \{Y_s\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathscr{S}_0}(X_i, Y_j).$$

Similar definitions apply to tow-$\mathscr{G}$ and tow-$\mathscr{A}$ where $\mathscr{G}$ is the category of groups, and $\mathscr{A}$ is the category of abelian groups.

For $n \geq 1$, the nth *homotopy pro-group* of $\{X_s\}$ is the pro-group $\{\pi_n X_s\}$. We say that two maps, f and g, from $\{X_s\}$ to $\{Y_s\}$ are *homotopic* if there is a map $h: \{X_s \times I\} \to \{Y_s\}$ such that the diagram

$$\begin{array}{ccc} \{X_s \vee X_s\} & \xrightarrow{f \vee g} & \{Y_s\} \\ \downarrow & \searrow & \uparrow h \\ \{X_s\} & \longleftarrow & \{X_s \times I\} \end{array}$$

Received March 24, 1975.

commutes where $\vee$ denotes pointed union, $X_s \times I$ is an abuse of notation for $X_s \times I/* \times I$, I is the standard 1-simplex, and the unlabeled maps are the natural ones. If $\{Y_s\}$ is (isomorphic in tow-$\mathscr{S}_0$ to) a tower of fibrations, then [6] "is homotopic to" is an equivalence relation on the set of maps from $\{X_s\}$ to $\{Y_s\}$, and we adopt the usual definitions of and notations for homotopy classes of maps, homotopy equivalence, etc.

Although homotopy pro-groups do not in general determine homotopy type, we do have the following result [4] for *fibrant* pro-spaces, that is, pro-spaces which are isomorphic to towers $\{X_s\}$ of fibrations such that for each s, there exists an n such that $\pi_k X_s = 0$ for $k > n$. (It is easy to turn a pro-space $\{X_s\}$ into a fibrant pro-space with the same homotopy pro-groups [5, Axiom CM5] by first making each X_s into a Kan complex X'_s, then forming $\{P_s X'_s\}$, where P_s denotes the sth Postnikov piece [7, p. 32], and finally turning $\{P_s X'_s\}$ into a tower of fibrations.)

THEOREM 1. *A map $\{X_s\} \to \{Y_s\}$ between two fibrant pro-spaces is a homotopy equivalence if and only if the induced map $\{\pi_n X_s\} \to \{\pi_n Y_s\}$ is an isomorphism of pro-groups for each $n \geq 1$.*

Finally we define the *pro-n-sphere* $\{\mathbf{S}^n_s\}$ by $\mathbf{S}^n_s = \bigvee_{k \geq s} S^n$, where S^n is the n-sphere, and the tower maps $\mathbf{S}^n_{s+1} \to \mathbf{S}^n_s$ are the obvious inclusions. The nth *homotopy group*, $n \geq 1$, of a tower of fibrations $\{X_s\}$, written $\pi_n\{X_s\}$, is then $[\{\mathbf{S}^n_s\}, \{X_s\}]$; the group operation is induced by the usual group operation in $[S^n, X_s]$ and is abelian for $n \geq 2$.

3. Pro-groups

In this section we adopt a more concrete point of view of pro-objects. Let $\mathscr{M}_*$ be the category of pointed sets, and again consider towers in $\mathscr{M}_*$,

$$\cdots \to X_{s+1} \to X_s \to \cdots \to X_1 \to X_0 = *,$$

written $\{X_s\}$. A *level map* from $\{X_s\}$ to $\{Y_s\}$, that is, a sequence of maps $\{X_s \to Y_s\}$ such that $X_{s+1} \to X_s \to Y_s$ equals $X_{s+1} \to Y_{s+1} \to Y_s$ for each s, is called a *pro-isomorphism* if for every $s \geq 1$, there is an $s' > s$ and a map $Y_{s'} \to X_s$ such that the following diagram commutes:

$$\begin{array}{ccc} X_{s'} & \to & Y_{s'} \\ \downarrow & \swarrow & \downarrow \\ X_s & \to & Y_s . \end{array}$$

It is not hard to see that any map in tow-$\mathscr{M}_*$ can be represented by a level map, and that a level map represents an isomorphism in tow-$\mathscr{M}_*$ if and only if it is a pro-isomorphism. Therefore it suffices to consider level maps. These definitions and remarks also apply, of course, to tow-$\mathscr{G}$ and tow-$\mathscr{A}$.

Now let X be a pointed set or a group. Let $I(X)$ be the direct product of a countable number of copies of X, modulo their direct sum. This $I(X)$ consists of sequences $(x_1, x_2, \ldots)$ of elements of X, where two sequences that agree almost everywhere are identified. Obviously I is functorial.

For $\{X_s\} \in$ tow-$\mathscr{M}_*$, define $P\{X_s\} = \varprojlim I(X_s)$. Clearly P is a functor from tow-$\mathscr{M}_*$ to $\mathscr{M}_*$; it is equivalent to Hom $(\{\mathbf{T}_s\}, \;)$, where $\mathbf{T}_s = \coprod_{k \geq s} T$, with the obvious injections $\mathbf{T}_{s+1} \to \mathbf{T}_s$, and T is a fixed set with two elements. We can similarly define

$$P : \text{tow-}\mathscr{G} \to \mathscr{G} \quad \text{and} \quad P : \text{tow-}\mathscr{A} \to \mathscr{A};$$

they are both equivalent to Hom $(\{\mathbf{Z}_s\}, \;)$, where $\mathbf{Z}_s = \coprod_{k \geq s} Z$ and Z is the infinite cyclic group.

LEMMA 1. *Let $\{f_s : X_s \to Y_s\}$ be a level map between towers of pointed sets [resp. groups]. Then $\{f_s\}$ is a pro-isomorphism if and only if $P\{f_s\}$ is an isomorphism.*

COROLLARY 1. *A map $\{X_s\} \to \{Y_s\}$ of pro-groups is an isomorphism if and only if the induced map $P\{X_s\} \to P\{Y_s\}$ is an isomorphism.*

Proof of Lemma 1. By [2, Proposition III.2.2], $\{f_s\}$ is a pro-isomorphism of towers of pointed sets if and only if it is a pro-isomorphism of towers of groups, so we only need to work with pointed sets. By definition, if $\{f_s : X_s \to Y_s\}$ is not a pro-isomorphism, then for some s *either* there are elements $y_{s+k} \in Y_{s+k}$ for each $k \geq 1$ such that the image of y_{s+k} in Y_s is not in the image of X_s in Y_s, *or* there are pairs of distinct elements $x_{s+k}, x'_{s+k} \in X_{s+k}$ for each $k \geq 1$ such that $f_{s+k}(x_{s+k}) = f_{s+k}(x'_{s+k})$ but the images of x_{s+k} and x'_{s+k} are distinct in X_s. In the former case we claim that $P\{f_s\}$ is not surjective. Indeed, elements of $P\{Y_s\}$ are sequences of sequences $(a_{i,j})$ such that $a_{i,j} \in Y_j$ and the image of $a_{i,j+1}$ in Y_j is equal to $a_{i,j}$ for almost all i. For $k = 1, 2, \ldots$, let $a_{k+1,s+k} = y_{s+k}$, and let $a_{1,s}$ and $a_{i,s+k}$ be arbitrary elements of Y_s and Y_{s+k} for $i \leq k$, respectively. Let $a_{k,j}$ be the image in Y_j of $a_{k,s+k-1}$ for $j < s + k - 1$, for $k = 1, 2, \ldots$. Then by the choice of y_{s+k}, this element $(a_{i,j})$ is not in the image of $P\{f_s\}$. Similarly in the latter case we can show that $P\{f_s\}$ is not injective.

We shall also need the following observation for the proof of Theorem 2.

LEMMA 2. *Let $\{G_s\}$ be a tower of abelian groups. Then* $\varprojlim^1 I(G_s) = 0$.

Proof. For a tower of abelian groups $\{H_s\}$, $\varprojlim^1 H_s$ is defined [8] as the cokernel of the map ϕ from $\prod H_s$ to itself which sends $(h_1, h_2, \ldots)$ to $(h_1 - ph_2, h_2 - ph_3, \ldots)$, where p denotes each of the bonding maps $H_{s+1} \to H_s$. In the present case, if $(a_{i,j})$ represents an element of $\prod I(G_j)$, then it is the image under ϕ of $(b_{i,j})$ defined inductively by $b_{i,j} = 0$ if $j > i$, $b_{i,i} = a_{i,i}$, and $b_{i,j} = a_{i,j} + p(b_{i,j+1})$ if $j < i$. Hence $\varprojlim^1 I(G_s) = 0$.

4. Main theorem

THEOREM 2. *Let $\{X_s\}$ be a tower of fibrations. Then $\pi_n\{X_s\}$ is naturally isomorphic to $P\{\pi_n X_s\}$. A map $\{X_s\} \to \{Y_s\}$ between two fibrant pro-spaces is a homotopy equivalence if and only if the induced map $\pi_n\{X_s\} \to \pi_n\{Y_s\}$ is an isomorphism of groups for each $n \geq 1$.*

Proof. It was shown in [4] that if $\{X_s\}$ is a tower of fibrations, then there is a natural short exact sequence

$$* \to \underset{\overleftarrow{j}}{\lim}{}^1 \underset{\overrightarrow{i}}{\lim} [SA_i, X_j] \to [\{A_s\}, \{X_s\}] \to \underset{\overleftarrow{j}}{\lim} \underset{\overrightarrow{i}}{\lim} [A_i, X_j] \to *$$

where SA_i is the reduced suspension of A_i. Letting $\{A_s\} = \{S^n_s\}$, we easily obtain a natural exact sequence of groups

$$* \to \underset{\leftarrow}{\lim}{}^1 I(\pi_{n+1}X_s) \to \pi_n\{X_s\} \to P\{\pi_n X_s\} \to *$$

for each $n \geq 1$. The first conclusion now follows from Lemma 2, and the second follows from Theorem 1 and Corollary 1.

Remark. E. M. Brown [3] first defined P in an equivalent way on the category of towers of groups and level maps. What he called the nth proper homotopy group of a complex is essentially the nth homotopy group of a pro-space representing the tower of inclusions of complements of an exhausting increasing sequence of compact subcomplexes. Brown proved Theorem 2 in this setting.

REFERENCES

1. M. ARTIN AND B. MAZUR, *Etale homotopy*, Lecture Notes in Math., vol. 100, Springer, New York, 1969.
2. A. K. BOUSFIELD AND D. M. KAN, *Homotopy limits, completions, and localizations*, Lecture Notes in Math., vol. 304, Springer, New York, 1972.
3. E. M. BROWN, *Proper homotopy theory in simplicial complexes*, Lecture Notes in Math., vol. 375, Springer, New York, 1974.
4. J. W. GROSSMAN, *Homotopy classes of maps between pro-spaces*, Michigan Math. J., vol. 21 (1974), pp. 355–362.
5. ———, *A homotopy theory of pro-spaces*, Trans. Amer. Math. Soc., vol. 201 (1975), pp. 161–176.
6. D. A. EDWARDS AND H. M. HASTINGS, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math., Springer, New York.
7. J. P. MAY, *Simplicial objects in algebraic topology*, Van Nostrand, New York, 1967.
8. J. W. MILNOR, *On axiomatic homology theory*, Pacific J. Math., vol. 12 (1962), pp. 337–341.

OAKLAND UNIVERSITY
ROCHESTER, MICHIGAN