SEMISIMPLICITY OF 2-GRADED LIE ALGEBRAS

BY

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1. Introduction

In a graded Lie algebra, the defining commutator identities involve signatures depending on the parity of the degrees. Being concerned only with the structural significance of these signatures, we consider only Z_2-gradings, and we call them 2-gradings.

With regard to the notion of semisimplicity, already a casual exploration of 2-graded Lie algebras brings up a surprise. Over any field of characteristic 0, there are 2-graded Lie algebras of arbitrarily high dimension which are *simple*, in the sense of having no proper ideals, but not *semisimple* in the module theoretic sense. In fact, we shall see in Section 5 that the most conventional and natural construction leads to precisely such algebras. Thus, already the first question about the existence of semisimple 2-graded Lie algebras other than the ordinary ones, in which the component of degree 1 is (0), does not have an immediate answer. However, we hasten to add that such 2-graded Lie algebras do exist.

In Section 2, we give a precise description of our setting, and we discuss the basic special features of the universal enveloping algebra of a 2-graded Lie algebra. In Section 3, we deal with the elementary facts concerning semisimple graded modules for 2-graded rings, which we need in Section 4 for proving that the direct sum of semisimple 2-graded Lie algebras is semisimple. Owing to the lack of an intrinsic characterization of semisimple 2-graded Lie algebras, this result, at present, is not as trivial as it ought to be. The other results of Section 4 aim at a reduction of the structure theory to the theory of ordinary semisimple Lie algebras and their representations. In fact, it seems to be most appropriate and promising to regard semisimple 2-graded Lie algebras as a superstructure to be built over classical Lie algebra theory. From this point of view, our only isolated specimen, exhibited in Section 6, appears to be of basic significance, resting upon and extending the representation theory of $sl(2)$.

I wish to thank Professor Murray Gell-Mann who drew my attention to 2-graded Lie algebras by inquiring about the literature on them and telling me that they are being contemplated for use in quantum theory. Leonard E. Ross, in his dissertation (Berkeley, 1964), has provided valuable hints for the material of Section 2 and its use later on. Finally, I thank David Goldschmidt for helping me read his mind in connection with Section 3, and Dragomir Djoković for his help in clarifying a number of issues as they arose while this work was in progress.

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2. Generalities

Let F be a field of characteristic 0. A 2-graded F-Lie algebra is a direct F-space sum $L = L_0 + L_1$, equipped with a bilinear composition $[*, *]$ satisfying the following conditions, where the indices are viewed as integers modulo 2. Let x belong to L_α and y to L_β, while z denotes any element of L. Then

$$[x, y] = -(-1)^{\alpha\beta}[y, x] \in L_{\alpha+\beta}$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{\alpha\beta}[y, [x, z]].$$

If $A = A_0 + A_1$ is a 2-graded associative algebra, we obtain a 2-graded Lie algebra $[A]$, with A as the underlying F-space, by defining the Lie algebra composition so that, for a in A_α and b in A_β, we have

$$[a, b] = ab - (-1)^{\alpha\beta}ba.$$

In particular, suppose that W is given as the direct F-space sum of two F-spaces U and V. Let $E(W)$ be the F-algebra of all linear endomorphisms of W. Let $E_0(W)$ denote the F-subalgebra of $E(W)$ consisting of the endomorphisms that stabilize each of U and V. Let $E_1(W)$ denote the F-subspace of $E(W)$ consisting of the endomorphisms that send U into V, and V into U. Then, as an F-space, $E(W)$ is the direct sum of $E_0(W)$ and $E_1(W)$, and this defines $E(W)$ as a 2-graded associative algebra. A morphism of 2-graded Lie algebras from L to $[E(W)]$ is called a representation of L on W, and W is called a *semigraded L-module*. The semigrading of W is understood to be the given decomposition of W as the direct sum of the subspaces U and V, where (U, V) is regarded as an *un*ordered pair. One obtains a 2-graded F-space and L-module by selecting one of the two possible orderings, putting $W_0 = U$ and $W_1 = V$, or $W_0 = V$ and $W_1 = U$. We shall often make such a selection in order to facilitate a computation or clarify a description.

The well-known facts concerning the universal enveloping algebra of an ordinary Lie algebra extend to 2-graded Lie algebras. In fact, with the help of [2], it is easy to adapt the treatment of the ordinary case, as given in [1], to the 2-graded case. The result is as follows. Let L be a 2-graded F-Lie algebra. There is a 2-graded associative F-algebra $\mathcal{U}(L)$ and a morphism $\mu: L \to [\mathcal{U}(L)]$ of 2-graded Lie algebras having the following universal mapping property. If A is a 2-graded associative algebra and $\rho: L \to [A]$ is a morphism of 2-graded Lie algebras then there is one and only one morphism $\rho^*: \mathcal{U}(L) \to A$ of 2-graded associative algebras such that $\rho^* \circ \mu = \rho$. Moreover, μ is injective, so that we may identify L with its image in $\mathcal{U}(L)$. The F-subalgebra of $\mathcal{U}(L)$ that is generated by $F + L_0$ may be identified with the ordinary universal enveloping algebra $\mathcal{U}(L_0)$.

If (a_i) is an ordered F-basis of L_1, then 1 and the ordered monomials $a_{i_1} \cdots a_{i_q}$, with $i_1 < \cdots < i_q$, constitute a free right and left $\mathcal{U}(L_0)$-basis of $\mathcal{U}(L)$. If V_0 is the F-subspace of $\mathcal{U}(L)$ spanned by 1 and the above monomials in which q is even, and if V_1 is the space spanned by the above monomials in which q is odd, then the homogeneous components of $\mathcal{U}(L)$ may be written

$$\mathcal{U}(L)_\alpha = V_\alpha \otimes_F \mathcal{U}(L_0) \quad (\alpha = 0 \text{ or } 1).$$

Later on, it will be convenient to refer to the subspace V_0^+ of V_0 that is spanned by the monomials in which q is even and $\neq 0$. We have the direct F-space decompositions

$$\mathcal{U}(L) = \mathcal{U}(L_0) + V_0^+ \otimes_F \mathcal{U}(L_0) + V_1 \otimes_F \mathcal{U}(L_0)$$

and

$$L\mathcal{U}(L) = L_0\mathcal{U}(L_0) + V_0^+ \otimes_F \mathcal{U}(L_0) + V_1 \otimes_F \mathcal{U}(L_0).$$

All of this follows easily from the "straightening" process in $\mathcal{U}(L)$: if x belongs to L_0, and a and b belong to L_1, then

$$xa = [x, a] + ax, \quad ab = [q, b] - ba \quad \text{and} \quad aa = \tfrac{1}{2}[a, a].$$

Actually, this gives more detailed information about the L_0-module structure, as follows.

Let V^q denote the F-subspace of $\mathcal{U}(L)$ that is spanned by the ordered monomials of degree q in the basis elements of L_1 ($V^0 = F$, $V^j = (0)$ for $j < 0$). Put

$$W_q = \sum_{i \geqq 0} V^{q-2i}\mathcal{U}(L_0).$$

It is easy to see that $\mathcal{U}(L_0)W_q \subset W_q$. As an F-space, V^q may be identified with the homogeneous component $\Lambda^q(L_1)$ of the exterior F-algebra built on the F-space L_1. The L_0-module structure of L_1 defines an L_0-module structure of $\Lambda^q(L_1)$ in the canonical fashion. Now let A be any L_0-module, viewed also as a $\mathcal{U}(L_0)$-module in the natural way. Let $\otimes_0$ indicate tensoring relative to $\mathcal{U}(L_0)$, and consider the L_0-modules $W_q \otimes_0 A$. By examining the straightening process in $\mathcal{U}(L)$, one sees that the factor L_0-module $(W_q \otimes_0 A)/(W_{q-2} \otimes_0 A)$ is isomorphic with the tensor product L_0-module $\Lambda^q(L_1) \otimes A$, where $\otimes$ indicates tensoring with respect to the base field F. If L and A are finite-dimensional, and L_0 is semisimple, it follows that there is an isomorphism of L_0-modules

$$W_q \otimes_0 A \approx \sum_{i \geqq 0} \Lambda^{q-2i}(L_1) \otimes A.$$

The notion of tensor product of semigraded L-modules requires some discussion. Suppose that $A = A_0 + A_1$ is a 2-graded L-module, and $B = B' + B^*$ is a semigraded L-module. We regard $A \otimes B$ as a semigraded F-space, with components $A_0 \otimes B' + A_1 \otimes B^*$ and $A_0 \otimes B^* + A_1 \otimes B'$. The L-module structure is defined so that, for a in A_α, b in B, and s in L_σ, we have

$$s \cdot (a \otimes b) = (s \cdot a) \otimes b + (-1)^{\alpha\sigma} a \otimes (s \cdot b).$$

If we select the other possible 2-grading of A, then we obtain another L-module structure on $A \otimes B$, which differs from the above in that α is replaced with

$\alpha + 1$. However, the semigraded L-module thus obtained is isomorphic with the above. In fact, an isomorphism from one to the other is η, where the restriction of η to $A \otimes B'$ is the identity map, while the restriction of η to $A \otimes B^*$ is the scalar multiplication by -1.

3. Semisimplicity

The elementary theory of semisimple modules for a ring can be extended to semigraded and 2-graded modules for a 2-graded ring. The required technique is known, but it is not "well-known," and we shall sketch it here.

Let $A = A_0 + A_1$ be a 2-graded ring, and let $M = M' + M^*$ be a semigraded A-module. Let $E(M) = E_0(M) + E_1(M)$ be the 2-graded ring of all additive endomorphisms of M (defined in the same way as was $E(W)$ at the beginning of Section 2). Let $\rho: A \to E(M)$ denote the A-module structure of M, so that ρ is a morphism of 2-graded rings. If $U = U_0 + U_1$ is any 2-graded ring, and $u \in U_\alpha$ and $v \in U_\beta$, we say that u and v *centralize* each other if $uv = (-1)^{\alpha\beta}vu$. If u and v are arbitrary elements of U, we say that they centralize each other if their 2-graded components centralize each other—which amounts to four conditions like the above. Referring to this notion of centralizing, we define the *M-commutant* of A as the centralizer of $\rho(A)$ in $E(M)$, in the element-wise sense. Clearly, this is a 2-graded subring of $E(M)$.

A semigraded A-module is called simple if its only homogeneous A-submodules are (0) and the whole module. It is called semisimple if it is the sum of simple homogeneous submodules. One sees exactly as in the ungraded case that a semigraded A-module M is semisimple if and only if every homogeneous A-submodule of M has a homogeneous A-module complement in M. It is easily seen from this, as in the ungraded case, that *if M is semisimple then every homogeneous A-submodule of M is stabilized by the M-bicommutant* (i.e., the commutant of the commutant) *of A*.

We shall need the following generalization of Jacobson's basic density theorem.

DENSITY THEOREM. *Let A be a 2-graded ring, and let M be a semisimple semigraded A-module. Let S be a finite subset of M, and let α be an element of the M-bicommutant of A. There is an element a in A such that $a \cdot s = \alpha(s)$ for every element s of S.*

Proof. Let U and V denote the components of the semigrading of M. Each element of S is the sum of an element of U and an element of V. Replacing S with the set of these summands, we may suppose that S is the union of a subset $(u_1, \ldots, u_p)$ of U and a subset $(v_1, \ldots, v_q)$ of V. Let W denote the direct sum of $p + q$ copies of the A-module M. We define a semigrading of W so that one component is the direct sum of, first, p copies of U and, second, q copies of V, while the other component is obtained by switching the places of U and V. Evidently, this makes W into a semigraded A-module. Let η denote the endomorphism of M defined by $\eta(u) = u$ for u in U and $\eta(v) = -v$ for v in V. For $i \leqq p$, let ρ_i denote the injection of M onto the ith direct summand of W, and let π_i denote the projection of W onto its ith direct summand. For $i > p$, let ρ_i denote the map $M \to W$ obtained by first applying η and then the injection, and let π_i denote the map $W \to M$ obtained by first applying the projection and then η.

In order to express the consequences of these definitions conveniently, we introduce some more notation.

Let $\varepsilon(i)$ stand for 0 if $i \leqq p$, and for 1 if $i > p$.

Let $\mu\colon A \to E(M)$ be the A-module structure of M, and let $\rho\colon A \to E(W)$ be the A-module structure of W.

Let a be an element of A_σ. Then we have

$$\rho_i \circ \mu(a) = (-1)^{\varepsilon(i)\sigma}\rho(a) \circ \rho_i \quad \text{and} \quad \mu(a) \circ \pi_i = (-1)^{\varepsilon(i)\sigma}\pi_i \circ \rho(a).$$

Also, if e is an element of $E_\sigma(W)$, we have

$$\pi_j \circ e \circ \rho_i \in E_{\sigma+\varepsilon(i)+\varepsilon(j)}(M).$$

Now let C denote the M-commutant of A, and let D denote the W-commutant of A. From the above, we find that, if $d \in D_\sigma$, then $\pi_j \circ d \circ \rho_i \in C_{\sigma+\varepsilon(i)+\varepsilon(j)}$.

Let C' and D' denote the M-bicommutant and the W-bicommutant of A, respectively. For α in C', define the endomorphism α^0 of W by

$$\alpha^0(x_1, \ldots, x_{p+q}) = (\alpha(x_1), \ldots, \alpha(x_{p+q})).$$

If α belongs to C'_σ we have, as above for $\rho(a)$ and $\mu(a)$,

$$\rho_i \circ \alpha = (-1)^{\varepsilon(i)\sigma}\alpha^0 \circ \rho_i \quad \text{and} \quad \alpha \circ \pi_i = (-1)^{\varepsilon(i)\sigma}\pi_i \circ \alpha^0.$$

Now let d be an element of D_σ, and put $d_{ij} = \pi_i \circ d \circ \rho_j$. We know from the above that d_{ij} is an element of $C_{\sigma+\varepsilon(i)+\varepsilon(j)}$. Also, we have

$$d = \sum_{i,j} \rho_i \circ d_{ij} \circ \pi_j.$$

From these facts, we verify directly that, if α belongs to C'_τ, we have $d \circ \alpha^0 = (-1)^{\sigma\tau}\alpha^0 \circ d$, showing that α^0 belongs to D'_τ.

Finally, the element $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ belongs to one of the homogeneous components of W, so that the A-submodule generated by it is a homogeneous A-submodule of W. By the remark just preceding the statement of the theorem, this submodule is stabilized by D'. In particular, there is an element a in A such that

$$\alpha^0(u_1, \ldots, u_p, v_1, \ldots, v_q) = \rho(a)(u_1, \ldots, u_p, v_1, \ldots, v_q),$$

and this is clearly the assertion of the theorem for the set of u_i's and v_j's. This completes the proof.

Just as in the ordinary theory, the density theorem enables one to establish the following result concerning semisimplicity of tensor products.

PRODUCT THEOREM. *Let F be a perfect field of characteristic $\neq 2$, and let A and B be 2-graded F-algebras. Let U and V be finite-dimensional semisimple semigraded modules for A and B, respectively. Then $U \otimes_F V$ is semisimple as a semigraded $A \otimes_F B$-module.*

Proof. We select one of the two possible 2-gradings of U, say $U = U_0 + U_1$, in order to define the semigraded $A \otimes_F B$-module structure of $U \otimes_F V$ (the remarks we made at the end of Section 2 apply also to the present situation, mutatis mutandis). Clearly, it suffices to deal with the case where U and V are simple. Moreover, just like in the ungraded theory, one sees (via an easy Galois descent argument, which is applicable because F is perfect) that it suffices to prove the theorem in the case where F is algebraically closed. Thus, we shall now assume that F is algebraically closed, and that U and V are simple.

Let C denote the U-commutant of A. Since U is simple and F algebraically closed, Schur's Lemma shows that C_0 consists of the scalar multiplications on U, i.e., $C_0 = F$. Let us choose an F-basis $(x_1, \ldots, x_p)$ of U_0 and an F-basis $(y_1, \ldots, y_q)$ of U_1.

First, let us consider the case where $C_1 = (0)$. In this case, we shall see that $U \otimes V$ is simple. Let

$$w = \sum_i x_i \otimes s_i + \sum_j y_j \otimes t_j$$

be any nonzero homogeneous element of $U \otimes V$, so that all the s_i's lie in one of the components of V, and all the t_j's lie in the other component of V. Now the U-bicommutant C' of A is the 2-graded F-algebra of all F-linear endomorphisms of U. Therefore, the density theorem shows that each $x_i \otimes s_i$ and each $y_j \otimes t_j$ belongs to the A-submodule of $U \otimes V$ that is generated by w. Since one of these is different from 0, it is now clear from the simplicity of U and V that the $A \otimes B$-submodule generated by w coincides with $U \otimes V$. Thus, $U \otimes V$ is simple as a semigraded $A \otimes B$-module.

Now suppose that $C_1 \neq (0)$. If c is a nonzero element of C_1, then $c(U)$ is a nonzero homogeneous A-submodule of U, so that it coincides with U. Therefore, c is an F-linear automorphism of U. Using this, and the facts that $C_1 C_1 \subset C_0 = F$ and F is algebraically closed, we see that there is an element c in C_1 such that $C_1 = Fc$ and $c^2 = 1$. Now C_0' consists of all linear endomorphisms e of U such that e stabilizes U_0 and U_1 and $ec = ce$, while C_1' consists of all linear endomorphisms f such that $f(U_0) \subset U_1, f(U_1) \subset U_0$, and $fc = -cf$. Clearly, $(c(x_1). \ldots, c(x_p))$ is an F-basis of U_1. We consider a nonzero homogeneous element w, written as above, but with $c(x_j)$ in the place of y_j. The density theorem now shows that the A_0-submodule of $U \otimes V$ that is generated by w contains each $x_i \otimes s_i + c(x_i) \otimes t_i$. For some index i, this is different from 0. Choosing such an index, we simplify the notation and assume, without loss of generality, that

$$w = x \otimes s + c(x) \otimes t$$

where x is a nonzero element of U_0, and s and t are homogeneous elements of V, belonging to different components of V, and not both being 0.

Now let D denote the V-commutant of B. If $D_1 = (0)$, we can proceed exactly as above, with the roles of U and V interchanged, to show that $U \otimes V$ is simple. It remains only to examine the case where $D_1 \neq (0)$, in which case we have, as above for C, that $D = F + Fd$, with $d^2 = 1$. Choose σ from F such that $\sigma^2 = -1$. Put

$$e = i_U \otimes i_V + \sigma c \otimes d \quad \text{and} \quad f = i_U \otimes i_V - \sigma c \otimes d.$$

Then e and f are homogeneous even $A \otimes B$-module endomorphisms, $e^2 = 2e$, $f^2 = 2f$, and $ef = 0 = fe$. Hence $U \otimes V$ is the direct sum of the homogeneous $A \otimes B$-submodules $e \cdot (U \otimes V)$ and $f \cdot (U \otimes V)$.

Now let p be a nonzero homogeneous element of $e \cdot (U \otimes V)$. As above, $(A \otimes B) \cdot p$ contains $w = x \otimes s + c(x) \otimes t$. If the elements t and $d(s)$ are linearly independent, it follows from the density theorem for V that there is an element b in B_0 such that $b \cdot t = 0$ and $b \cdot d(s) = d(s)$, whence $b \cdot s = s$. Hence the B_0-module generated by w contains $x \otimes s$, and therefore, as above, $(A \otimes B) \cdot p$ coincides with $U \otimes V$. The same conclusion holds if one of s or t is 0.

In the remaining case, we have

$$w = x \otimes s + \tau c(x) \otimes d(s)$$

where τ is a nonzero element of F. By the density theorem, there is an element a in A_1 such that $a \cdot u = c(u)$ for every u in U_0, and $a \cdot u = -c(u)$ for every u in U_1. Similarly, there is an element b in B_1 such that $b \cdot v = d(v)$ for every v in the homogeneous component of V containing s, and $b \cdot v = -d(v)$ for every v in the other homogeneous component of V. Now we have

$$(a \otimes b) \cdot w = c(x) \otimes d(s) - \tau x \otimes s$$

and

$$w - \tau(a \otimes b) \cdot w = (1 + \tau^2)x \otimes s.$$

If $\tau^2 \neq -1$ this gives $(A \otimes B) \cdot p = U \otimes V$. Otherwise $\tau = \pm\sigma$, so that w is either $e \cdot (x \otimes s)$ or $f \cdot (x \otimes s)$. Since w belongs to $e \cdot (U \otimes V)$, we must therefore have $w = e \cdot (x \otimes s)$ and hence $(A \otimes B) \cdot p = e \cdot (U \otimes V)$. This proves that $e \cdot (U \otimes V)$ is simple (or (0)). Similarly, $f \cdot (U \otimes V)$ is simple (or (0)). In any case, $U \otimes V$ is semisimple, so that the product theorem is proved.

4. Semisimple 2-graded Lie algebras

We begin with an application of the product theorem of the last section.

THEOREM 4.1. *Let F be a field of characteristic 0, and let S and T be 2-graded F-Lie algebras. Let M be a finite-dimensional semigraded module for the direct sum of S and T. If M is semisimple as an S-module and as a T-module, then M is semisimple as an $(S + T)$-module.*

Proof. Let I denote the annihilator of M in the universal enveloping algebra $\mathcal{U}(S)$ of S. Then I is a homogeneous ideal of $\mathcal{U}(S)$, and the 2-graded factor algebra $\mathcal{U}(S)/I$ may be viewed, in the natural way, as a 2-graded finite-dimensional $\mathcal{U}(S)$-module. As such, it is isomorphic with a $\mathcal{U}(S)$-submodule of a finite direct sum of copies of M (suitably 2-graded). Therefore, $\mathcal{U}(S)/I$ is semisimple as a 2-graded $\mathcal{U}(S)$-module.

Let M_T denote M with its structure as a semigraded $\mathcal{U}(T)$-module, and consider the semigraded $\mathcal{U}(S) \otimes_F \mathcal{U}(T)$-module $(\mathcal{U}(S)/I) \otimes_F M_T$. By the product theorem of Section 3, it is semisimple. From the $\mathcal{U}(S)$-module structure of M, we have a surjective F-linear map π: $(\mathcal{U}(S)/I) \otimes_F M_T \to M$. As in the ungraded case, $\mathcal{U}(S + T)$ is naturally identifiable with $\mathcal{U}(S) \otimes_F \mathcal{U}(T)$. If, accordingly, M is regarded as a semigraded $\mathcal{U}(S) \otimes_F \mathcal{U}(T)$-module, then π is clearly a morphism of semigraded $\mathcal{U}(S) \otimes_F \mathcal{U}(T)$-modules. Consequently, M is semisimple as a semigraded module for $\mathcal{U}(S) \otimes_F \mathcal{U}(T)$, which means that it is semisimple as a semigraded $(S + T)$-module. This completes the proof of Theorem 4.1.

We shall say that a finite-dimensional 2-graded Lie algebra L is semisimple if every semigraded finite-dimensional L-module is semisimple. It is clear from Theorem 4.1 that *a direct sum of semisimple 2-graded Lie algebras is semisimple. Obviously, a homomorphic image of a semisimple 2-graded Lie algebra is semisimple.*

The following lemma records an elementary fact concerning ordinary Lie algebras. For use here and later on, we introduce a notational device. If M is any module, and S is a set of endomorphisms of M, then M^S denotes the S-annihilated part of M.

LEMMA 4.2. *Let L be an ordinary finite-dimensional F-Lie algebra, where F is a field of characteristic* 0. *If L is not semisimple, there exists a finite-dimensional L-module A such that $(L \cdot A)^L \neq (0)$.*

Proof. Write $L = R + S$, where R is the radical of L, and S is semisimple. Let T denote the S-module dual to the S-module $R/[R, R]$. Let A be the direct F-space sum $T + F$, made into an L-module as follows. First, $L \cdot F = (0)$. Next, for τ in T, r in R, and s in S, put

$$(r + s) \cdot \tau = s \cdot \tau + \tau(r + [R, R]).$$

It suffices to verify that $s \cdot (r \cdot \tau) - r \cdot (s \cdot \tau) = [s, r] \cdot \tau$, and this is seen immediately. Since $R \neq [R, R]$, we have $T \neq (0)$, whence $F \subset L \cdot A$, so that Lemma 4.2 is established.

THEOREM 4.3. *If L is a semisimple 2-graded Lie algebra, then $[L_0, L] = L$, and L_0 is semisimple as an ordinary Lie algebra.*

Proof. Suppose that L_0 is not semisimple. Let A be as in Lemma 4.2 (applied to L_0), and choose a nonzero element b in $(L_0 \cdot A)^{L_0}$. Let $\otimes_0$ indicate tensoring with respect to $\mathcal{U}(L_0)$. We consider the L-module $M = \mathcal{U}(L) \otimes_0 A$. From Section 2, we know that M is finite-dimensional. It may be viewed as a 2-graded L-module, with $\mathcal{U}(L)_0 \otimes_0 A = M_0$ and $\mathcal{U}(L)_1 \otimes_0 A = M_1$. Let N denote the homogeneous L-submodule $\mathcal{U}(L) \otimes_0 Fb$ of M. Suppose that it has a homogeneous L-module complement K in M.

We may write $b = \sum_i x_i \cdot a_i$, with each x_i in L_0 and each a_i in A. We must have $1 \otimes a_i = k_i + h_i$, where k_i lies in K_0 and h_i in N_0. Applying the endomorphism corresponding to x_i and summing for i, we obtain $1 \otimes b = k + h$, where $k = \sum_i x_i \cdot k_i$ belongs to K_0 and $h = \sum_i x_i \cdot h_i$ belongs to $L_0 \cdot N_0$. This shows that k belongs to $K_0 \cap N_0$, so that we must have $k = 0$ and $1 \otimes b = h$. Using the notation of Section 2, we have $N_0 = V_0 \otimes_F Fb$. From the description of $L\mathcal{U}(L)$ given in Section 2 and from $L_0 \cdot b = (0)$, we see immediately that $L_0 \cdot N_0 \subset V_0^+ \otimes_F Fb$. Hence we have from the above that $1 \otimes b$ belongs to $V_0^+ \otimes_F Fb$, i.e., we have the contradiction that 1 belongs to V_0^+. We have shown that N cannot have a homogeneous L-module complement in M. Therefore, if L is semisimple, L_0 must be semisimple.

Now let us view L as a semigraded L-module, via the adjoint representation. Clearly, L^L is a homogeneous L-submodule of L. Since L is semisimple, there is a homogeneous L-module complement, P say, for L^L in L. We have $[L, L] = L \cdot P \subset P$. Next, the homogeneous L-submodule $L \cdot P$ of P has a homogeneous L-module complement Q in P. Now $L \cdot Q \subset Q \cap L \cdot P = (0)$, so that $Q \subset L^L$. Therefore, $Q = (0)$ and $L \cdot P = P$. Thus, L is the direct L-module sum of $[L, L]$ and L^L, which implies that L^L is semisimple as a 2-graded Lie algebra (being a homomorphic image of L). As in the case of an ordinary abelian Lie algebra, we see that this can be the case only if $L^L = (0)$. Hence we have $[L, L] = L$, whence $L = L_0 + [L_0, L_1]$. From the fact that L_0 is semisimple, it now follows that $[L_0, L] = L$, which completes the proof of Theorem 4.3.

THEOREM 4.4. *If L is a semisimple 2-graded Lie algebra, then L is a direct sum of 2-graded Lie algebras S and T, where $T = T_0$ is an ordinary semisimple Lie algebra, and S is a semisimple 2-graded Lie algebra such that $[S_1, S_1] = S_0$.*

Proof. Put $S = L_1 + [L_1, L_1]$. Evidently, S is a homogeneous ideal of L. The semisimplicity of L implies that there is a complementary homogeneous ideal T, and clearly we must have $T = T_0 \subset L_0$. As homomorphic images of L, both S and T are semisimple 2-graded Lie algebras, and

$$[S_1, S_1] = [L_1, L_1] = S_0.$$

This completes the proof of Theorem 4.4.

The structure of a 2-graded Lie algebra L becomes more transparent when viewed as follows. The composition from $L_1 \times L_1$ may be regarded as a morphism of L_0-modules from the homogeneous component $S^2(L_1)$ of the symmetric algebra built on L_1 to the (adjoint) L_0-module L_0. Denoting this by

η, so that, for x and y in L_1, we have $\eta(xy) = [x, y]$, the remaining identity of the 2-graded Lie algebra structure says that, for all x, y, z in L_1, we have $[\eta(xy), z] + [\eta(yz), x] + [\eta(zx), y] = 0$.

Let T denote the subspace of $S^2(L_1)$ that is spanned by the elements of the form $\eta(xy)\cdot(uv) + \eta(uv)\cdot(xy)$, where x, y, u, v range over L_1. Evidently, T is contained in the kernel of η. A straightforward computation shows that T is an L_0-submodule of $S^2(L_1)$.

PROPOSITION 4.5. *If L is semisimple then the L_0-submodule T of $S^2(L_1)$ defined above coincides with the kernel of the L_0-module homomorphism $\eta\colon S^2(L_1) \to L_0$, and $S^2(L_1)^{L_0} = (0)$.*

Proof. Let M_0 denote the L_0-module $S^2(L_1)/T$, and let M_1 denote the L_0-module L_1. Put $M = M_0 + M_1$. We define an action of L_1 on M such that M becomes a 2-graded L-module. Let x be an element of L_1. The action of x on $M_1 = L_1$ is defined by putting $x\cdot y = xy + T \in M_0$. Since T lies in the kernel of η, there is a linear map from $S^2(L_1)/T$ to L_1 that sends each element $uv + T$ onto $[x, \eta(uv)] = [x, [u, v]]$. Thus we may define an action of x on M_0 such that $x\cdot w = [x, \eta(w)]$. Now one can verify directly that this makes M into a 2-graded L-module. The definition of T enters in this verification as follows.

Let x and y be elements of L_1. We verify that, for w in M_0, we have

$$x\cdot(y\cdot w) + y\cdot(x\cdot w) = [x, y]\cdot w.$$

It suffices to consider an element w of the form $uv + T$. We have

$$\begin{aligned} x\cdot(y\cdot(uv + T)) + y\cdot(x\cdot(uv + T)) &= x\cdot[y, [u, v]] + y\cdot[x, [u, v]] \\ &= x[y, [u, v]] + y[x, [u, v]] + T \\ &= -[u, v]\cdot(xy) + T \\ &= [x, y]\cdot(uv) + T \\ &= [x, y]\cdot(uv + T). \end{aligned}$$

Leaving the remaining parts of the verification to the reader, we observe that the map $M \to L$ whose restriction to M_0 is the map induced by η, and whose restriction to M_1 is the identity map $M_1 \to L_1$, is a morphism of 2-graded L-modules. Let $K = K_0 + K_1$ be its kernel. Evidently, $K \subset M_0$, so that $K = K_0$ and $K_1 = (0)$. Since L is semisimple, K has a homogeneous L-module complement $P = P_0 + P_1$ in M. Clearly, we must have $P_1 = M_1$, and M_0 is the direct sum of K_0 and P_0. Now $L_1\cdot P_1 \subset P_0$, i.e., $L_1\cdot M_1 \subset P_0$. On the other hand, it is clear from the definition that $L_1\cdot M_1 = M_0$. Therefore, we have $M_0 \subset P_0$, whence $K_0 = (0)$, so that $K = (0)$. This means that the kernel of η coincides with T. Since L_0 is semisimple, we have $L_0^{L_0} = (0)$. Since η is a morphism of L_0-modules, it follows that $S^2(L_1)^{L_0}$ lies in the kernel T of η. Since $T \subset L_0 \cdot S^2(L_1)$, this implies that $S^2(L_1)^{L_0} = (0)$, so that Proposition 4.5 is proved.

5. Examples

The standard example of a 2-graded Lie algebra is the 2-graded Lie algebra $[E(W)]$, defined in Section 2, starting from a direct F-space sum $W = U + V$ (we assume that F is a field of characteristic 0). Call this L, so that $L_0 = E_0(W)$ and $L_1 = E_1(W)$. Let us choose an F-basis $(x_1, \ldots, x_{m+n})$ of W such that $(x_1, \ldots, x_m)$ is a basis of U, and $(x_{m+1}, \ldots, x_{m+n})$ is a basis of V, and let us suppose that $mn \neq 0$. Using the standard notation e_{pq} for the linear endomorphism of W sending x_q onto x_p and annihilating the other basis elements, we see that L_0 is spanned by the e_{pq}'s where p and q are either both $\leqq m$ or both $> m$, and that L_1 is spanned by the remaining e_{pq}'s. It is easy to verify that $[L_0, L_1] = L_1$, and that $[L_1, L_1]$ consists precisely of all those elements of L_0 whose restrictions to U and V have equal traces. This gives $[L, L]_0 = [L_1, L_1]$, and $[L, L]_1 = L_1$.

In general, let us call a 2-graded Lie algebra *odd* (or oddly generated) if it is generated by its homogeneous component of degree 1. In view of Theorem 4.4, we are primarily interested in odd semisimple 2-graded Lie algebras. We have just seen that the above $[L, L]$ is an odd 2-graded Lie algebra. If u and v denote the linear projections $W \to U$ and $W \to V$, respectively, that correspond to the decomposition $W = U + V$, then $[L, L]_0$ contains $nu + mv$ as a central element, so that it is *not* a semisimple ordinary Lie algebra. Thus, in contrast to the ungraded analog, $[L, L]$ is *not* a semisimple 2-graded Lie algebra, by virtue of Theorem 4.3. It is somewhat startling that, at the same time, if $n \neq m$, then $[L, L]$ is actually simple, in the sense that its only ideals are (0) and $[L, L]$.

In order to verify this fact, let us consider a nonzero ideal I of $[L, L]$. In writing matrix units, let a roman index be understood to belong to $(1, \ldots, m)$, and a greek index to $(m + 1, \ldots, m + n)$. Let a be a nonzero element of I, and write $a = a_0 + a_1$, with a_0 in $[L, L]_0$ and a_1 in $[L, L]_1 = L_1$. Then we have $[nu + mv, a] = [nu + mv, a_1]$. Moreover,

$$[nu + mv, e_{\alpha i}] = (m - n)e_{\alpha i} \quad \text{and} \quad [nu + mv, e_{i\alpha}] = (n - m)e_{i\alpha}.$$

Hence it is clear that $[nu + mv, a_1]$ is 0 only if $a_1 = 0$, and we conclude that I contains a nonzero homogeneous ideal $J = J_0 + J_1$. The Lie algebra L_0 is the direct sum of the three simple ideals $[E(U), E(U)]$, $[E(V), E(V)]$, and $F(nu + mv)$, and we must consider each one of the following four cases:

(1) $(0) \neq [E(U), E(U)] \subset J_0$,

(2) $(0) \neq [E(V), E(V)] \subset J_0$,

(3) $F(nu + mv) \subset J_0$,

(4) $J_0 = (0)$.

In the first three cases, it is easy to check that $[L_1, J_0] = L_1$, which implies that $J = [L, L]$. In case (4), we have $J_1 \neq (0)$. Let

$$0 \neq p = \sum_{\alpha, i} c_{\alpha i} e_{\alpha i} + \sum_{j, \beta} c_{j\beta} e_{j\beta} \in J_1.$$

Then we have $[e_{i\alpha}, p] = c_{\alpha i}(e_{ii} + e_{\alpha\alpha})$ and $[e_{\beta j}, p] = c_{j\beta}(e_{\beta\beta} + e_{jj})$, which shows that, contrary to assumption, $J_0 \neq (0)$. This completes the proof of our assertion that $[L, L]$ is simple.

In the case where $n = m$, our 2-graded Lie algebra $[L, L]$ contains the identity map $u + v$, so that it is not simple. Let M denote the 2-graded factor Lie algebra $[L, L]/F(u + v)$. Then M is an odd 2-graded Lie algebra, and M_0 is the semisimple Lie algebra $[E(U), E(U)] + [E(V), E(V)]$. Without examining this further in the general case, we observe that, in the case $m = 2 = n$, this algebra M is *not* semisimple as a 2-graded Lie algebra. In order to see this, let a_{ij} denote the canonical image of e_{ij} in M. Then it can be verified directly that the following element of $S^2(M_1)$ is annihilated by M_0:

$$(a_{13} - a_{42})(a_{31} - a_{24}) + (a_{41} + a_{23})(a_{14} + a_{32}).$$

By Proposition 4.5, it follows that M is not semisimple.

6. The $sl(2)$ model

From our above discussion, one might incline to the (false) belief that the only 2-graded semisimple Lie algebras are the ordinary semisimple Lie algebras. Fortunately, it is possible to show by direct calculation that the odd candidate of the lowest possible dimension 5 is indeed semisimple. In the notation used in Section 5, this candidate is a subalgebra of $[L, L]$, in the case where $m = 1$ and $n = 2$. Put $x = e_{23}$, $y = e_{32}$, $z = e_{22} - e_{33}$, and put $S_0 = Fx + Fy + Fz$. Then S_0 is the split 3-dimensional simple Lie algebra:

$$[z, x] = 2x, \quad [z, y] = -2y, \quad [x, y] = z.$$

Now put $u = e_{21} - e_{13}$, $v = e_{12} + e_{31}$, $S_1 = Fu + Fv$. The representation of S_0 on S_1 is the natural 2-dimensional representation of $sl(2)$:

$$[x, u] = 0, \quad [y, u] = v, \quad [z, u] = u,$$

$$[x, v] = u, \quad [y, v] = 0, \quad [z, v] = -v.$$

The remaining part of the 2-graded Lie algebra structure of $S = S_0 + S_1$ is given by

$$[u, u] = -2x, \quad [v, v] = 2y, \quad [u, v] = z.$$

We must show that every finite-dimensional semigraded S-module is semisimple, in the semigraded sense. Let M be such an S-module. As before, let $\otimes_0$ indicate tensoring with respect to $\mathscr{U}(S_0)$. The S-module structure of M gives a surjective morphism of semigraded S-modules from $\mathscr{U}(S) \otimes_0 M$ to M. Since S_0 is semisimple, we can decompose each of the two homogeneous components of M into a direct sum of simple S_0-modules. Therefore, it will suffice to prove that, for every finite-dimensional simple S_0-module A, the semigraded S-module $\mathscr{U}(S) \otimes_0 A$ is semisimple.

Recall that, for every $k = 1, 2, \ldots,$ there is one and only one isomorphism class of simple S_0-modules of dimension k. Denote a representative of this class by A_k. We choose a standard basis $(f_0, \ldots, f_{k-1})$ of A_k such that the representation of S_0 on A_k is as follows:

$$x \cdot f_j = jf_{j-1}, \quad y \cdot f_j = (k - 1 - j)f_{j+1}, \quad z \cdot f_j = (k - 1 - 2j)f_j.$$

Now let us fix k, and put $N = \mathscr{U}(S) \otimes_0 A_k$. We use the 2-grading of N coming from that of $\mathscr{U}(S)$, i.e.,

$$N_0 = \mathscr{U}(S)_0 \otimes_0 A_k \quad \text{and} \quad N_1 = \mathscr{U}(S)_1 \otimes_0 A_k.$$

First, let us examine the S_0-module structures of N_0 and N_1.

From Section 2, we know that, as S_0-modules, N_0 is isomorphic with $\Lambda^2(S_1) \otimes A_k + \Lambda^0(S_1) \otimes A_k$, and N_1 is isomorphic with $\Lambda^1(S_1) \otimes A_k$ (where $\otimes$ indicates tensoring with respect to our base field F). Here, S_1 may be identified with A_2, so that N_0 is isomorphic with the direct sum of two copies of A_k, while N_1 is isomorphic with $A_2 \otimes A_k \approx A_{k+1} + A_{k-1}$ (by the Clebsch-Gordan formula). Let G denote the S_0-component of N_1 that is isomorphic with A_{k+1}, and let H denote the S_0-component of N_1 that is isomorphic with A_{k-1} (here, A_0 is to be interpreted as the 0-module, so that, in the case $k = 1$, we have $H = (0)$).

Note that Fz is a Cartan subalgebra of S_0, and that Fx may be taken as the positive root space, and Fy as the negative root space. With reference to this choice, the component G must be generated as an S_0-module by an element of the 1-dimensional highest weight subspace of N_1, whose weight is k. Noting that, in $\mathscr{U}(S)$, we have $zu = [z, u] + uz = u + uz$, we find that $z \cdot (u \otimes f_0) = k(u \otimes f_0)$, showing that the element $u \otimes f_0$ of N_1 is of weight k. We put $g_0 = k(u \otimes f_0)$, and we calculate a standard basis for G by operating with y on g_0. Precisely, we calculate a standard basis $(g_0, \ldots, g_k)$, where $g_{j+1} = (k - j)^{-1} y \cdot g_j$ (cf. the above formulas for the standard basis of A_k, now replacing k with $k + 1$). We claim that this gives, for every $j = 0, \ldots, k$,

$$g_j = (k - j)u \otimes f_j + jv \otimes f_{j-1}$$

where f_k and f_{-1} may be interpreted as 0.

Clearly, this holds for $j = 0$. Suppose it has already been established for some $j < k$. In $\mathscr{U}(S)$, we have $yu = v + uy$ and $yv = vy$. Using this and our inductive hypothesis, we find

$$\begin{aligned}(k - j)^{-1} y \cdot g_j &= y \cdot (u \otimes f_j) + (k - j)^{-1} jy \cdot (v \otimes f_{j-1}) \\ &= v \otimes f_j + u \otimes y \cdot f_j + (k - j)^{-1} jv \otimes y \cdot f_{j-1} \\ &= (k - j - 1)u \otimes f_{j+1} + (j + 1)v \otimes f_j\end{aligned}$$

so that our claim is proved. This identifies G as the F-space spanned by $g_0, \ldots, g_k$.

Next, we proceed similarly to determine H explicitly. We wish to construct a standard basis $h_0, \ldots, h_{k-2}$ for H (recall that $H = (0)$ if $k = 1$). Using the weight theory as before, we know in advance that, up to a scalar multiple, h_0 is determined by the property of having weight $k - 2$. Putting $h_0 = v \otimes f_0 - u \otimes f_1$, one can verify directly that $z \cdot h_0 = (k - 2)h_0$. Proceeding as we did above, we find, for all $j = 0, 1, \ldots, (k - 2)$,

$$h_j = v \otimes f_j - u \otimes f_{j+1}.$$

Next, we calculate the transforms of the g_j's and h_j's under the actions of u and v. Using that, in $\mathscr{U}(S)$, we have $uu = -x$, $vv = y$, and $vu = z - uv$, we obtain the following results:

$$u \cdot g_j = j(j - k + uv) \otimes f_{j-1}, \quad v \cdot g_j = (k - j)(k - 1 - j - uv) \otimes f_j,$$
$$u \cdot h_j = (j + 1 + uv) \otimes f_j, \quad v \cdot h_j = (j + 2 + uv) \otimes f_{j+1}.$$

Now observe that

$$v \cdot g_{j-1} = -(k + 1 - j)j^{-1}u \cdot g_j \quad \text{and} \quad v \cdot g_k = 0 = u \cdot g_0.$$

It follows that $S_1 \cdot G$ is the k-dimensional subspace of N_0 that is spanned by the elements $u \cdot g_1, \ldots, u \cdot g_k$. Since S_1 is an S_0-submodule of S with respect to the adjoint representation, and since G is an S_0-submodule of N_1, it is clear that $S_1 \cdot G$ is actually an S_0-submodule of N_0, and therefore is isomorphic with A_k. In fact, it is easy to verify directly that the elements $j^{-1}u \cdot g_j$, with $j = 1, \ldots, k$, constitute a standard basis of the k-dimensional simple S_0-module $S_1 \cdot G$.

We claim that $S_1 \cdot (S_1 \cdot G) = G$. We have $u \cdot (u \cdot g_j) = -x \cdot g_j \in G$. Next, $v \cdot (u \cdot g_j) = z \cdot g_j - u \cdot (v \cdot g_j)$, and $v \cdot g_j$ is a rational multiple of $u \cdot g_{j+1}$, so that $u \cdot (v \cdot g_j) \in G$, and $v \cdot (u \cdot g_j) \in G$. Now we have seen that $S_1 \cdot (S_1 \cdot G)$ is contained in G, and therefore is an S_0-submodule of G. Since G is simple as an S_0-module, and since $S_1 \cdot (S_1 \cdot G) \neq (0)$, it follows that $S_1 \cdot (S_1 \cdot G) = G$.

Now it is clear that $S_1 \cdot G + G$ is a simple homogeneous S-submodule of N. Let us call this P_k, with components $(P_k)_0 = S_1 \cdot G$ and $(P_k)_1 = G$.

In the case where $k = 1$, we have $H = (0)$, and $S_1 \cdot G$ is the 1-dimensional S_0-submodule of N_0 that is spanned by $uv \otimes f_0$. Here, we have $S_0 \cdot N_0 = (0)$, so that the subspace spanned by $(1 + uv) \otimes f_0$ is an S_0-module complement of $S_1 \cdot G$ in N_0. In $\mathscr{U}(S)$, we have

$$u(1 + uv) = u - xv = -vx$$

and

$$v(1 + uv) = v + vuv = v + zv - uvv = v + zv - uy = vz - uy.$$

This shows, since f_0 is annihilated by S_0, that S_1 annihilates $(1 + uv) \otimes f_0$. Thus, N is the direct S-module sum of the homogeneous simple S-submodule P_1 and the trivial 1-dimensional homogeneous S-submodule spanned by $(1 + uv) \otimes f_0$.

Now let us consider the case $k > 1$. The S_0-submodule $S_1 \cdot H$ of N_0 is spanned by the elements $u \cdot h_j$ and $v \cdot h_j$. We have $u \cdot h_{j+1} = v \cdot h_j$, so that $S_1 \cdot H$ is spanned by $u \cdot h_0$ and the $v \cdot h_j$'s, i.e., by the elements $(j + 1 + uv) \otimes f_j$, with $j = 0, \ldots, k - 1$. It is easy to see from this and our above description of $S_1 \cdot G$, that N_0 is the direct S_0-module sum of $S_1 \cdot H$ and $S_1 \cdot G$, each of these summands being isomorphic with A_k. We have

$$u \cdot (u \cdot h_0) = -x \cdot h_0,$$

$$v \cdot (u \cdot h_0) = z \cdot h_0 - u \cdot (v \cdot h_0) = z \cdot h_0 - u \cdot (u \cdot h_1) = z \cdot h_0 + x \cdot h_1,$$

$$u \cdot (v \cdot h_j) = u \cdot (u \cdot h_{j+1}) = -x \cdot h_{j+1} \quad \text{and} \quad v \cdot (v \cdot h_j) = y \cdot h_j.$$

Thus, $S_1 \cdot (S_1 \cdot H) \subset H$. Since H is simple as an S_0-module, and since $S_1 \cdot (S_1 \cdot H) \neq (0)$, it follows that $S_1 \cdot (S_1 \cdot H) = H$.

Now it is clear that $S_1 \cdot H + H$ is a simple homogeneous S-submodule Q_k of N, and that N is the direct S-module sum of P_k and Q_k. This holds also for $k = 1$, if we define Q_1 as the trivial 1-dimensional submodule spanned by $(1 + uv) \otimes f_0$. This completes the proof that S is semisimple.

It is clear from the above that P_k and Q_{k+1} are isomorphic as S_0-modules. We shall show that they are actually isomorphic as semigraded S-modules. In the notation used above, $(P_k)_1$ has the standard S_0-module basis $(g_0, \ldots, g_k)$, where $g_j = (k - j)u \otimes f_j + jv \otimes f_{j-1}$. The component $(P_k)_0$ has a basis $(v \cdot g_0, \ldots, v \cdot g_{k-1})$, where $v \cdot g_j = (k - j)(k - 1 - j - uv) \otimes f_j$. The elements $(k - j)^{-1} v \cdot g_j$ constitute a standard S_0-module basis of $(P_k)_0$, as is seen by noting that, in $\mathcal{U}(S)$, we have $yuv = (1 + uv)y$.

In order to give a description of Q_{k+1} that is notationally compatible with the above description of P_k, we introduce a standard basis $(t_0, \ldots, t_k)$ of the S_0-module A_{k+1} to replace the basis $(f_0, \ldots, f_{k-1})$ used originally in our description of Q_k. Then we see from the above that $(Q_{k+1})_0$ has a basis

$$(j + 1 + uv) \otimes t_j,$$

with $j = 0, \ldots, k$, while $(Q_{k+1})_1$ has a basis $v \otimes t_j - u \otimes t_{j+1}$, with $j = 0, \ldots, k - 1$. Again, each of these is a standard basis with respect to the S_0-module structure.

Now we define a linear isomorphism $\eta: P_k \to Q_{k+1}$ sending $(P_k)_0$ onto $(Q_{k+1})_1$ and $(P_k)_1$ onto $(Q_{k+1})_0$ by prescribing the images of basis elements as follows.

$$\eta(g_j) = (j + 1 + uv) \otimes t_j, \quad \eta(v \cdot g_j) = (k - j)(v \otimes t_j - u \otimes t_{j+1}).$$

In order to prove that η is an isomorphism of semigraded S-modules, it suffices to verify that $\eta(u \cdot p) = u \cdot \eta(p)$ and $\eta(v \cdot p) = v \cdot \eta(p)$ for every basis element p of P_k, because it is already clear from the above that η is an isomorphism of S_0-modules. We leave the details of this verification to the reader.

We have now shown that the Q_k's are precisely all the simple finite-dimensional semigraded S-modules, up to isomorphisms. Let us display this result in an independent notation.

There are no even-dimensional simple semigraded S-modules.

For every positive integer k, there is precisely one isomorphism class of simple semigraded S-modules of dimension $2k - 1$. *Let* $R(k)$ *denote a representative of this class. As an* S_0*-module,* $R(k)$ *is the direct sum of* $R(k)_0 \approx A_{k-1}$ *and* $R(k)_1 \approx A_k$, *which are the homogeneous components of* $R(k)$. *There are standard* S_0*-module bases*

$$(p_0, \ldots, p_{k-2}) \quad \textit{and} \quad (q_0, \ldots, q_{k-1})$$

of $R(k)_0$ *and* $R(k)_1$, *respectively, such that the actions of* u *and* v *are as follows:*

$$u \cdot p_j = q_j, \quad v \cdot p_j = q_{j+1},$$

$$u \cdot q_j = -jp_{j-1}, \quad v \cdot q_j = (k - 1 - j)p_j.$$

We observe that it is actually not necessary to check that this agrees with our above description of Q_k. It is more illuminating to verify directly that $R(k)$ is indeed a simple semigraded (or 2-graded) S-module.

Finally, there is a straightforward extension of the Clebsch-Gordan formula for tensor products. In exhibiting this, it is best to select 2-gradings for all the constituents. We have already given a 2-grading to each $R(k)$ in the above description. In order to indicate the correct matching of homogeneous components in a direct sum, we make the following notational convention. If i is an even integer, ${}_iR(k)$ stands for the S-module $R(k)$ with its 2-grading as given above. If i is an odd integer, ${}_iR(k)$ stands for $R(k)$ with the other 2-grading. Of course, as semigraded S-modules, these two are the same.

Now the *Clebsch-Gordan formula* is as follows:

$$\text{If } m \geqq n \text{ then } R(m) \otimes R(n) \approx \sum_{i=1}^{2n-1} {}_iR(m + n - i).$$

In order to prove this, we recall that, in our notation, the classical Clebsch-Gordan formula reads, with $p \geqq q$,

$$A_p \otimes A_q \approx \sum_{i=1}^{q} A_{p+q+1-2i}.$$

We know that $R(m) \otimes R(n)$ is isomorphic with a direct sum of $R(k)$'s, suitably 2-graded. As an S_0-module, $R(k)$ is isomorphic with $A_k + A_{k-1}$. The following display shows that only the sum written in the above formula gives the correct S_0-module.

$$R(m)_1 \otimes R(n)_1 \approx A_m \otimes A_n \approx A_{m+n-1} + A_{m+n-3} + \cdots$$

$$R(m)_0 \otimes R(n)_0 \approx A_{m-1} \otimes A_{n-1} \approx A_{m+n-3} + A_{m+n-5} + \cdots$$

$$R(m)_1 \otimes R(n)_0 \approx A_m \otimes A_{n-1} \approx A_{m+n-2} + A_{m+n-4} + \cdots$$

$$R(m)_0 \otimes R(n)_1 \approx A_{m-1} \otimes A_n \approx A_{m+n-2} + A_{m+n-4} + \cdots$$

The highest dimensional simple component of $R(m) \otimes R(n)$ must be ${}_1R(m + n - 1)$ so that A_{m+n-1} appears in the homogeneous component of degree 0. This supplies one copy of A_{m+n-2} in the component of degree 1. The display shows that there remains another copy of A_{m+n-2} in the component of degree 1. This can be supplied only by adding ${}_2R(m + n - 2)$, which also adds a copy of A_{m+n-3} to the component of degree 0, etc. The total number of simple components is dictated by the dimension.

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