SOLUTION OF THE BURNSIDE PROBLEM FOR EXPONENT SIX[1]

In commemoration of G. A. Miller

BY

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1. Introduction

In 1902 Burnside [1] raised the question as to whether a finitely generated group G of exponent n is necessarily finite. G is said to be of exponent n if $g^n = 1$ for every element g of G. For k generators $x_1, \cdots, x_k$ there is a group $B(n, k)$ such that every group of exponent n with k generators is a homomorphic image of $B(n, k)$. Here $B(n, k)$ is easily seen to be F_k/F_k^n where F_k is the free group with k generators, and F_k^n is the fully invariant subgroup of F_k generated by all n^{th} powers of elements of F_k.

It is trivial that the Burnside group $B(2, k)$ is Abelian and of order 2^k. In his original paper Burnside showed that $B(3, k)$ is finite, but did not find the true order of $B(3, k)$. This value is 3^K, $K = k + \binom{k}{2} + \binom{k}{3}$ and was obtained by Levi and van der Waerden [5]. Burnside showed that $B(4, 2)$ is of order at most 2^{12}, and Sanov [6] showed that $B(4, k)$ is finite, but the order of $B(4, k)$ is not known.

In this paper it is shown that $B(6, k)$ is finite. The order of $B(6, k)$ is

$$(1.1) \qquad 2^a 3^{b+\binom{b}{2}+\binom{b}{3}}, \qquad a = 1 + (k-1)\cdot 3^{k+\binom{k}{2}+\binom{k}{3}}, \qquad b = 1 + (k-1)\, 2^k.$$

This follows from a result of Philip Hall and Graham Higman [3]. Their results apply to what is known as the restricted Burnside problem. This is the question as to whether there exists a largest finite group $R(n, k)$ of exponent n generated by k elements. If it can be shown that there is a largest finite group $R(n, k)$, then either $B(n, k)$ is infinite or $B(n, k) = R(n, k)$. They have shown that the existence of a largest finite group for each prime power exponent dividing n, and any number of generators, implies the existence of a largest finite solvable group of exponent n and any number of generators. The requirement of solvability is superfluous if n is divisible by only two distinct primes, since any such finite group must be solvable. From their theorems and the result of Levi and van der Waerden they obtained the order above for $R(6, k)$. Graham Higman [4] has solved the restricted Burnside problem for exponent five.

2. Theorems on groups of exponent three

THEOREM 2.1. *If a group G is generated by elements $x_1, x_2, \cdots, x_n$, and if any four of the x's generate a group of exponent three, then G is of exponent three.*

Received November 13, 1958.

[1] This research was supported by the Office of Ordnance Research, U. S. Army.

Proof. We shall suppose that G is generated by $x_1, \cdots, x_n$ with the defining relations $z^3 = 1$ for every z in a subgroup generated by four of the x's. Every further group satisfying the hypotheses of the theorem is a homomorphic image of G and so of exponent three if G is. In particular the Burnside group $B(3, n)$ generated by $x_1, \cdots, x_n$ with defining relations $z^3 = 1$ for every z of the group is a homomorphic image of G.

We shall use the notation $(x, y) = x^{-1}y^{-1}xy$ for a commutator and also write $((x, y), z) = (x, y, z)$, $((x, y, z), w) = (x, y, z, w)$. In a group of exponent three, Levi and van der Waerden [5] have shown that the following relations hold for any elements:

$$(2.1) \qquad \begin{gathered} (x^{-1}, y) = (x, y^{-1}) = (x, y)^{-1} = (y, x), \\ (x, y, y) = 1, \qquad (x, y, z) = (y, z, x) = (z, x, y), \\ (x, y, z, w) = 1, \qquad ((x, y), (z, w)) = 1. \end{gathered}$$

In our group G it will follow that these relations will hold if x, y, z, w are any four elements in a subgroup generated by four of the x's.

Any element of G is of the form

$$(2.2) \qquad g = a_1 a_2 \cdots a_t,$$

where each a_i is an x_j or x_j^{-1}. Let us apply the collecting process of Philip Hall [2] to this expression altering a string by the rule

$$(2.3) \qquad \cdots RS \cdots = \cdots SR(R, S) \cdots,$$

this being an identity by the definition of the commutator $(R, S) = R^{-1}S^{-1}RS$. Now for fixed a_i, a_j, a_k of (2.2)

$$(2.4) \qquad (a_i, a_j, a_k, x_u) = 1, \qquad u = 1, \cdots, n,$$

since the next to last relation of (2.1) applies. Thus (a_i, a_j, a_k) permutes with every x_u, $u = 1, \cdots, n$, and so is in the center of G. Hence if we apply the collecting process to (2.2) first moving x's to the left, following these by x_2's, x_3's, $\cdots$, x_n's, g takes the form

$$(2.5) \qquad g = x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}c_1 \cdots c_s,$$

where $e_i = 0, 1, 2$, $i = 1, \cdots, n$, and each c_u is a commutator of the form (a_i, a_j) or (a_i, a_j, a_k) since by (2.4) any longer commutator is the identity. But as the commutators (a_i, a_j, a_k) are in the center of G, and the commutators (a_i, a_j) permute with each other by the last relations of (2.1), we may rearrange $c_1, \cdots, c_s$ in (2.5) and use the first three relations of (2.1) so that we have only commutators (x_i, x_j), $i < j$, or (x_i, x_j, x_k) with $i < j < k$. Hence g may be put in the form

$$(2.6) \qquad g = x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n} \prod_{i<j} (x_i, x_j)^{f_{ij}} \prod_{i<j<k} (x_i, x_j, x_k)^{h_{ijk}}.$$

Here each of the exponents takes only the values 0, 1, 2, and so the order of

G is at most

$$3^N, \qquad N = n + \binom{n}{2} + \binom{n}{3}. \tag{2.7}$$

But Levi and van der Waerden have shown that 3^N is the order of the Burnside group $B(3, n)$. And as $B(3, n)$ is a homomorphic image of G, it follows that $G = B(3, n)$, proving our theorem.

THEOREM 2.2. *If G is the group $\{a, b, c, d\}$ generated by a, b, c, d, and if each of the subgroups $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ is of exponent three, then G is finite. If further G is of exponent six, then G is in fact of exponent three.*

COROLLARY. *If G is of exponent six and generated by $x_1, \cdots, x_n$, and if any three x's generate a group of exponent three, then G is of exponent three.*

The corollary is an immediate consequence of the two theorems.

Proof. For the first part of the theorem we assume the defining relation of G to be $g^3 = 1$ for every g in each of the four subgroups given, and for the second part we assume also $g^6 = 1$ for every g of G. Thus G has 24 automorphisms permuting a, b, c, d according to the symmetric group on four letters and 16 automorphisms replacing one or more of a, b, c, d by their inverses. Other groups satisfying the hypotheses of the theorem will be homomorphic images of G as given by these defining relations, and the conclusions will follow.

We shall use the following notation for elements of G':

$$\begin{array}{lll}
u_1 = (a, b) & v_1 = (a, b, c) & z_1 = (a, b, c, d) \\
u_2 = (a, c) & v_2 = (a, b, d) & z_2 = (a, b, c, d^{-1}) \\
u_3 = (a, d) & v_3 = (a, c, d) & z_3 = (a, b, d, c) \\
u_4 = (b, c) & v_4 = (b, c, d) & z_4 = (a, b, d, c^{-1}) \\
u_5 = (b, d) & & z_5 = (a, c, d, b) \\
u_6 = (c, d) & & z_6 = (a, c, d, b^{-1}) \\
 & & z_7 = (b, c, d, a) \\
 & & z_8 = (b, c, d, a^{-1})
\end{array} \tag{2.8}$$

The relations (2.1) are valid in the four subgroups generated by any three of a, b, c, d. Thus

$$(b, a, c) = ((a, b)^{-1}, c) = (a, b, c)^{-1}. \tag{2.9}$$

We note the following:

$$\begin{aligned} z_1 = (a, b, c, d) &= (a, b, c)^{-1} d^{-1}(a, b, c) d, \\ (a, b, c)z_1 &= d^{-1}(a, b, c) d, \\ z_1^{-1}(a, b, c)^{-1} &= d^{-1}(a, b, c)^{-1} d = d^{-1}(b, a, c) d, \\ (b, a, c)^{-1}z_1^{-1}(a, b, c)^{-1} &= (b, a, c)^{-1} d^{-1}(b, a, c) d = (b, a, c, d), \\ v_1 z_1^{-1} v_1^{-1} &= (b, a, c, d). \end{aligned} \tag{2.10}$$

This gives the following relations:

$$\begin{aligned} (b, a, c, d) &= v_1 z_1^{-1} v_1^{-1}, & (b, a, c, d^{-1}) &= v_1 z_2^{-1} v_1^{-1}, \\ (b, a, d, c) &= v_2 z_3^{-1} v_2^{-1}, & (b, a, d, c^{-1}) &= v_2 z_4^{-1} v_2^{-1}, \\ (c, a, d, b) &= v_3 z_5^{-1} v_3^{-1}, & (c, a, d, b^{-1}) &= v_3 z_6^{-1} v_3^{-1}, \\ (c, b, d, a) &= v_4 z_7^{-1} v_4^{-1}, & (c, b, d, a^{-1}) &= v_4 z_8^{-1} v_4^{-1}. \end{aligned} \tag{2.11}$$

We list in tabular form the results of transforming the u's and v's by a, b, c, d and their inverses.

(2.12)

U	$a^{-1}Ua$	$b^{-1}Ub$	$c^{-1}Uc$	$d^{-1}Ud$
u_1	u_1	u_1	$u_1 v_1$	$u_1 v_2$
u_2	u_2	$u_2 v_1^{-1}$	u_2	$u_2 v_3$
u_3	u_3	$u_3 v_2^{-1}$	$u_3 v_3^{-1}$	u_3
u_4	$u_4 v_1$	u_4	u_4	$u_4 v_4$
u_5	$u_5 v_2$	u_5	$u_5 v_4^{-1}$	u_5
u_6	$u_6 v_3$	$u_6 v_4$	u_6	u_6
v_1	v_1	v_1	v_1	$v_1 z_1$
v_2	v_2	v_2	$v_2 z_3$	v_2
v_3	v_3	$v_3 z_5$	v_3	v_3
v_4	$v_4 z_7$	v_4	v_4	v_4

(2.13)

U	aUa^{-1}	bUb^{-1}	cUc^{-1}	dUd^{-1}
u_1	u_1	u_1	$u_1 v_1^{-1}$	$u_1 v_2^{-1}$
u_2	u_2	$u_2 v_1$	u_2	$u_2 v_3^{-1}$
u_3	u_3	$u_3 v_2$	$u_3 v_3$	u_3
u_4	$u_4 v_1^{-1}$	u_4	u_4	$u_4 v_4^{-1}$
u_5	$u_5 v_2^{-1}$	u_5	$u_5 v_4$	u_5
u_6	$u_6 v_3^{-1}$	$u_6 v_4^{-1}$	u_6	u_6
v_1	v_1	v_1	v_1	$v_1 z_2$
v_2	v_2	v_2	$v_2 z_4$	v_2
v_3	v_3	$v_3 z_6$	v_3	v_3
v_4	$v_4 z_8$	v_4	v_4	v_4

These relations all are from given properties of the subgroups or by definition of the z's.

From the relations holding in the four subgroups we may derive further relations by transformation. Thus from

$$u_6^{-1} u_5 u_6 = u_5$$

holding in $\{b, c, d\}$ if we transform by a, using (2.12) we get

(2.14) $$v_3^{-1} u_6^{-1} u_5 v_2 u_6 v_3 = u_5 v_2 ,$$

whence

(2.15) $$u_6^{-1} v_2 u_6 = u_5^{-1} v_3 u_5 v_2 v_3^{-1}.$$

Similarly from

(2.16) $$u_6^{-1} u_3 u_6 = u_3$$

holding in $\{a, c, d\}$ and transforming by b^{-1}, using (2.13) we get

(2.17) $$v_4 u_6^{-1} u_3 v_2 u_6 v_4^{-1} = u_3 v_2 ,$$

whence

(2.18) $$u_6^{-1} v_2 u_6 = u_3^{-1} v_4^{-1} u_3 v_2 v_4 .$$

Transforming

(2.19) $$u_5^{-1} u_3 u_5 = u_3$$

by c^{-1} gives

(2.20) $$v_4^{-1} u_5^{-1} u_3 v_3 u_5 v_4 = u_3 v_3 ,$$

whence

(2.21) $$u_5^{-1} v_3 u_5 = u_3^{-1} v_4 u_3 v_3 v_4^{-1}.$$

From (2.15) and (2.18) we have

(2.22) $$u_5^{-1} v_3 u_5 v_2 v_3^{-1} = u_3^{-1} v_4^{-1} u_3 v_2 v_4 .$$

Substituting in this from (2.21) we have

(2.23) $$u_3^{-1} v_4 u_3 v_3 v_4^{-1} v_2 v_3^{-1} = u_3^{-1} v_4^{-1} u_3 v_2 v_4 .$$

From this we get using $v_4^{-2} = v_4$

(2.24) $$v_3 v_4^{-1} v_2 v_3^{-1} v_4^{-1} v_2^{-1} = u_3^{-1} v_4 u_3 .$$

In (2.24) replace a by a^{-1}, b by b^{-1}, and d by d^{-1}. This gives

(2.25) $$v_3 v_4^{-1} v_2^{-1} v_3^{-1} v_4^{-1} v_2 = u_3^{-1} v_4 u_3 .$$

(2.24) and (2.25) give together

(2.26) $$v_2 v_3^{-1} v_4^{-1} v_2^{-1} = v_2^{-1} v_3^{-1} v_4^{-1} v_2 ,$$

and so

$$v_2^{-1}v_3^{-1}v_4^{-1}v_2 = v_3^{-1}v_4^{-1}. \tag{2.27}$$

Substituting this in (2.25) we have

$$v_3 v_4^{-1}v_3^{-1}v_4^{-1} = u_3^{-1}v_4 u_3 . \tag{2.28}$$

In (2.28) replacing a by a^{-1}, c by c^{-1}, and d by d^{-1} gives

$$v_3^{-1}v_4^{-1}v_3 v_4^{-1} = u_3^{-1}v_4 u_3 . \tag{2.29}$$

From the left-hand sides of (2.28) and (2.29) we have

$$v_3 v_4^{-1}v_3^{-1}v_4^{-1} = v_3^{-1}v_4^{-1}v_3 v_4^{-1}, \tag{2.30}$$

whence

$$v_3 v_4 = v_4 v_3 , \tag{2.31}$$

and from (2.29)

$$v_4 = u_3^{-1}v_4 u_3 . \tag{2.32}$$

We already had from $\{b, c, d\}$ the relation

$$v_4 = u_4^{-1}v_4 u_4 . \tag{2.33}$$

Permuting a, b, c, d in (2.31), (2.32), and (2.33) in all ways, we find

$$v_i v_j = v_j v_i , \qquad i, j = 1, 2, 3, 4, \tag{2.34}$$

and

$$v_i u_j = u_j v_i , \qquad i = 1, 2, 3, 4, \quad j = 1, \cdots, 6. \tag{2.35}$$

Now from

$$v_4 v_i = v_i v_4 , \qquad i = 1, 2, 3, \tag{2.36}$$

transforming by a we get

$$v_4 z_7 v_i = v_i v_4 z_7 , \qquad i = 1, 2, 3, \tag{2.37}$$

whence

$$z_7 v_i = v_i z_7 , \qquad i = 1, 2, 3. \tag{2.38}$$

Similarly from

$$v_4 u_i = u_i v_4 , \qquad i = 1, 2, 3, \tag{2.39}$$

transforming by a we get

$$v_4 z_7 u_i = u_i v_4 z_7 , \qquad i = 1, 2, 3, \tag{2.40}$$

whence

$$z_7 u_i = u_i z_7 , \qquad i = 1, 2, 3. \tag{2.41}$$

Also from

$$v_4 u_4 = u_4 v_4 \tag{2.42}$$

transforming by a we get

$$v_4 z_7 u_4 v_1 = u_4 v_1 v_4 z_7 , \tag{2.43}$$

and using (2.42) and (2.38) we have

$$z_7 u_4 = u_4 z_7 . \tag{2.44}$$

Similarly we find

$$z_7 u_5 = u_5 z_7 \quad \text{and} \quad z_7 u_6 = u_6 z_7 . \tag{2.45}$$

In (2.12) we find

$$a^{-1} v_4 a = v_4 z_7 . \tag{2.46}$$

Transform by b and use (2.12). This gives

$$b^{-1} a^{-1} b v_4 b^{-1} a b = v_4 b^{-1} z_7 b. \tag{2.47}$$

By definition of u_1, $b^{-1}ab = au_1$, and so

$$u_1^{-1} a^{-1} v_4 a u_1 = v_4 b^{-1} z_7 b, \tag{2.48}$$

and by using (2.12) this becomes

$$u_1^{-1} v_4 z_7 u_1 = v_4 b^{-1} z_7 b. \tag{2.49}$$

By (2.39) and (2.41), u_1 commutes with both v_4 and z_7, and so

$$z_7 = b^{-1} z_7 b. \tag{2.50}$$

We also take the relation

$$v_4 = a(a^{-1} v_4 a) a^{-1} = a(v_4 z_7) a^{-1} = v_4 z_8 a z_7 a^{-1}, \tag{2.51}$$

whence

$$a z_7 a^{-1} = z_8^{-1}. \tag{2.52}$$

Also

$$\begin{aligned} a^{-2} v_4 a^2 &= a v_4 a^{-1} = v_4 z_8 = a^{-1}(a^{-1} v_4 a) a \\ &= a^{-1}(v_4 z_7) a = v_4 z_7 a^{-1} z_7 a, \end{aligned} \tag{2.53}$$

whence

$$a^{-1} z_7 a = z_7^{-1} z_8 . \tag{2.54}$$

Substituting in (2.50), (2.52), and (2.54) in all ways, we obtain the following tables:

(2.55)

z_i	$a^{-1}z_i a$	$b^{-1}z_i b$	$c^{-1}z_i c$	$d^{-1}z_i d$
z_1	z_1	z_1	z_1	$z_1^{-1}z_2$
z_2	z_2	z_2	z_2	z_1^{-1}
z_3	z_3	z_3	$z_3^{-1}z_4$	z_3
z_4	z_4	z_4	z_3^{-1}	z_4
z_5	z_5	$z_5^{-1}z_6$	z_5	z_5
z_6	z_6	z_5^{-1}	z_6	z_6
z_7	$z_7^{-1}z_8$	z_7	z_7	z_7
z_8	z_7^{-1}	z_8	z_8	z_8

(2.56)

z_i	$az_i a^{-1}$	$bz_i b^{-1}$	$cz_i c^{-1}$	$dz_i d^{-1}$
z_1	z_1	z_1	z_1	z_2^{-1}
z_2	z_2	z_2	z_2	$z_2^{-1}z_1$
z_3	z_3	z_3	z_4^{-1}	z_3
z_4	z_4	z_4	$z_4^{-1}z_3$	z_4
z_5	z_5	z_6^{-1}	z_5	z_5
z_6	z_6	$z_6^{-1}z_5$	z_6	z_6
z_7	z_8^{-1}	z_7	z_7	z_7
z_8	$z_8^{-1}z_7$	z_8	z_8	z_8

These tables, together with (2.12) and (2.13) show that $u_1, \cdots, u_6$, $v_1, \cdots, v_4$, $z_1, \cdots, z_8$ generate a normal subgroup of G, which, since it includes the commutators of pairs of the generators, $u_1, \cdots, u_6$, must be G'.

If we now take the relation from (2.45)

$$z_7 u_6 = u_6 z_7 \tag{2.57}$$

and transform by b, we get

$$z_7 u_6 v_4 = u_6 v_4 z_7 , \tag{2.58}$$

whence

$$z_7 v_4 = v_4 z_7 . \tag{2.59}$$

Adjoining this to (2.38)–(2.45) we find that z_7 permutes with all u's and v's. On substituting we have

$$z_i u_j = u_j z_i , \qquad i = 1, \cdots, 8, \quad j = 1, \cdots, 6, \tag{2.60}$$

$$z_i v_j = v_j z_i , \qquad i = 1, \cdots, 8, \quad j = 1, \cdots, 4. \tag{2.61}$$

Also from $v_4^3 = 1$, transforming by a and using (2.59), we have

$$(2.62) \qquad (v_4 z_7)^3 = 1 = v_4^3 z_7^3,$$

whence $z_7^3 = 1$, and on substituting,

$$(2.63) \qquad z_i^3 = 1, \qquad i = 1, \cdots, 8.$$

From the definition of u_1 we have

$$(2.64) \qquad a^{-1}b^{-1}u_6 ba = u_1 b^{-1}a^{-1}u_6 abu_1^{-1}.$$

By using (2.12) this becomes

$$(2.65) \qquad u_6 v_3 v_4 z_7 = u_1 u_6 v_4 v_3 z_5 u_1^{-1}.$$

From this we find, using (2.34), (2.35), (2.60), and (2.61),

$$(2.66) \qquad z_7 z_5^{-1} = u_6^{-1} u_1 u_6 u_1^{-1}.$$

Transforming this by a^{-1} we obtain, using (2.13) and (2.56),

$$(2.67) \qquad z_8^{-1} z_5^{-1} = v_3 u_6^{-1} u_1 u_6 v_3^{-1} u_1^{-1} = u_6^{-1} u_1 u_6 u_1^{-1},$$

the last being from (2.35). Comparing (2.66) and (2.67) we find

$$(2.68) \qquad z_8 = z_7^{-1},$$

and on substituting also

$$(2.69) \qquad z_6 = z_5^{-1}, \qquad z_4 = z_3^{-1}, \qquad z_2 = z_1^{-1};$$

note that (2.55) now shows that the z's are in the center of G. On making the appropriate substitutions in (2.66) we find

$$(2.70) \qquad \begin{aligned} u_6^{-1} u_1 u_6 u_1^{-1} &= z_7 z_5^{-1} = z_1 z_3^{-1}, \\ u_5^{-1} u_2 u_5 u_2^{-1} &= z_3^{-1} z_7^{-1} = z_1^{-1} z_5^{-1}, \\ u_4^{-1} u_3 u_4 u_3^{-1} &= z_3^{-1} z_5 = z_1^{-1} z_7. \end{aligned}$$

Our relations now show that, modulo the group z_1, z_3, z_5, z_7 (which is elementary Abelian of order 27 or a divisor of 27), G is the Burnside group $B(3, 4)$ of exponent three and order 3^{14}. This shows that G is finite and of exponent nine, whence if we assume that G is of exponent six, then G is necessarily of exponent three, and our theorem is proved.

This last is however proved directly if we calculate that

$$(2.71) \qquad (u_1 cd)^3 = u_1^3 z_1 z_3 = z_1 z_3.$$

Hence if $(u_1 cd)^6 = 1$, we have $(z_1 z_3)^2 = 1$, but since $(z_1 z_3)^3 = 1$, this gives $z_1 z_3 = 1$, and on substitution we have

$$(2.72) \qquad z_1 z_3 = 1, \qquad z_7 z_1 = 1, \qquad z_7 z_5 = 1, \qquad z_7 z_3^{-1} = 1.$$

Then from (2.70) and (2.72) we have

$$z_1 z_3^{-1} z_5 z_7^{-1} = 1, \qquad z_1^4 = z_1 = 1, \tag{2.73}$$

and so all z's are 1, and G is of order at most 3^{14}, and so $G = B(3, 4)$. This completes the proof of our theorem.

3. The main theorem

Our main theorem is of course the proof of the Burnside conjecture for exponent six.

THEOREM 3.1. *A finitely generated group G of exponent six is necessarily finite.*

Proof. Philip Hall and Graham Higman [3] have shown that there is a finite group $R(6, k)$ generated by $x_1, \cdots, x_k$ of exponent six such that every other finite group of exponent six generated by k elements is a homomorphic image of $R(6, k)$. Its order is

$$2^a 3^{b+\binom{b}{2}+\binom{b}{3}}, \qquad a = 1 + (k-1)\, 3^{k+\binom{k}{2}+\binom{k}{3}}, \quad b = 1 + (k-1)\, 2^k. \tag{3.1}$$

Thus, once the finiteness of the Burnside group $G = B(6, k)$ is established, its order is given by (3.1). $G = B(6, k)$ is of course the group generated by $x_1, \cdots, x_k$ with defining relations $z^6 = 1$ for every element z of the group G.

The proof of this theorem does not depend on the Hall-Higman results, though in order to get the exact order of the Burnside group their results must be used. The motivation for the proof does, however, come from their work. They have shown that a finite group H of exponent six has 2-length one. This means that H has a normal series

$$1 \subseteqq U \subseteqq V \subset H, \tag{3.2}$$

where U is a maximal normal subgroup of order prime to 2, V/U is a 2-group, and H/V is of order prime to 2.

Our proof will follow this idea. We show the existence of a normal subgroup M of G such that

$$G \supset M \supset M', \tag{3.3}$$

where G/M is finite of exponent three, M/M' is finite of exponent two. M' is easily seen to be finitely generated, and the main difficulty will be in showing that M' is of exponent three and hence finite by the results of Levi and van der Waerden.

The proof is given by a succession of lemmas.

LEMMA 1. *A group G of exponent six generated by k elements $x_1, \cdots, x_k$ has a subgroup M, generated by the cubes of elements of G of index dividing 3^K, $K = k + \binom{k}{2} + \binom{k}{3}$.*

This is a direct consequence of the results of Levi and van der Waerden.

LEMMA 2. *M is generated by a finite number of elements of order 2. The derived group M' of M is of index a power of 2 in M, and M' is generated by a finite number of elements of the form abab where $a^2 = 1$, $b^2 = 1$.*

Proof. M being of finite index in a finitely generated group is itself finitely generated, say by $\alpha_1, \alpha_2, \cdots, \alpha_m$. M is also generated by elements of order 2, namely the cubes of the elements of G. Thus each α can be expressed in terms of a finite number of elements of order 2, and the finite number of elements of order 2 needed to express $\alpha_1, \cdots, \alpha_m$ will be a set of generators for M. If M is generated by $x_1, \cdots, x_t$ with $x_i^2 = 1$, $i = 1, \cdots, t$, then M' is of index at most 2^t in M. Also M' is generated by the commutators $x_i^{-1}x_j^{-1}x_ix_j$ and their conjugates and so by a finite set of these. Hence M' is generated by a finite number of elements of the form $abab$ where $a^2 = 1$, $b^2 = 1$.

Now M' is of finite index in G and has a finite number of generators of the form $abab$ where $a^2 = 1$, $b^2 = 1$. If it can be shown that M' is of exponent three, then by the results of Levi and van der Waerden it will follow that M' is finite and so also G, proving our theorem. From the corollary to Theorem 2.2 it will be enough to prove the following lemma.

LEMMA 3. *If $a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 1$ in a group of exponent six, then the subgroup $\{abab, cdcd, efef\}$ is of exponent three.*

The rest of the proof consists of steps leading to the proof of this lemma This lemma might be attacked by a high speed computer, but would probably be a very long problem.

To motivate the rest of our proof we observe that if H is a finite group of exponent six, and if H/H' is a 2-group, then H' must be of exponent three. For by the Hall-Higman results H must have 2-length one, that is, H has normal subgroups R and S such that $H \supseteqq R \supseteqq S \supseteqq 1$, where H/R is a 3-group, R/S is a 2-group, and S is a 3-group. If $H \neq R$, then H has a maximal normal subgroup T of index 3, and as H/T is the cyclic group of order 3, $T \supseteqq H'$ and so H/H' would contain an element of order 3, contrary to assumption. Hence $H = R$. As $H/S = R/S$ is a 2-group of exponent two, H/S is Abelian and so $S \supseteqq H'$. But as H/H' is a 2-group and H/S is the maximal factor group which is a 2-group, $H' \supseteqq S$. Hence $H' = S$ is of exponent three.

LEMMA 4. *If $H = \{x, a, b\}$ is of exponent six, and if $x^2 = 1$, $a^3 = 1$, $b^3 = 1$, $xax = a^{-1}$, $xbx = b^{-1}$, then $\{a, b\}$ is of exponent three.*

This lemma is critical since we note that $[H:H'] = 2$, and so if H is finite, then $H' = \{a, b\}$ must be of exponent three and so of order 27 (or naturally a divisor of 27). Thus if H is finite, its order divides 54.

We assume that H is given by the generators x, a, b, and defining relations

$x^2 = 1$, $a^3 = 1$, $b^3 = 1$, $xax = a^{-1}$, $xbx = b^{-1}$, and relations $u^6 = 1$ for every $u \in H$. Then in $A = \{a, b\}$ there are automorphisms obtained by replacing a or b by its inverse and interchanging a and b.

The derived group A' of A is generated by z_1, z_2, z_3, z_4, where

$$z_1 = a^{-1}b^{-1}ab, \qquad z_2 = a^{-1}bab^{-1},$$
(3.4)
$$z_3 = ab^{-1}a^{-1}b, \qquad z_4 = aba^{-1}b^{-1}.$$

Here the z's are transformed by a and b in the following way:

(3.5)

z_i	$a^{-1}z_i a$	$b^{-1}z_i b$
z_1	$z_3 z_1^{-1}$	$z_1^{-1}z_2$
z_2	$z_4 z_2^{-1}$	z_1^{-1}
z_3	z_1^{-1}	$z_3^{-1}z_4$
z_4	z_2^{-1}	z_3^{-1}

We also have

$$xz_1x = z_4, \qquad xz_2x = z_3,$$
(3.6)
$$xz_3x = z_2, \qquad xz_4x = z_1.$$

Replacing the generators has the following effect on the z's:

(3.7)

z_i	$\begin{pmatrix}a, b\\ b, a\end{pmatrix}$	$\begin{pmatrix}a, b\\ a^{-1}, b\end{pmatrix}$	$\begin{pmatrix}a, b\\ a, b^{-1}\end{pmatrix}$
z_1	z_1^{-1}	z_3	z_2
z_2	z_3^{-1}	z_4	z_1
z_3	z_2^{-1}	z_1	z_4
z_4	z_4^{-1}	z_2	z_3

Now $[A:A'] = 9$. If we can show that $z_1^3 = 1$, $z_2 = z_1^{-1}$, $z_3 = z_1^{-1}$, $z_4 = z_1$, then it will follow that A' is of order 3 and so A is of order 27, and easily seen to be of exponent three as we wish to prove.

For our first relation

$$(3.8) \qquad 1 = (xab)^6 = (xabxab)^3 = (a^{-1}b^{-1}ab)^3 = z_1^3.$$

Replacing a and b by their inverses in turn we have

$$(3.9) \qquad z_1^3 = z_2^3 = z_3^3 = z_4^3 = 1.$$

We find

$$(z_3 z_1^{-1} z_2)^2 = ((ab^{-1})^3)^2 = (ab^{-1})^6 = 1,$$
$$(z_3^{-1}z_2)^2 = (b^{-1}(ab)^3 b)^2 = b^{-1}(ab)^6 b = 1,$$
$$1 = (z_1 x)^6 = (z_1 x z_1 x)^3 = (z_1 z_4)^3,$$

whence

$$(a^{-1}z_1 z_4 a)^3 = 1 \quad \text{or} \quad (z_3 z_1^{-1} z_2^{-1})^3 = 1.$$

Also

$$(z_2 z_1^{-1} z_2^{-1} z_1)^2 = ((a^{-1}bab)^3)^2 = (a^{-1}bab)^6 = 1.$$

Thus we have found the following four relations on the z's:

$$(3.10) \qquad (z_3 z_1^{-1} z_2)^2 = 1,$$

$$(3.11) \qquad (z_3^{-1} z_2)^2 = 1,$$

$$(3.12) \qquad (z_3 z_1^{-1} z_2^{-1})^3 = 1,$$

$$(3.13) \qquad (z_2 z_1^{-1} z_2^{-1} z_1)^2 = 1.$$

From (3.10) we find

$$(3.14) \qquad z_3 z_1^{-1} z_2 z_3 = z_2^{-1} z_1 .$$

From (3.11)

$$(3.15) \qquad z_3^{-1} z_2 = z_2^{-1} z_3 .$$

By combining (3.14) and (3.15)

$$(3.16) \qquad z_3 z_1^{-1} z_2^{-1} = z_3 z_1^{-1} z_2 z_3 \cdot z_3^{-1} z_2 = z_2^{-1} z_1 z_2^{-1} z_3 .$$

From (3.12)

$$(3.17) \qquad z_3 z_1^{-1} z_2^{-1} z_3 z_1^{-1} z_2^{-1} z_3 z_1^{-1} z_2^{-1} = 1.$$

Substituting from (3.16) into (3.17) we have

$$(3.18) \qquad z_2^{-1} z_1 z_2^{-1} z_3 \cdot z_3 z_1^{-1} z_2^{-1} z_3 z_1^{-1} z_2^{-1} = 1,$$

whence

$$(3.19) \qquad z_3^{-1} z_1^{-1} z_2^{-1} z_3 = z_2 z_1^{-1} z_2 \cdot z_2 z_1 = z_2 z_1^{-1} z_2^{-1} z_1 .$$

Squaring and using (3.13) we have

$$(3.20) \qquad z_3^{-1}(z_1^{-1} z_2^{-1})^2 z_3 = (z_2 z_1^{-1} z_2^{-1} z_1)^2 = 1.$$

Hence

$$(3.21) \qquad (z_1^{-1} z_2^{-1})^2 = 1, \quad \text{and so} \quad (z_1 z_2)^2 = 1.$$

In (3.21) interchange a and b. This gives

$$(3.22) \qquad (z_1 z_3)^2 = 1.$$

In (3.21) replacing a by a^{-1} gives

$$(3.23) \qquad (z_3 z_4)^2 = 1.$$

In (3.22) replacing b by b^{-1} gives

$$(3.24) \qquad (z_2 z_4)^2 = 1.$$

In (3.11) replacing a by a^{-1} gives

$$(3.25) \qquad (z_1^{-1}z_4)^2 = 1.$$

If we write $w_1 = z_1 z_2$, $w_2 = z_1 z_3$, $w_3 = z_1 z_4^{-1}$, we have from (3.21), (3.22), and (3.25)

$$(3.26) \qquad w_1^2 = 1, \quad w_2^2 = 1, \quad w_3^2 = 1.$$

We have from (3.11), (3.23), and (3.24)

$$(3.27) \qquad (w_2^{-1}w_1)^2 = 1, \quad (w_2^{-1}w_3)^2 = 1, \quad (w_1^{-1}w_3)^2 = 1.$$

From (3.26) and (3.27) the w's are of order 2 and permute pairwise. Now

$$(3.28) \qquad 1 = (z_1 z_2 x)^6 = (z_1 z_2 x z_1 z_2 x)^3 = (z_1 z_2 z_4 z_3)^3.$$

But

$$(3.29) \qquad (z_1 z_2 z_4 z_3)^2 = (w_1 w_3^{-1} w_2)^2 = w_1^2 w_3^2 w_2^2 = 1.$$

From (3.28) and (3.29) we have

$$(3.30) \qquad z_1 z_2 z_4 z_3 = 1 \quad \text{or} \quad z_3 z_1 z_2 z_4 = 1.$$

Transforming (3.30) by a gives

$$(3.31) \qquad z_3 z_1^{-1} z_4 z_2^{-1} \cdot z_2^{-1} z_1^{-1} = 1 \quad \text{or} \quad z_2 z_1^{-1} z_3 z_1^{-1} z_4 = 1.$$

In (3.30) replacing a by a^{-1} gives

$$(3.32) \qquad z_3 z_4 z_2 z_1 = 1 \quad \text{or} \quad z_2 z_1 z_3 z_4 = 1.$$

From (3.31) and (3.32) we have

$$(3.33) \qquad z_4^{-1} = z_2 z_1^{-1} z_3 z_1^{-1} = z_2 z_1 z_3,$$

whence

$$(3.34) \qquad z_1^{-1} z_3 z_1^{-1} = z_1 z_3,$$

and so

$$(3.35) \qquad z_1 z_3 = z_3 z_1.$$

But then

$$(3.36) \qquad (z_1 z_3)^3 = z_1^3 z_3^3 = 1.$$

while from (3.22), $(z_1 z_3)^2 = 1$, and so

$$(3.37) \qquad z_1 z_3 = 1 \quad \text{or} \quad z_3 = z_1^{-1}.$$

In this, interchanging a and b gives

$$(3.38) \qquad z_2 = z_1^{-1}.$$

From (3.33) we now find

$$z_4^{-1} = z_2 . \tag{3.39}$$

We now have shown $z_1^3 = 1$, $z_2 = z_1^{-1}$, $z_3 = z_1^{-1}$, $z_4 = z_1$, proving A' of order 3, A of order 27, and so our lemma is true.

The next lemma is similar.

LEMMA 5. *If $H = \{x, a, b\}$ is of exponent six, and if $x^2 = 1$, $a^3 = 1$, $b^3 = 1$, $xax = a^{-1}$, $xbx = b$, then $\{a, b\}$ is of exponent three.*

Proof. With $A = \{a, b\}$, as in the previous lemma A' is generated by $z_1 = a^{-1}b^{-1}ab$, $z_2 = a^{-1}bab^{-1}$, $z_3 = ab^{-1}a^{-1}b$, and $z_4 = aba^{-1}b^{-1}$. Here automorphisms of A include replacing a or b by its inverse, but not an interchange of a and b. Here

$$\begin{aligned} xax = a^{-1}, \quad x(bab^{-1})x = ba^{-1}b^{-1} = (bab^{-1})^{-1}, \quad \text{and} \\ x(b^{-1}ab)x = b^{-1}a^{-1}b = (b^{-1}ab)^{-1}. \end{aligned} \tag{3.40}$$

Hence by Lemma 4 both $\{a, bab^{-1}\}$ and $\{a, b^{-1}ab\}$ are of exponent three. Thus

$$(a^{-1}\cdot b^{-1}ab)^3 = 1 \quad \text{or} \quad z_1^3 = 1, \tag{3.41}$$

and similarly

$$z_2^3 = z_3^3 = z_4^3 = 1. \tag{3.42}$$

Also

$$z_4 z_2^{-1} z_4^{-1} z_2 = (a\cdot bab^{-1})^3 = 1, \quad \text{or} \quad z_4 z_2 = z_2 z_4 . \tag{3.43}$$

In this replacing b by b^{-1} gives

$$z_3 z_1^{-1} z_3^{-1} z_1 = 1 \quad \text{or} \quad z_3 z_1 = z_1 z_3 . \tag{3.44}$$

Also

$$1 = ((ab^{-1})^3)^2 = (z_3 z_1^{-1} z_2)^2, \tag{3.45}$$

and

$$1 = (b^{-1}(ab)^3 b)^2 = (z_3^{-1} z_2)^2. \tag{3.46}$$

Also as $z_1^3 = 1$, we have

$$1 = (b^{-1} z_1^{-1} b)^3 = (z_2^{-1} z_1)^3. \tag{3.47}$$

From (3.45) we have

$$z_3 z_1^{-1} z_2 z_3 z_1^{-1} z_2 = 1, \tag{3.48}$$

and by using (3.44)

$$z_3 z_2 z_3 = z_1 z_2^{-1} z_1 . \tag{3.49}$$

From (3.46)

$$z_3 z_2^{-1} = z_2 z_3^{-1} \quad \text{and} \quad z_2^{-1} z_3 = z_3^{-1} z_2 . \tag{3.50}$$

Thus using (3.49) and (3.50), we have

$$z_2 z_3 z_2 = z_2 z_3^{-1} \cdot z_3^{-1} z_2 = z_3 z_2^{-1} \cdot z_2^{-1} z_3 = z_3 z_2 z_3 = z_1 z_2^{-1} z_1 . \tag{3.51}$$

Hence

$$z_3 = z_2^{-1} z_1 z_2^{-1} z_1 z_2^{-1}. \tag{3.52}$$

But from (3.47) this gives

$$z_3 = (z_2^{-1} z_1)^3 \cdot z_1^{-1} = z_1^{-1}. \tag{3.53}$$

In (3.53) replace b by b^{-1};

$$z_4 = z_2^{-1}. \tag{3.54}$$

Transform (3.53) by b, and we have

$$z_3^{-1} z_4 = z_2^{-1} z_1 , \tag{3.55}$$

whence from (3.53) and (3.54)

$$z_1 z_2^{-1} = z_2^{-1} z_1 , \tag{3.56}$$

and so z_1 and z_2 permute. Substitute $z_3 = z_1^{-1}$ in (3.46), and we have

$$(z_1 z_2)^2 = 1. \tag{3.57}$$

But as z_1 and z_2 permute,

$$(z_1 z_2)^3 = z_1^3 z_2^3 = 1. \tag{3.58}$$

From (3.57) and (3.58) we get

$$z_1 z_2 = 1. \tag{3.59}$$

Combining (3.53), (3.54), and (3.59) we have

$$z_2 = z_1^{-1}, \qquad z_3 = z_1^{-1}, \qquad z_4 = z_1 , \qquad z_1^3 = 1. \tag{3.60}$$

Thus A' is of order 3, and $A = \{a, b\}$ is of order 27 and exponent three.

LEMMA 6. *If* $H = \{x, a, b, c\}$ *is of exponent six and* $x^2 = 1$, $a^3 = b^3 = c^3 = 1$, $xax = a^{-1}$, $xbx = b^{-1}$, $xcx = c^{-1}$, *then* $\{a, b, c\}$ *is of exponent three.*

Proof. By Lemma 4 $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ are of exponent three. Since the rules (2.1) apply to groups of exponent three, we have $x(a, b)x = (a^{-1}, b^{-1}) = (a, b^{-1})^{-1} = (a, b)$. Hence by Lemma 5, $\{c, (a, b)\}$ is of exponent three and also $\{a, (b, c)\}$, $\{b, (c, a)\}$. Let us write

$$\begin{aligned} u_1 &= (a, b), & u_2 &= (c, a), & u_3 &= (b, c), \\ v_1 &= (a, b, c), & v_2 &= (c, a, b), & v_3 &= (b, c, a). \end{aligned} \tag{3.61}$$

Then since $\{(a, b), c\}$ is of exponent three, we have $(a, b, c^{-1}) = (a, b, c)^{-1}$. Similarly we have the following relations:

$$(3.62)\qquad \begin{array}{lll} v_1^{-1} = (a, b, c^{-1}), & v_1^{-1} = (b, a, c), & v_1 = (b, a, c^{-1}), \\ v_2^{-1} = (c, a, b^{-1}), & v_2^{-1} = (a, c, b), & v_2 = (a, c, b^{-1}), \\ v_3^{-1} = (b, c, a^{-1}), & v_3^{-1} = (c, b, a), & v_3 = (c, b, a^{-1}). \end{array}$$

We now calculate $a^{-1}v_1 a$.

$$(3.63)\qquad \begin{aligned} a^{-1}v_1 a &= a^{-1}(a, b)^{-1}c^{-1}(a, b)ca \\ &= (a, b)^{-1}a^{-1}c^{-1}(a, b)ca \\ &= (a, b)^{-1}a^{-1}c^{-1}ac\cdot c^{-1}a^{-1}(ab)ac\cdot c^{-1}a^{-1}ca \\ &= (a, b)^{-1}(a, c)c^{-1}(a, b)c\cdot (c, a) \\ &= (a, b)^{-1}(a, c)(a, b)(a, b, c)(c, a) \\ &= u_1^{-1}u_2^{-1}u_1 v_1 u_2 . \end{aligned}$$

Here we noted that since $\{a, b\}$ is of exponent three, a permutes with (a, b). In (3.63) let us replace b by b^{-1} and c by c^{-1}. This gives

$$(3.64)\qquad a^{-1}v_1 a = u_1 u_2 u_1^{-1}v_1 u_2^{-1}.$$

From (3.63) and (3.64) we have

$$(3.65)\qquad u_1^{-1}u_2^{-1}u_1 v_1 u_2\cdot u_2 v_1^{-1}u_1 u_2^{-1}u_1^{-1} = 1,$$

whence

$$(3.66)\qquad v_1 u_2^{-1}v_1^{-1} = u_1^{-1}u_2 u_1 u_1 u_2 u_1^{-1},$$

and so

$$(3.67)\qquad u_2 v_1 u_2^{-1}v_1^{-1} = (u_2 u_1^{-1})^3,$$

whence

$$(3.68)\qquad (u_2 v_1 u_2^{-1}v_1^{-1})^2 = (u_2 u_1^{-1})^6 = 1.$$

As $xv_1 x = v_1^{-1}$, $xu_2 x = u_2$, by Lemma 5, u_2 and v_1 generate a group of exponent three, and so

$$(3.69)\qquad (u_2 v_1 u_2^{-1}v_1^{-1})^3 = 1.$$

Combining (3.68) and (3.69) we have

$$(3.70)\qquad u_2 v_1 u_2^{-1}v_1^{-1} = 1 \quad \text{or} \quad u_2 v_1 = v_1 u_2 .$$

Also using (3.67) we find

$$(3.71)\qquad (u_2 u_1^{-1})^3 = 1.$$

We have $u_2 v_2 = v_2 u_2$ since $\{u_2, b\}$ is of exponent three, and this with (3.70) and substitution gives

$$u_i v_j = v_j u_i, \qquad i, j = 1, 2, 3. \tag{3.72}$$

From $xcx = c^{-1}$, $x(aba)x = (aba)^{-1}$ we may apply Lemma 4 and conclude that $\{aba, c\}$ is of exponent three. In particular $(abac)^3 = 1$ or

$$\begin{aligned} &abacabacabac = 1 \\ &aba^{-1}b^{-1} \cdot ba^{-1}cac^{-1}b^{-1} \cdot bcbacabac = 1. \end{aligned} \tag{3.73}$$

Here since $aca = c^{-1}a^{-1}c^{-1}$, we have

$$\begin{aligned} &u_1 b(c, a)b^{-1} \cdot bcbc^{-1}b \cdot b^{-1}a^{-1}ba(a^{-1}b^{-1}c^{-1}bca)a^{-1}c^{-1}ac = 1, \\ &u_1(c, a)(c, a, b^{-1})u_3^{-1}u_1^{-1}a^{-1}(b, c)a \cdot u_2^{-1} = 1, \\ &u_1 u_2 v_2^{-1}u_3^{-1}u_1^{-1}(b, c)(b, c, a)u_2^{-1} = 1, \\ &u_1 u_2 v_2^{-1}u_3^{-1}u_1^{-1}u_3 v_3 u_2^{-1} = 1. \end{aligned} \tag{3.74}$$

This with (3.72) gives

$$v_2^{-1}v_3 = u_2 u_3^{-1}u_1 u_3 u_2^{-1}u_1^{-1}. \tag{3.75}$$

In this replacing a by a^{-1} and c by c^{-1} gives

$$v_2^{-1}v_3 = u_2 u_3 u_1^{-1}u_3^{-1}u_2^{-1}u_1. \tag{3.76}$$

From (3.75) and (3.76) we get

$$u_2^{-1}u_1^{-1}u_1^{-1}u_2 = u_3^{-1}u_1^{-1}u_3^{+1}u_3^{+1}u_1^{-1}u_3^{-1}, \tag{3.77}$$

whence

$$u_1^{-1}u_2^{-1}u_1 u_2 = (u_1^{-1}u_3^{-1})^3. \tag{3.78}$$

But if in (3.71) we replace $\begin{pmatrix} a, b, c \\ b, c, a^{-1} \end{pmatrix}$ we get

$$(u_1^{-1}u_3^{-1})^3 = 1, \tag{3.79}$$

and so from (3.78)

$$u_1^{-1}u_2^{-1}u_1 u_2 = 1 \quad \text{or} \quad u_1 u_2 = u_2 u_1. \tag{3.80}$$

Substituting we have

$$u_i u_j = u_j u_i, \qquad i, j = 1, 2, 3. \tag{3.81}$$

Then (3.75) becomes

$$v_2^{-1}v_3 = 1 \quad \text{or} \quad v_2 = v_3. \tag{3.82}$$

Substituting we have

$$v_1 = v_2 = v_3. \tag{3.83}$$

Writing v for the common value of v_1, v_2, v_3 we have $a^{-1}va = a^{-1}v_3 a = v_3 = v$ and $a^{-1}u_1 a = u_1$, $a^{-1}u_2 a = u_2$, $a^{-1}u_3 a = u_3 v$. Similarly $\{u_1, u_2, u_3, v\}$ is transformed into itself by b and c and thus is the derived group of $A = \{a, b, c\}$. This shows that A is of order dividing 3^7, which is the order of the Burnside group $B(3, 3)$. Thus $A = B(3, 3)$ is of exponent three, proving the lemma.

LEMMA 7. *If* $H = \{x, a_1, a_2, \cdots, a_n\}$ *is of exponent six and* $x^2 = 1$, $a_i^3 = 1$, $i = 1, \cdots, n$, $xa_i x = a_i^{-1}$, $i = 1, \cdots, n$, *then* $A = \{a_1, \cdots, a_n\}$ *is of exponent three.*

Proof. By Lemma 6 any three of the a_i generate a group of exponent three, and by the corollary to Theorem 2.2 this proves that A is of exponent three.

LEMMA 8. *If* $H = \{a, b, c\}$ *is of exponent six and* $a^2 = b^2 = c^2 = 1$, *then* H' *is of exponent three.*

Proof. The following transformation table shows that α_1, α_2, α_3, α_4, α_5 generate H'.

(3.84)

	$a\alpha_i a$	$b\alpha_i b$	$c\alpha_i c$
$\alpha_1 = abab$	α_1^{-1}	α_1^{-1}	$\alpha_2^{-1}\alpha_5^{-1}\alpha_4\alpha_3$
$\alpha_2 = acac$	α_2^{-1}	$\alpha_1^{-1}\alpha_4$	α_2^{-1}
$\alpha_3 = bcbc$	α_5	α_3^{-1}	α_3^{-1}
$\alpha_4 = abcacb$	α_4^{-1}	$\alpha_1^{-1}\alpha_2$	$\alpha_2^{-1}\alpha_5^{-1}\alpha_1\alpha_3$
$\alpha_5 = abcbca$	α_3	$\alpha_1^{-1}\alpha_5^{-1}\alpha_1$	$\alpha_2^{-1}\alpha_5^{-1}\alpha_2$

Here $\alpha_1^3 = (ab)^6 = 1$, $\alpha_2^3 = (ac)^6 = 1$, and $a\alpha_1 a = \alpha_1^{-1}$, $a\alpha_2 a = \alpha_2^{-1}$, whence by Lemma 4, $\{\alpha_1, \alpha_2\}$ is of exponent three. Hence

$$b^{-1}\{\alpha_1, \alpha_2\}b = \{\alpha_1^{-1}, \alpha_1^{-1}\alpha_4\}$$

is also of exponent three, and so in particular $\alpha_4^3 = 1$. Thus since $a\alpha_4 a = \alpha_4^{-1}$ from Lemma 6, $\{\alpha_1, \alpha_2, \alpha_4\}$ is of exponent three. This is the group $\{abab, acac, abcacb\}$. If we interchange a and b, the corresponding group $\{baba, bcbc, bacbca\}$ must also be of exponent three. But this is $\{\alpha_1^{-1}, \alpha_3, \alpha_1^{-1}\alpha_5\} = \{\alpha_1, \alpha_3, \alpha_5\}$. In particular the subgroup $\{\alpha_3, \alpha_5\}$ is of exponent three, and so we have

$$(3.85)\qquad (\alpha_3\alpha_5^{-1})^3 = 1, \qquad (\alpha_3^{-1}\alpha_5)^3 = 1.$$

But from (3.84)

$$(3.86)\qquad \begin{aligned} a(\alpha_3\alpha_5^{-1})a &= \alpha_5\alpha_3^{-1} = (\alpha_3\alpha_5^{-1})^{-1}, \\ a(\alpha_3^{-1}\alpha_5)a &= \alpha_5^{-1}\alpha_3 = (\alpha_3^{-1}\alpha_5)^{-1}. \end{aligned}$$

Hence from Lemma 7 the following elements

$$(3.87) \qquad \alpha_1, \quad \alpha_2, \quad \alpha_4, \quad \alpha_3\alpha_5^{-1}, \quad \alpha_3^{-1}\alpha_5$$

are all of order 3 and transformed into their inverses by a, whence they generate a group of exponent three. In particular

$$(3.88) \qquad (\alpha_i\alpha_3\alpha_5^{-1})^3 = 1, \qquad (\alpha_i\alpha_3^{-1}\alpha_5)^3 = 1, \qquad i = 1, 2, 4,$$

whence

$$(3.89) \qquad (\alpha_5^{-1}\alpha_i\alpha_3)^3 = 1, \qquad (\alpha_5\alpha_i\alpha_3^{-1})^3 = 1, \qquad i = 1, 2, 4.$$

It follows that the group K given by

$$(3.90) \qquad K = \{\alpha_i, \alpha_3\alpha_5^{-1}, \alpha_3^{-1}\alpha_5, \alpha_5^{-1}\alpha_i\alpha_3, \alpha_5\alpha_i\alpha_3^{-1}\}, \qquad i = 1, 2, 4,$$

is of exponent three by Lemma 7 since each of the elements is of order 3 and is transformed into its inverse by a.

Noting that $\alpha_3^{-1}\alpha_5\cdot\alpha_5^{-1}\alpha_i\alpha_3 = \alpha_3^{-1}\alpha_i\alpha_3$ and $\alpha_3\alpha_5^{-1}\cdot\alpha_5\alpha_i\alpha_3^{-1} = \alpha_3\alpha_i\alpha_3^{-1}$, and also that $\alpha_3^{-1}(\alpha_3\alpha_5^{-1})\alpha_3 = (\alpha_3^{-1}\alpha_5)^{-1}$ and $\alpha_3^{-1}(\alpha_3^{-1}\alpha_5)\alpha_3 = \alpha_3\alpha_5^{-1}(\alpha_3^{-1}\alpha_5)^{-1}$, we see that K is normalized by α_3. Since K is trivially normalized by α_1, α_2, α_4, and $\alpha_3^{-1}\alpha_5$, we see that K is normal in $H' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Further we note in H'

$$(3.91) \qquad \begin{aligned} &\alpha_1 \equiv 1 \pmod{K}, \qquad \alpha_2 \equiv 1 \pmod{K}, \qquad \alpha_3 \equiv \alpha_3 \pmod{K}, \\ &\alpha_4 \equiv 1 \pmod{K}, \qquad \alpha_5 \equiv \alpha_3 \pmod{K}. \end{aligned}$$

Thus K is of index 3 in H'. Hence for an arbitrary $z \in H'$ we have, since K is of exponent three,

$$(3.92) \qquad z^3 \in K, \qquad z^9 = (z^3)^3 = 1.$$

But as H was of exponent six, we have

$$(3.93) \qquad z^9 = 1, \quad z^6 = 1 \quad \text{whence} \quad z^3 = 1.$$

Thus H' is of exponent three, proving our lemma.

LEMMA 9. *If* $H = \{a, b, c, d\}$ *is of exponent six and* $a^2 = b^2 = c^2 = d^2 = 1$ *and* $\alpha = abab$, $\beta = cdcd$, *then* $\{\alpha, \beta\}$ *is of exponent three.*

Proof. Write $\beta = \beta_1 = cdcd$, $\beta_2 = acdcda$. Then $\{\beta_1, \beta_2\}$ is in the derived group of $\{a, c, d\}$ and so by Lemma 8 is of exponent three. In particular

$$(3.94) \qquad (\beta_1\beta_2^{-1})^3 = 1, \qquad (\beta_1^{-1}\beta_2)^3 = 1.$$

Thus the group

$$(3.95) \qquad U = \{\alpha, \beta_1\beta_2^{-1}, \beta_1^{-1}\beta_2\}$$

is generated by elements of order 3 and $a\alpha a = baba = \alpha^{-1}$,

$$a(\beta_1 \beta_2^{-1})a = \beta_2 \beta_1^{-1} = (\beta_1 \beta_2^{-1})^{-1},$$
$$a(\beta_1^{-1}\beta_2)a = \beta_2^{-1}\beta_1 = (\beta_1^{-1}\beta_2)^{-1},$$

whence by Lemma 6, U is of exponent three. Thus also

$$(3.96) \qquad (\alpha\beta_1 \beta_2^{-1})^3 = 1, \qquad (\alpha\beta_1^{-1}\beta_2)^3 = 1,$$

and so

$$(3.97) \qquad (\beta_2^{-1}\alpha\beta_1)^3 = 1, \qquad (\beta_2 \alpha\beta_1^{-1})^3 = 1,$$

whence by Lemma 7 the group V

$$(3.98) \qquad V = \{\alpha, \beta_1 \beta_2^{-1}, \beta_1^{-1}\beta_2, \beta_2^{-1}\alpha\beta_1, \beta_2 \alpha\beta_1^{-1}\}$$

is of exponent three, being generated by elements of order 3 which are transformed into their inverses. But we readily see that V is normal of index 3 in $A = \{\alpha, \beta_1, \beta_2\}$, whence A is of exponent nine, but by hypothesis being of exponent six, must be of exponent three. But $\{\alpha, \beta_1\} = \{\alpha, \beta\}$ is a subgroup of A and so of exponent three, as we wished to prove.

Now for Lemma 3 and the proof of the main theorem!

LEMMA 3. *If* $H = \{a, b, c, d, e, f\}$ *is o, exponent six and* $a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 1$ *and* $\alpha = abab$, $\beta = cdcd$, $\gamma = efef$, *then* $\{\alpha, \beta, \gamma\}$ *is of exponent three.*

Proof. Write $\beta_1 = \beta = cdcd$, $\beta_2 = acdcda$, $\gamma_1 = \gamma = efef$, $\gamma_2 = aefefa$. Then by Lemma 8, $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are of exponent three. Thus the elements

$$(3.99) \qquad \alpha, \quad \beta_1 \beta_2^{-1}, \quad \beta_1^{-1}\beta_2, \quad \gamma_1 \gamma_2^{-1}, \quad \gamma_1^{-1}\gamma_2$$

are of order 3 and transformed into their inverses by a. Hence by Lemma 7 they generate a group of exponent three. We assert that if $W(u, v)$ is an arbitrary word in elements u, v and their inverses, then the two elements of H

$$(3.100) \qquad W(\beta_1, \gamma_1)W(\beta_2, \gamma_2)^{-1}, \qquad W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1}$$

are of order 3 and are transformed into their inverses by a. Since $a\beta_1 a = \beta_2$, $a\beta_2 a = \beta_1$, $a\gamma_1 a = \gamma_2$, $a\gamma_2 a = \gamma_1$, $a\alpha a = \alpha^{-1}$, we have surely

$$(3.101) \qquad \begin{aligned} a(W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1})a &= W(\beta_2, \gamma_2)\alpha^{-1}W(\beta_1, \gamma_1)^{-1} \\ &= (W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1})^{-1}, \end{aligned}$$

and similarly without α. Thus the elements of (3.100) are all transformed into their inverses by a. Hence by Lemma 7, those elements of (3.100) which are of order 3 generate a group of exponent three. To prove they are of order 3 we proceed by induction on the length of W, this being trivially true if $W = 1$. Now suppose this true for a particular $W(u, v)$. Then the elements

$$W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1}, \qquad W(\beta_1, \gamma_1)W(\beta_2, \gamma_2),$$
$$\beta_2^{-1}\beta_1, \qquad \beta_2\beta_1^{-1}, \qquad \gamma_2^{-1}\gamma_1, \qquad \gamma_2\gamma_1^{-1}$$

by Lemma 7 generate a group of exponent three. Thus

$$(\beta_2^{-1}\beta_1 W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1})^3 = 1, \tag{3.102}$$

whence

$$(\beta_1 W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1}\beta_2^{-1})^3 = 1, \tag{3.103}$$

and similarly without the α. Thus the statement is also true for $uW(u, v)$, and in exactly the same way true for $u^{-1}W(u, v)$, $vW(u, v)$, and $v^{-1}W(u, v)$. But we may build up any word $W(u, v)$ by successively multiplying on the left by u, u^{-1}, v, *or* v^{-1}. By Lemma 9, $\{\beta_1, \gamma_1\}$ is of exponent three and so of order dividing 27. Thus with 27 words $W(u, v)$ we obtain all distinct elements of (3.100). By Lemma 7 the elements of (3.100) generate a group R of exponent three. We note that

$$\beta_1^{-1}W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1}\beta_1 = \beta_1^{-1}W(\beta_1, \gamma_1)\alpha W(\beta_2, \gamma_2)^{-1}\beta_2\cdot\beta_2^{-1}\beta_1 \in R, \tag{3.104}$$

and similarly without α. Thus R is normalized by β_1. Similarly R is normalized by γ_1. But as R contains α, $\beta_1^{-1}\beta_2$, $\gamma_1^{-1}\gamma_2$, R is normal in $A = \{\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2\}$. Furthermore in A we have

$$\begin{aligned} &\alpha \equiv 1 \pmod R, \qquad \beta_1 \equiv \beta_1 \pmod R, \qquad \beta_2 \equiv \beta_1 \pmod R, \\ &\gamma_1 \equiv \gamma_1 \pmod R, \qquad \gamma_2 \equiv \gamma_1 \pmod R. \end{aligned} \tag{3.105}$$

Thus A/R is a homomorphic image of the group $\{\beta_1, \gamma_1\}$ which by Lemma 9 is of exponent three and order 27. Hence for an arbitrary $z \in A$ we have $z^3 \in R$, and $(z^3)^3 = z^9 = 1$. But as $z^6 = 1$ by hypothesis, we have $z^3 = 1$, whence A is of exponent three, and consequently $\{\alpha, \beta, \gamma\}$ which is a subgroup of A is also of exponent three. This proves Lemma 3. The proof of the main theorem is now immediate.

Proof of main theorem. M' by Lemmas 1 and 2 is of finite index in G. By Lemma 2, M' is generated by a finite number of elements of the form $abab$ with $a^2 = b^2 = 1$. By Lemma 3 any three of these generate a group of exponent three. By the corollary to Theorem 2.2 it follows that M' is of exponent three. By the results of Levi and van der Waerden it follows that M' is finite. Since M' is finite and of finite index in G, it follows that G is finite. This proves our theorem.

Bibliography

1. W. Burnside, *On an unsettled question in the theory of discontinuous groups*, Quarterly Journal of Pure and Applied Mathematics (London), vol. 33 (1902), pp. 230–238.

2. P. HALL, *A contribution to the theory of groups of prime-power order*, Proc. London Math. Soc. (2), vol. 36 (1933), pp. 29–95.
3. P. HALL AND G. HIGMAN, *On the p-length of p-soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3), vol. 6 (1956), pp. 1–42.
4. G. HIGMAN, *On finite groups of exponent five*, Proc. Cambridge Philos. Soc., vol. 52 (1956), pp. 381–390.
5. F. LEVI AND B. L. VAN DER WAERDEN, *Über eine besondere Klasse von Gruppen*, Abh. Math. Sem. Univ. Hamburg, vol. 9 (1933), pp. 154–158.
6. I. N. SANOV, *Solution of Burnside's problem for exponent 4*, Leningrad State Univ. Annals (Uchenye Zapiski) Mat. Ser., vol. 10 (1940), pp. 166–170 (Russian).

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