GROUP ALGEBRAS AND MODEL THEORY

BY

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This paper explores some uses of model theory in classical problems of group algebras. Some apparently new results are obtained, especially bearing on the von Neumann finiteness condition $xy = 1 \rightarrow yx = 1$. We begin with a brief discussion of basic model-theoretic methods.

One of the most fundamental theorems of mathematical logic is the completeness theorem for first-order logic (originally due to Godel). It has various applications to algebra, via the concept of a model. We shall develop here some necessary terminology in abbreviated form (for a thorough exposition, see Mendelson [1]). A first-order theory is a structure consisting of a (first-order) language, axioms, and rules (for constructing proofs). Typically, the rules are fixed, and the language and axioms are varied to suit one's purposes. A model M of a theory T is another structure containing objects corresponding to various of the symbols of the language of T, such that the axioms of T are true when interpreted via this correspondence. Thus, any group can be regarded as a model of the theory whose axioms are the usual group axioms as well as the customary axioms of logic and equality.

Now, the completeness theorem can be stated in several ways. Given a (first-order) theory T:

(1) If a sentence s of the language of T is true in all models of T, then s is a theorem of T.

(2) If T is consistent (i.e., no logical contradiction is provable in T) then T has a model.

(3) If every finite subtheory of T (i.e., one using only finitely many of T's axioms) has a model, then so does T.

Usually the third version is the one most readily applicable here, and in this form has its own special name: the compactness theorem for first-order logic. One must exercise caution in using this result. The theory must be specified carefully, and one should know something about its possible models. A typical example is the following.

Let $\mathfrak{F}$ stand for the theory whose language includes the symbols $+$, $\cdot$, 0, and 1, as well as the usual logical symbols $\rightarrow$, $\wedge$, $\vee$, $\forall$, $\exists$, $\sim$, variables, and parentheses and $=$, and whose axioms, in addition to the standard ones of first-order logic with equality, include the field axioms, i.e.,

$$\forall x\, \forall y\, (x + y = y + x), \quad \forall x\, \forall y\, (x \cdot y = y \cdot x),$$

etc. Any model of $\mathfrak{F}$ is necessarily a field (if elements related by "$=$" are

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identified) and conversely every field can be seen as a model of $\mathfrak{F}$. Thus the theorems of $\mathfrak{F}$, by the completeness theorem, are precisely the sentences of $\mathfrak{F}$ which hold in all fields. Now if we add to $\mathfrak{F}$ the axiom $1 + \cdots + 1 = 0$ (where in the sum 1 appears p times, p a prime), this new theory $\mathfrak{F}_p$, is the theory of fields of characteristic p (its models are precisely those fields). Since it is trivial to construct finite fields of characteristic p, indeed of arbitrarily large order, it follows that we have models not only of $\mathfrak{F}_p$, but of the theory

$$\mathfrak{F}_p \cup \{\exists x_1 \cdots \exists x_k (\sim x_1 = x_2 \wedge \sim x_1 = x_3 \wedge \cdots \wedge \sim x_{k-1} = x_k)\},$$

namely any fields of characteristic p and order at least k for arbitrary fixed k. We immediately conclude that there exist *infinite* fields of characteristic p, since we have just seen that every finite subtheory of the theory

$$\mathfrak{F}_p \cup \{c_2\} \cup \{c_3\} \cup \cdots$$

has a model, where c_k is the axiom added to $\mathfrak{F}_p$ above, for $k = 2, 3, \ldots,$ and so the entire theory has a model which of course must be a field of characteristic p and with infinitely many elements. This obviously is not a new result, nor even an original application of our method; it is solely to show the kind of argument that will be used below.

Now we turn to group algebras. First we construct an appropriate theory. We want to include the axioms of $\mathfrak{F}$ above, but only for *field* elements, so we introduce the symbol F into the language, that we will use by writing $F(x)$ to mean x is in the field of the group algebra in question. Similarly $G(x)$ will mean x is in the group of the group algebra. Thus, we want all the axioms of $\mathfrak{F}$ but modified so as to apply to field elements,

$$\forall x\, \forall y\, (F(x) \wedge F(y) \rightarrow x + y = y + x),$$

etc., as well as the closure rules

$$F(x) \wedge F(y) \rightarrow (F(x + y) \wedge F(x \cdot y)).$$

Similarly, for group axioms we use the normal ones restricted to group elements, e.g.,

$$G(1) \wedge \forall y\, (G(y) \rightarrow y \cdot 1 = 1 \cdot y = y),$$

and also the closure rule $G(x) \wedge G(y) \rightarrow G(x \cdot y)$.

Let us call the theory whose language then is that of $\mathfrak{F}$ together with the symbols F and G, and whose axioms are the above field and group axioms, TGA (tentative theory of group algebras). Since $+$ and $\cdot$ are symbols of the language, the usual formal sums found in group algebras will appear in sentences, e.g., the sum $f_1 \cdot g_1 + f_2 \cdot g_2 + \cdots + f_n \cdot g_n$. We would like, if possible, to include axioms that require there to be no elements (of models) that are *not* of this form. This would help a great deal to insure that all models are group algebras. However various practical and theoretical difficulties stand in the way, and indeed we will show there is no genuine first-order theory of group algebras (i.e., one whose models are precisely the group algebras); a main stumbling block is the need to express the fact that we want *finite* formal sums, which presupposes a satisfactory formalization of arithmetic.

Nevertheless we can include the very important axiom that finite sums are formal in the sense that they are to be considered equal only when formally identical; this actually requires infinitely many axioms given as follows, for each $k = 1, 2, 3, \ldots$:

$$
\begin{aligned}
[F(f_1) \wedge \cdots \wedge F(f_k) \wedge F(\bar{f}_1) \wedge \cdots \wedge F(\bar{f}_k) \\
&\wedge G(g_1) \wedge \cdots \wedge G(g_k) \wedge G(\bar{g}_1) \wedge \cdots \wedge G(\bar{g}_k) \\
&\wedge \sim g_1 = g_2 \wedge \sim g_1 = g_3 \wedge \cdots \wedge \sim g_{k-1} = g_k \\
&\wedge \sim \bar{g}_1 = \bar{g}_2 \wedge \sim \bar{g}_1 = \bar{g}_3 \wedge \cdots \wedge \sim \bar{g}_{k-1} = \bar{g}_k] \\
&\quad [f_1 \cdot g_1 + \cdots + f_k \cdot g_k = \bar{f}_1 \cdot \bar{g}_1 + \cdots + \bar{f}_k \cdot \bar{g}_k \\
\leftrightarrow \bigvee_{i_1, \ldots, i_k} f_1 = \bar{f}_{i_1} \wedge f_2 = \bar{f}_{i_2} \wedge \cdots \wedge f_k = \bar{f}_{i_k} \wedge g_1 = \bar{g}_{i_1} \wedge \cdots \wedge g_k = \bar{g}_{i_k}]
\end{aligned}
$$

where the disjunction is over all permutations $(i_1, \ldots, i_k)$ of the integers $1, 2, \ldots, k$.

Finally, we need the product rule for formal sums, and the general associative (for $+$) and distributive laws:

$$F(f_1) \wedge F(f_2) \wedge G(g_1) \wedge G(g_2) \to (f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2) \cdot (g_1 \cdot g_2)$$

$$\forall x \, \forall y \, \forall z \, [(x + (y + z)) = ((x + y) + z)]$$

$$\forall x \, \forall y \, \forall z \, ((y + z) \cdot x = (y \cdot x) + (z \cdot x) \wedge x \cdot (y + z) = (x \cdot y) + (x \cdot z)).$$

There is some redundancy with earlier axioms, but this is unimportant.

We shall then take TGA augmented by these further axioms to be our basic theory $\mathfrak{GA}$ for our study of group algebras, even though as already observed whereas every group algebra can be seen as a model of $\mathfrak{GA}$, not every model of $\mathfrak{GA}$ is necessarily a group algebra.

We now turn to the von Neumann finiteness condition, $xy = 1 \to yx = 1$, i.e., inverses commute. It is a long-standing conjecture that this condition holds in all group algebras, where x and y are arbitrary formal sums. This is already known to be true when the field is of characteristic 0 (due to Kaplansky; see Passman [2, p. 98]); however all proofs of this to date have relied on analytic methods, relating the field to the complex numbers and utilizing properties of operators. Our first result here will show that, in a well-defined sense, algebraic proofs of this must exist.

DEFINITION. Let $\mathfrak{GA}_0$ be $\mathfrak{GA} \cup \{\sim 1 + 1 = 0\} \cup \{\sim 1 + 1 + 1 = 0\} \cup \cdots$ where all sums of 1 a prime number of times appear, so that in the new theory all finite (i.e., positive) characteristics are ruled out.

LEMMA. *Let n be a positive integer. Then the sentence* V_n,

$$[(F(f_1) \wedge \cdots \wedge F(f_n) \wedge F(\bar{f}_1) \wedge \cdots \wedge F(\bar{f}_n) \wedge G(g_1) \wedge \cdots \wedge G(g_n) \wedge G(\bar{g}_1) \wedge \cdots \wedge G(\bar{g}_n) \wedge (f_1 \cdot g_1 + \cdots + f_n \cdot g_n) \cdot (\bar{f}_1 \cdot \bar{g}_1 + \cdots + \bar{f}_n \cdot \bar{g}_n) = 1] \rightarrow (\bar{f}_1 \cdot \bar{g}_1 + \cdots + \bar{f}_n \cdot \bar{g}_n) \cdot (f_1 \cdot g_n + \cdots + f_n \cdot g_n) = 1,$$

is a theorem of $\mathfrak{G}\mathfrak{A}_0$, *i.e., there is a proof entirely in the language of* $\mathfrak{G}\mathfrak{A}_0$, *and using only its axioms and rules, of the fact that formal sum inverses of length* $\leq n$ *commute.*

Proof. We will show that the sentence V_n above holds in all models of $\mathfrak{G}\mathfrak{A}_0$, which will suffice. Let M be a model of $\mathfrak{G}\mathfrak{A}_0$. Then M contains a group G^* corresponding to the symbol G and a field F^* of characteristic 0 corresponding to the symbol F, and M thus contains the group algebra $F^*(G^*)$. But the sentence when interpreted in M refers only to formal sums arising from F^* and G^*, i.e., refers only to $F^*(G^*)$. Indeed V_n refers only to sums of length $\leq n$. And now Kaplansky's result for characteristic 0 tells us that formal sum inverses (of any finite length, i.e., any elements of $F^*(G^*)$) commute, so that V_n is true in M. This completes the proof.

One might think that since the above argument works for all n, that we could instead simply apply it directly to the statement $xy = 1 \rightarrow yx = 1$. However this would no longer necessarily refer, in a model, to formal sums, and so Kaplansky's work would not apply; and to refer to formal sums we must take them one length at a time, unless we alter our formalism to include reference to finiteness which is very troublesome. Still, we see in the above proof that every model of $\mathfrak{G}\mathfrak{A}_0$ *contains* a group algebra.

It is then not necessarily accurate to say that the von Neumann condition is provable in $\mathfrak{G}\mathfrak{A}_0$. That is, quite different proofs may be needed for the cases, say, of length $L \leq 12$ and of $L \leq 1{,}039$; to be sure, the latter would include the former, but neither may do for $L \leq 1{,}040$, etc. At this time, to my knowledge, no one has actually discovered such proofs in $\mathfrak{G}\mathfrak{A}_0$, for any lengths except in the extremely simple cases of $n = 1$ or $n = 2$ where in fact the characteristic is not important at all.

THEOREM. *Let n be a positive integer. Then there is a prime* $p = p(n)$ *such that for any field F of characteristic* $\geq p$ *and for any group G the von Neumann finiteness condition holds for all formal sums of length* $\leq n$ *in the group algebra* $F(G)$.

Proof. The earlier sentence V_n, being a theorem of $\mathfrak{G}\mathfrak{A}_0$, must have a proof in $\mathfrak{G}\mathfrak{A}_0$, which can use only finitely many axioms of $\mathfrak{G}\mathfrak{A}_0$, by the very meaning of proof in a formal theory, and hence V_n must be a theorem of

$$\mathfrak{G}\mathfrak{A} \cup \{\sim 1 + 1 = 0\} \cup \{\sim 1 + 1 + 1 = 0\} \cup \cdots \cup \{\sim 1 + \cdots + 1 = 0\}$$

where the last sum has 1 appearing, say, q times for some prime q. But then if F has characteristic $p > q$ and G is any group, $F(G)$ will be a model of this theory and so its theorem V_n will be true in $F(G)$.

Clearly, in this case too there are algebraic proofs yet to be discovered, in particular the formal proofs of V_n in the above theories for each n and associated p. This is actually no different from the task of finding proofs in $\mathfrak{GA}_0$, of course. Unfortunately it is in general quite difficult to find proofs in formal theories unless very similar informal proofs are already known, even though in our case we have a very definite proof that such proofs (i.e., "algebraic") must exist!

The theorem suggests a function, $p(n)$, where $p(n)$ is the least prime such that V_n is true in all group algebras of characteristic $\geq p$. We already know of course that $p(1) = p(2) = 2$. An immediate question is whether we can show $p(n)$ to be bounded, which if true would say that all group algebras of sufficiently high characteristic are von Neumann finite; but this would appear to depend on a far deeper knowledge of the proof structure of $\mathfrak{GA}_0$ than presently exists. Another direction is to suppose that in general $p(n)$ is increasing and thus one should look for counterexamples using small characteristic and progressively longer sums. In the simplest case, taking $F = \{0, 1\}$ so that sums are merely sums of group elements, for an arbitrary group G, already sums of length 3 become a calculational displeasure, although I believe I have been able to show that after all $xy = 1 \rightarrow yx = 1$ for this case. Others are invited to consider lengths 5, 7, 9, etc. (only odd lengths are possible candidates for this simplest of all fields, as is easily seen) in a search for counterexamples. If $p(n)$ has a genuine dependence on n, one might conjecture that $p(n) =$ the least prime larger than n, since a sum of 1 n times might reasonably appear in a product of two formal sums of length n, whereas it is more difficult to see how longer sums of 1's, and hence higher characteristic, would be significant.

Concerning von Neumann finiteness, one further model-theoretic illustration, although trivial, may be of interest: the above corollary immediately implies that if G is finite then there is a p such that $F(G)$ is von Neumann finite if F is any field of characteristic $\geq p$. This of course is well known (if G is finite F can have *any* characteristic) but the usual argument relies on matrix representations of $F(G)$. In general the approach of models avoids deeper probing of the algebraic structure itself but draws out latent information from the formal aspects of given results.

These techniques are similarly applicable to certain other matters in group algebras, such as zero-divisors which are briefly treated below. However when the problems depend crucially on set-theoretic concepts (such as semisimplicity which requires use of ideals) then the language must be expanded and the models become correspondingly more intractable.

Our final application here concerns the conjecture that "G torsion-free implies $F(G)$ has no zero-divisors." This is readily seen to be equivalent to the following assertion:

"For each fixed n, there cannot be zero-divisors of length $\leq n$ in group algebras of arbitrarily large torsion" where by the "torsion" of $F(G)$ I mean the least prime p such that G has an element of order p, if such exists, and 0 otherwise. Or, equivalently, "for each n, there is a $p(n)$ such that $F(G)$ has no zero-divisors of length $\leq n$ if G has no elements of order $< p$ other than the identity." It follows that $p(n)$ would have to be greater than n if these equivalent statements are true, since it is known that zero-divisors of length n can be constructed in $F(G)$ if G has an element of order n.

The above assertions of equivalence follow immediately from consideration of the theory

$$\mathfrak{G}\mathfrak{A} \cup (G(x) \wedge \sim x = 1 \rightarrow \sim x^2 = 1) \cup \cdots \cup (G(x) \wedge \sim x = 1 \rightarrow \sim x^k = 1) \cup \cdots$$

and the sentence that says there exist two nonzero sums of length no greater than a fixed n, whose product is zero.

Unfortunately the nontrivial direction of the equivalence is one that sheds no light on how one might try to prove the torsion-free conjecture. On the other hand, it does suggest that one look for zero-divisors of bounded length in group algebras of ever-greater torsion (recall the above definition) to try to disprove the conjecture.

If $F(G)$ has *no* zero-divisors then it is von Neumann finite, interestingly: $xy = 1$ implies yx idempotent so $yx(yx - 1) = 0$. (Kaplansky's proof shows that $e^2 = e$ and tr $(e) = 1$ implies $e = 1$ in char 0. And we can thus say that for every n there is a p such that $L(e) \leq n$, with $e^2 = 1$ and tr $(e) = 1$, implies $e = 1$ in char $\geq p$.)

Now, in a negative vein, we point out some limitations of these methods. Returning first to von Neumann finiteness, we observe that in its general form $xy = 1 \rightarrow yx = 1$, it is *not* a theorem of $\mathfrak{G}\mathfrak{A}_0$. This is so since any ring R containing the rationals Q is a model of $\mathfrak{G}\mathfrak{A}_0$, where $F = Q$ and G is trivial. But since we can always embed Q in $R_0 \times Q \equiv R$ for any ring R_0 with identity, then von Neumann finiteness in models of $\mathfrak{G}\mathfrak{A}_0$ will in general fail.

Next we shall prove the aforementioned result that there is no first-order theory T whose models are precisely the group algebras. The referee has pointed out the following very simple proof: we add the axioms (to any supposed T as above) that a fixed constant c is distinct from every finite sum of length n, for each $n = 1, 2, \ldots$. Compactness gives immediately that this is consistent, yet any model of this extension is *not* a group algebra due to the constant c, contradicting the condition on T's models.

However, we will also present a longer proof, which has an apparently wider range of application to other cases than the above argument. Roughly, we shall show that any theory T as above necessarily can be extended so that it contains a complete characterization of a fixed infinite cardinal. But this will be seen to contradict the following model-theoretic result:

THEOREM. *Let T be a first-order theory whose language includes a predicate symbol N. If T has a model in which the set $\bar{N}$ of elements having the property corresponding to N is of cardinality $\kappa \geq \aleph_0$ then $\forall\lambda > \kappa$ there is a model of T in which the corresponding set $\bar{N}$ has cardinality λ.*

This theorem is an extension of Tarski's Cardinality Theorem, and since its proof is only a variation of that of the latter it will be omitted.

Now suppose T is a theory whose models are the group algebras, as above. Extend T to T' by adding the predicate symbols N and A and a function symbol L, as well as the following axioms:

$$F(f) \to A(f), \tag{1}$$

$$G(g) \to A(g), \tag{2}$$

$$A(x) \wedge A(y) \to A(xy) \wedge A(x + y), \tag{3}$$

$$N(y) \to \exists x\,(A(x) \wedge y = L(x)), \tag{4}$$

and for all $n = 1, 2, \ldots$

$$[g_1 \neq g_2 \wedge \cdots \wedge \bar{g}_{n-1} \neq \bar{g}_n \wedge f_1 \neq 0 \wedge \cdots \wedge \bar{f}_n \neq 0] \to L(f_1 g_1 + \cdots + f_n g_n) = L(\bar{f}_1 \bar{g}_1 + \cdots + \bar{f}_n \bar{g}_n) \tag{5}$$

Intuitively $A(x)$ means x is an element of the algebra, and $N(y)$ that y is an integer giving the length $L(x)$ of an algebra element x. Now clearly there are models of T' in which N corresponds to an infinite set $\bar{N}$, namely any group algebra $F(G)$ where G is infinite and $\bar{N}$ is the set of nonnegative integers. It follows from the theorem above that there are models of T' in which the corresponding $\bar{N}$ is *uncountable*. Our conclusion can now be easily reached:

THEOREM. *If T is a first-order theory whose language includes that of $\mathfrak{G}\mathfrak{A}_0$, then the models of T cannot be precisely the group algebras.*

Proof. Suppose otherwise. Extend T to T' as above. Any model of T' now consists in part of a set $\bar{A}$ corresponding to A which is an actual group algebra (by hypothesis), and in part of a set $\bar{N}$ each of whose elements is associated (via L) with an element of $\bar{N}$. But elements of $\bar{A}$ of the same length correspond to the same element of $\bar{N}$. Since there are only countably many lengths in $\bar{A}$, $\bar{N}$ must be countable, a contradiction.

Since this argument relies on cardinality rather than a specific finitary property, it can be similarly applied to uncountable structures or structures in which infinite countable formal sums appear. In particular we easily can determine that there is no theory whose models are precisely the group algebras *and* certain natural analytic structures that come to mind as models of $\mathfrak{G}\mathfrak{A}_0$, namely "ultra" group algebras allowing countably infinite formal sums and obeying the same finitary combinatorial rules but having a norm as well. The norm can be used to introduce an unambiguous set of lengths as above. Numerous variations on this theme are similarly handled. Thus not merely nonfinitely expressible elements can occur in models of $\mathfrak{G}\mathfrak{A}_0$, but ones not even countably expressible. Also, we get trivial proofs, for instance, that there are no theories of precisely the, say, real vector spaces, nor those over any given field or fields of bounded cardinality.

It should be pointed out that with a different kind of formal language these negative results need not hold. In particular in an *infinitary* language, one can adequately characterize the integers and indeed successfully require that every element of the algebra be a (finite) formal sum. This opens up possibilities for further study and in fact infinitary model theory is an active subject; however it also brings up new complications: for instance the compactness theorem is no longer true in general in the nice form that we have used. The reader is referred to [3] for further information along these lines.

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