A FACTORIZATION THEOREM FOR COMPACT OPERATORS

BY

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1. Notation and definitions

If X and Y are Banach spaces, let $K(Y, X)$ denote the compact operators from Y to X with the operator norm, let $F_0(Y, X)$ denote the bounded operators from Y to X with finite-dimensional range, and let $F(Y, X)$ denote the closure of $F_0(Y, X)$ in $K(Y, X)$. If $X = Y$, we write simply $K(X)$, etc.

A Banach space X has the *approximation property* if the identity operator on X can be approximated uniformly on compact subsets of X by operators in $F_0(X)$. If these operators can be taken to have norm less than or equal λ, then X has the *λ-metric approximation property*. Finally, X has the *bounded approximation property* if it has the λ-metric approximation property for some λ.

By "subspace" we mean "closed subspace," and by "isomorphic" we mean "linearly homeomorphic".

2. Statement of results

A theorem of Grothendieck [5, Proposition 35] states that X has the approximation property if and only if $F(Y, X) = K(Y, X)$ for all Banach spaces Y. If $F(Y, X) \neq K(Y, X)$ and $Z = X \oplus Y$, then one easily shows that $F(Z) \neq K(Z)$. However, it is an open question whether $F(X) = K(X)$ implies X has the approximation property. In this paper we prove the following:

THEOREM 1. *If E is a Banach space with the bounded approximation property, and E has a subspace X which fails the approximation property, then E has a subspace Y such that $F(Y, X) \neq K(Y, X)$.*

If in addition E is isomorphic to $E \oplus E$, then E has a subspace S such that $F(S) \neq K(S)$.

For examples of Banach spaces failing the approximation property, the reader is referred to [1], [3], and [7].

The above theorem generalizes a result of Freda Alexander [1]. The author would like to thank Dr. Alexander for making a preprint of [1] available to him.

We note that, if E and X are as above, then the Y produced by the proof of Grothendieck's theorem for which $F(Y, X) \neq K(Y, X)$ is not *a priori* isomorphic to a subspace of E.

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Theorem 1 is essentially a consequence of Grothendieck's theorem and the following factorization theorem for compact operators:

THEOREM 2. *Suppose E is a Banach space with the λ-metric approximation property, that X is a subspace of E, and that Z is an arbitrary Banach space.*

Then given $T \in K(Z, X)$ and $\delta > 0$, there exists a subspace Y of E and operators $U \in K(Y, X)$, $V \in K(Z, Y)$ such that

(i) $T = UV$

(ii) $\|U\| \leq \lambda$

(iii) $V \in \overline{F(E)T}$

(iv) $\|T - V\| \leq \delta$.

Only conclusion (i) of Theorem 2 is used in establishing Theorem 1; this part of the theorem can be restated as follows:

THEOREM 2′. *If E is a Banach space with the bounded approximation property, then any compact operator from a Banach space into a subspace of E can be factored compactly through some subspace of E.*

When $E = L^p(\mu)$, Theorem 2′ is a special case of a theorem of Figiel [4, Theorem 7.4].

Theorem 2 follows from the generalized Cohen factorization theorem for Banach modules.

A Banach space E has a commuting π_λ system if there exists a net of projections $\{P_\gamma\}$ in $F_0(E)$ such that $\|P_\gamma\| \leq \lambda$, $P_\gamma P_{\gamma'} = P_{\gamma'} P_\gamma = P_\gamma$, $\gamma \leq \gamma'$, and $P_\gamma x \to x$ uniformly on compact subsets of E. It is an open question whether every Banach space with the bounded approximation property has a commuting π_λ system for some λ. If E has a commuting π_λ system then the full force of the generalized Cohen theorem is not needed for Theorem 2. Rather, one only needs the following easily proved special case:

THEOREM 3. *Let A be a Banach algebra and B a left Banach A-module such that the linear span of $A \cdot B$ is dense in B. Suppose that $\{p_\gamma\}_{\gamma\in\Gamma}$ is a net in A such that $\|p_\gamma\| \leq \lambda$, $p_\gamma p_{\gamma'} = p_{\gamma'} p_\gamma = p_\gamma$, $\gamma \leq \gamma'$, and $p_\gamma a \to a$ for all $a \in A$.*

Then given $b \in B$, $\delta > 0$, and $0 < \beta < 1$, there exists an increasing sequence $\{\gamma_n\}_{n=1}^\infty$ in Γ such that, setting $\mu = (1 - \beta)/(1 + \beta)$ and $p_{\gamma_0} = 0$, we have

(i) $u = \mu \sum_{n=0}^{\infty} \beta^n (p_{\gamma_{n+1}} - p_{\gamma_n}) \in A$

(ii) $v = \mu^{-1} \sum_{n=0}^{\infty} \beta^{-n} (p_{\gamma_{n+1}} - p_{\gamma_n}) b \in B$

(iii) $b = uv$

(iv) $\|u\| \leq \lambda$

(v) $\|b - v\| \leq \delta$, *for β sufficiently small.*

One feature of Theorem 3 not present in the general theorem is that it gives an explicit description of both terms in the factorization.

3. Proof of results

We proceed first with the proof of Theorem 2. We observe that $F(E)$ is a Banach algebra and that $F(Z, E)$ is a left Banach $F(E)$-module. We now prepare to apply the generalized Cohen factorization theorem [6, Theorem 32.22]: First, since E has the λ-metric approximation property, there exists a net $\{T_\gamma\}$ in $F_0(E)$ such that $\|T_\gamma\| \leq \lambda$ and $T_\gamma x \to x$ uniformly on compact subsets of E. One easily verifies that $\{T_\gamma\}$ is a bounded left approximate identity for $F(E)$. Second, $T \in K(Z, X) \subset K(Z, E) = F(Z, E)$, since E has the approximation property. Third, since $F_0(E)F_0(Z, E) = F_0(Z, E)$, we have that $F(E)F(Z, E)$ is dense in $F(Z, E)$. Therefore, by the generalized Cohen theorem, there exist $U_1 \in F(E)$, $V \in F(Z, E)$ such that $T = U_1 V$, $\|U_1\| \leq \lambda$, $V \in \overline{F(E)T}$, and $\|T - V\| \leq \delta$. Now $U_1 V(Z) = T(Z) \subset X$, so $V(Z) \subset U_1^{-1}(X)$. Let Y be a subspace of E such that $V(Z) \subset Y \subset U_1^{-1}(X)$ and let $U = U_1 \mid Y$. Then $U \in K(Y, X)$ and $\|U\| \leq \lambda$. Also $V \in K(Z, Y)$ and $T = UV$. This establishes Theorem 2.

If $T \in K(Z, X)$ and T is factored as in Theorem 2, then $U \in F(Y, X)$ implies $T \in F(Z, X)$. Thus we see that the first part of Theorem 1 follows from Theorem 2 and Grothendieck's theorem.

For the second part, we observe that if E is isomorphic to $E \oplus E$ and $F(X \oplus Y) \neq K(X \oplus Y)$, where X and Y are subspaces of E, then $X \oplus Y$ is isomorphic to a subspace S of E and $F(S) \neq K(S)$.

Proof of Theorem 3. One first verifies that $p_\gamma b \to b$. Thus there exists an increasing sequence $\{\gamma_n\}_{n=1}^{\infty}$ in Γ such that

$$\|p_{\gamma_n} b - b\| < \beta^n \delta / 2^{n+2}, \quad n = 1, 2, \ldots .$$

Then

$$\|(p_{\gamma_{n+1}} - p_{\gamma_n}) b\| < \beta^n \delta / 2^{n+1}, \quad n = 1, 2, \ldots .$$

It is easy to see that $b = uv$, provided that the series defining u and v converge. Now

$$\begin{aligned} \|u\| &\leq \mu\left(\lambda + \sum_{n=1}^{\infty} \beta^n \cdot 2\lambda\right) \\ &= \mu\left(\lambda + \frac{2\lambda\beta}{1 - \beta}\right) \\ &= \mu\lambda\mu^{-1} \\ &= \lambda, \end{aligned}$$

and

$$\|v - b\| \leq \|\mu^{-1}p_{\gamma_1}b - p_{\gamma_1}b\| + \|p_{\gamma_1}b - b\| + \mu^{-1}\sum_{n=1}^{\infty}\beta^{-n}\frac{\beta^n\delta}{2^{n+1}}$$
$$< (\mu^{-1} - 1)\lambda\|b\| + \frac{\beta\delta}{8} + \mu^{-1}\frac{\delta}{2}$$
$$\leq \delta \quad \text{for } \beta \text{ sufficiently small.}$$

Thus Theorem 3 is established.

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