ON THE HOMOTOPY TYPE OF NON-SIMPLY-CONNECTED CO-H-SPACES

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A co-H-space is a space X which admits a co-multiplication, i.e. a map $\phi: X \to X \vee X$ such that $p \circ \phi \sim q \circ \phi \sim 1_X$, where p and q are the natural projections of $X \vee X$ into its two factors. Important particular cases of co-H-spaces are all suspensions. For the latter it is known, and can be seen by a direct geometrical argument that there exists a homotopy equivalence

$$X \sim Y \vee S \tag{0.1}$$

where Y is 1-connected and S is a wedge of circles or a point. In [3], T. Ganea asked whether the same splitting exists for arbitrary co-H-spaces. In this note we show that the answer is affirmative provided that X is associative, i.e. $(\phi \vee 1) \circ \phi \sim (1 \vee \phi) \circ \phi$. While we were unable to settle the question in the non-associative case we have obtained a necessary and sufficient condition for a space X to admit a decomposition (0.1) in terms of "associative co-operation" (see Section 1 below).

In the first section we state and prove the main result while the second section is devoted to the proof of an important lemma used in Section 1. All spaces considered have the based homotopy type of CW-complexes. We are indebted to Alexander Zabrodsky for several fruitful conversations.

1. The main result

In this section we introduce and prove by a sequence of lemmas the main result of the paper which gives a characterization of the class of spaces which have the homotopy type of a wedge (0.1) between a simply connected space and a 1-dimensional complex. This characterization is given in terms of "co-operation" of a wedge of circles S on X.

DEFINITION 1.1. We say that a wedge of circles S co-operates on X if there is a map $\psi: X \to X \vee S$ such that $p \circ \psi \sim 1_X$ and that $q \circ \psi$ induces an isomorphism on fundamental groups.

Recall that p and q are the natural projections on the first and second factors of a wedge. In particular $G = \pi_1 X$ must be free.

DEFINITION 1.2. We say that a co-operation is associative if

$$(\psi \vee 1) \circ \psi \sim (1 \vee \phi) \vee \psi$$

where ϕ is some associative co-multiplication on S.

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Example 1.3. Let X be a co-H-space. Then it is well known that $\pi_1 X$ is free. Let $\psi\colon X \to X \vee X$ be the co-multiplication; let $m\colon X \to S$ be any map which induces an isomorphism of fundamental groups. Then the composition $\bar{\psi}\colon X \to X \vee X \to X \vee S$ defines a co-operation of S on X. This co-operation is associative if X is associative and if we can define an associative co-multiplication ϕ on S such that the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "m"'] & X \vee X \arrow[d, "m \vee m"] \\ S \arrow[r, "\phi"] & S \vee S \end{tikzcd}$$

homotopy commutes. Such a co-multiplication can easily be found (see 2.4 below).

Example 1.4. Let $X = Y \vee S$ where Y is a 1-connected and let S be a wedge of circles. By viewing for example S as a suspension of a discrete set of points we can define a co-multiplication ϕ on S which is associative. Then the map $\psi = 1 \vee \phi\colon Y \vee S \to Y \vee S \vee S$ is an associative co-operation on X. Our main theorem asserts that the converse of Example 1.4 is also true.

THEOREM 1.5. *A necessary and sufficient condition for a space X to admit a representation* (0.1) *as a wedge between a simply connected space and a wedge of circles is the existence of an associative co-operation of S on X.*

The proof is given by a sequence of lemmas of which the most important ones are the following two lemmas:

LEMMA 1.6. *If X admits an associative co-operation by S then there are maps*

$$X \xrightarrow{k} Y \vee S \xrightarrow{l} X \tag{1.7}$$

where k and l induce isomorphisms of integral homology and of fundamental groups, $l \circ k \sim 1_X$ and Y is simply connected.

Proof. The proof is given in Section 2 below.

LEMMA 1.8. *Let $m\colon F' \to F''$ be a homomorphism of free R-modules, $R = A(G)$ where A is a principal ideal domain and G a free group. If*

$$m \otimes_R 1\colon F' \otimes_R A \to F'' \otimes_R A$$

(*where A is a trivial R-module*) *is a monomorphism then so is also m.*

Proof. Let I denote as usual the augmentation ideal. The following sequence

$$0 \to I \to R \to A \to 0 \tag{1.8.1}$$

is exact. Denote by FI the image of the composition

$$F \otimes I \to F \otimes R \to F$$

where $\otimes$ denotes tensor over R. If F is free then we have $F \otimes I = FI$. For a free group G, the ideal I is free as an R-module [1, p. 192] and $I^n = I^{n-1} \cdot I = I^{n-1} \otimes I$ are free R-modules. Tensoring in (1.8.1) by I^{n-1} we conclude that $I^{n-1}/I^n \approx I^{n-1} \otimes A$, i.e. that I^{n-1}/I^n is a direct sum of copies of A. That together with the assumption on m implies that in the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F'I^n & \longrightarrow & F'I^{n-1} & \longrightarrow & F' \otimes I^{n-1}/I^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle m\otimes 1} & & \\ 0 & \longrightarrow & F''I^n & \longrightarrow & F''I^{n-1} & \longrightarrow & F'' \otimes I^{n-1}/I^n & \longrightarrow & 0 \end{array}$$

in which the two maps on the left are induced by m, the map $m \otimes 1$ is a monomorphism. We have the inclusions $F'I^n \subset F'I^{n-1} \subset F'$. It follows by chasing the above diagram that any element of ker m which lies in $F'I^{n-1}$ must lie already in $F'I^n$ and so

$$\ker m \subseteq \bigcap_{n=1}^{\infty} F'I^n.$$

But $\bigcap_n I^n = 0$ [2] which implies that $\bigcap_n F'I^n = 0$ so that $\ker m = 0$.

The following consequence of the above follows also from the freeness of projectives over a free group.

COROLLARY 1.9. *The assertion of Lemma 1.8 remains valid if we replace in it free modules by projectives.*

Proof. Given any projective module, its direct sum with some free module (in general infinitely generated) is free. This means that if $m: P' \to P''$ is a map of projectives which satisfies the assumptions of Lemma 1.8, we can add a free module F to both P' and P'', such that

$$m \oplus 1 : P' \oplus F \to P'' \oplus F$$

is a map of free modules with the same properties, hence a monomorphism; then so is m.

PROPOSITION 1.10. *Let X be a space, with $\pi_1(X)$ free such that all homology modules of the universal covering of X in positive dimensions, with respect to a principal ideal domain A are projective $A(\pi_1(X))$-modules. Then $H_p(X) = H_p(\tilde{X}) \otimes A$ ($p \neq 1$, the tensor product is over the group ring).*

Proof. Follows immediately from the collapse of the spectral sequence of the covering, since all the groups $H_p(G, H_q(\tilde{X}))$ vanish unless $p = 0$ or $q = 0$ and $p = 1$; here $G = \pi_1 X$.

The following fact is essentially known and "obvious", although we do not know of any published proof.

LEMMA 1.11. *If $X = Y \vee S$, where Y is simply connected and S is a wedge of circles, then the homology groups of the universal covering space $\tilde{X}$ over a coefficient field K, are free $K(\pi_1(X))$-modules.*

Outline of proof. $\tilde{X}$ can be constructed in the following way. Let $\tilde{S}$ be a universal covering space of S. It is a tree, whose vertices are in 1-1-correspondence with elements g of the group $G = \pi_1(X)$. We obtain a simply connected covering space $\tilde{X}$ of X by attaching at each vertex of $\tilde{S}$ a copy Y_g of Y and by extending the covering projection to the identity map of Y_g onto Y. Then $H_*(\tilde{X}) = H_*(Y) \otimes K(G)$ and, since $H_*(Y)$ is K-free, $H_*(\tilde{X})$ is free over $K(G)$.

We now come to the proof of the main result.

Proof of 1.5. Make use of the diagram 1.7. In

$$X \xrightarrow{k} Y \vee S \xrightarrow{l} X$$

one has $l \circ k \sim 1$; let Z be the universal cover of $Y \vee S, \tilde{l}\colon Z \to \tilde{X}$ be a lifting of l and $\tilde{k}$ lifting of k, so that $\tilde{l} \circ \tilde{k} \sim 1_{\tilde{X}}$. In the sequence

$$H_q(\tilde{X}) \xrightarrow{\tilde{k}_*} H_q(Z) \xrightarrow{\tilde{l}_*} H_q(\tilde{X})$$

the composition is the identity so that by Lemma 1.11, $H_q(\tilde{X}, K)$, $q > 0$ is projective over $K(G)$ for any coefficient field K. Since on the other hand both k and l induce isomorphisms of homology, by applying Proposition 1.10 and Corollary 1.9, we obtain that $\tilde{l}_*$ and $\tilde{k}_*$ are isomorphisms for any field K. By the universal coefficient theorem the same is true for the integers. Since k induces an isomorphism of fundamental groups it is a homotopy equivalence. This completes the proof.

2. Spaces with associative co-operation

The purpose of this section is to prove Lemma 1.6. For the sake of convenience in certain arguments we introduce the ad-hoc notion of a *triple* and reformulate Definitions 1.1 and 1.2 in terms of triples: *A* triple (X, G, u) consists of a space X and an isomorphism $u\colon G \to \pi_1 X$. If (X, G, u) and (Y, H, v) are triples a map $f\colon X \to Y$ is said to induce the homomorphism $f_*\colon G \to H$ if the diagram

$$\begin{tikzcd} G \arrow[r, "u"] \arrow[d, "f_*"'] & \pi_1 X \arrow[d, "\pi_1 f"] \\ H \arrow[r, "v"] & \pi_1 Y \end{tikzcd}$$

commutes.

We shall use the following:

LEMMA 2.1. *If (X, G, u) and (Y, H, v) are triples and at least one of the spaces X or Y is a wedge of circles then the correspondence between homotopy classes of maps $f\colon X \to Y$ and the induced homomorphisms $f_*\colon G \to H$ is bijective.*

Proof. If X is the wedge of circles the homotopy classes of maps $X \to Y$ are described by the induced homomorphism; this is still true for Y a wedge of circles by a theorem of Hurewicz, since Y is a space of type $K(\pi, 1)$. Now notice that by the van Kampen Theorem,

$$\pi_1(X \vee Y) \simeq \pi_1(X) * \pi_1(Y)$$

where $*$ denotes the free-product (co-product) of groups. Therefore if X is a co-H-space with co-multiplication $\phi\colon X \to X \vee X$ the group $G = \pi_1 X$ is a co-H-object in the category of groups by the induced map $\phi_*\colon G \to G * G$, and G is a free group [4]. Clearly, if ϕ is associative, then ϕ_* is also associative in the sense that $(\phi_* * 1)\cdot\phi = (1 * \phi_*)\cdot\phi_*$. Keeping these remarks in mind we would like to reformulate the notion of co-operation and say that the group G itself co-operates on the space X. Thus we make Definitions 2.2 and 2.3 which are obviously equivalent to 1.1 and 1.2.

DEFINITION 2.2. If (X, G, u) is a triple, the group G co-operates on X if there is a map $\psi\colon X \to X \vee S$, where (S, G, v) is a wedge of circles such that $p \circ \psi \sim 1_X$, and $q \circ \psi$ induces the identity on G. Here $p\colon X \vee S \to X$ and $q\colon X \vee S \to S$ are the canonical projections. If (X, G, u) and (Y, H, v) are triples, then the van Kampen isomorphism defines a unique triple $(X \vee Y, G * H, w)$. Now if ψ is a co-operation, we have $p_* \circ \psi_* = 1$ and $q_* \circ \psi_* = 1$, so that ψ_* is a co-multiplication. By Lemma 2.1, ψ_* is induced by a unique co-multiplication $\phi\colon S \to S \vee S$.

Remark 2.3. A co-operation $\psi\colon X \to X \vee S$ is associative if the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\psi"'] & X \vee S \arrow[d, "\psi \vee 1"] \\ X \vee S \arrow[r, "(1 \vee \phi)"] & X \vee S \vee S \end{tikzcd}$$

(homotopy) commutes. It then follows from Lemma 2.1 that we have an induced diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\phi"] & S \vee S \arrow[d, "\phi \vee 1"] \\ S \vee S \arrow[r, "1 \vee \phi"] & S \vee S \vee S \end{tikzcd}$$

also commutes. Thus, we have:

Remark 2.4. If G co-operates associatively on X, then the corresponding S is given the structure of an associative co-H-space.

Proof of 1.6. We now get to the proof of the main result of this section. In the proof we shall repeatedly use Lemma 2.1, without mentioning it. In view of this lemma, if the domain or the target of maps is a wedge of circles, then the commutativity of a diagram can be verified by simply checking it on the groups. We need also the following:

LEMMA 2.5. *If $\phi: S \to S \vee S$ is an associative co-multiplication then there exists an inverse $\lambda: S \to S$ such that $F \circ (\lambda \vee 1) \circ \phi$ is trivial, where $F: S \vee S \to S$ is the folding map.*

Proof. We want to mention that when we are speaking here of a wedge of circles we mean in fact any 1-dimensional CW-complex (the cell structure is not given in advance). Since the map $\phi_*: G \to G * G$ is an associative co-multiplication, a theorem of Kan [4] shows the existence of a set of preferred generators $\{x_i\}$ such that the map is defined on them simply as $x_i \to x_i' x_i''$, where x_i' and x_i'' in $G * G$ are copies of x_i in the two factors of the free product. We define λ by requiring it to induce λ_* such that for all i, $\lambda_*(x_i) = x_i^{-1}$. We verify that on generators we have indeed

$$F_* \circ (1 \vee \lambda_*) \circ (x_i) = x_i x_i^{-1} = e$$

and so the same is true for any element of G; this implies that λ is an inverse of ϕ.

Consider the following diagram

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "g"] & S \vee S \arrow[r, "1 \vee \lambda"] \arrow[d, "t"] & S \vee S \arrow[r, "F"] \arrow[d, "t"] & S \arrow[d, "g"] \\
X \vee S \arrow[r, "\psi \vee 1"] \arrow[d, "\psi"] \arrow[drr, "k'"] & (X \vee S) \vee S \arrow[r, "h"] & (X \vee S) \vee S \arrow[r, "f"] & X \vee S \arrow[d, "f'"] \\
X & & (X/S) \vee S \arrow[ur, "l'"] & X
\end{tikzcd} \tag{2.6}$$

where $g = i \circ g'$, with i the inclusion $X \to X \vee S$ and g' any map $S \to X$ inducing the identity on G. Further, $t = \pi \circ (g \vee 1)$, where π permutes the two copies of S in $X \vee S \vee S$, $h = 1 \vee \lambda \vee 1$ and finally F and f' are the foldings

$$F: S \vee S \to S \quad \text{and} \quad f': X \vee S \to X \vee X \to X, f = f' \vee 1.$$

The corresponding diagram induced on groups

$$\begin{tikzcd}
G \arrow[r, "\psi_*"] \arrow[d, "g_*"] & G * G \arrow[r, "1 * \lambda_*"] \arrow[d, "t_*"] & G * G \arrow[r, "F_*"] \arrow[d, "t_*"] & G \arrow[d, "g_*"] \\
G * G \arrow[r, "\psi_* \vee 1"] & (G * G) * G \arrow[r, "h_*"] & (G * G) * G \arrow[r, "f_*"] & G * G
\end{tikzcd} \tag{2.7}$$

clearly commutes and therefore the corresponding diagram of spaces also commutes. Since λ is the inverse for ϕ, the composition in the upper row is null-homotopic. Therefore $f \circ (h \circ (\psi \vee 1) \circ g) \sim 0$ and the canonical map k' lifts by l' to $X \vee S$. Define $k = k' \circ \psi$ and $l = f' \circ l'$. Now it remains to show that the composition

$$f' \circ f \circ h \circ (\psi \vee 1) \circ \psi$$

is homotopic to the identity on X. It follows from the associativity that $(\psi \vee 1) \circ \psi \sim (1 \vee \phi) \circ \psi$ and we also have $f' \circ f = f' \circ (1 \vee F)$ and therefore the composition is homotopic to

$$f' \circ (1 \vee F) \circ (1 \vee \lambda \vee 1) \circ (1 \vee \phi) \circ \psi. \tag{2.8}$$

But $F \circ (\lambda \vee 1) \circ \phi \sim 0$ and thus (2.8) is homotopic to $f' \circ (1 \vee 0) \circ \psi = 1$. It is a matter of routine to check that for instance $k = k' \circ \psi$ induces an isomorphism of all integral homology groups.

References

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