REMARKS ON QUASI-FROBENIUS RINGS

BY
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1.

The theory of Frobenius or quasi-Frobenius rings has, from the start, been connected with the idea of "duality". But in most papers on that theory, by "duality" one understands, either the lattice-theoretic duality [1], [10], or, when the ring A under consideration is an algebra over a commutative ring K, the duality of *K-modules* [3], [8]. Actually, it seems to me that the kind of duality which is most closely related to these questions is the duality of *A-modules*; and I propose to show in this paper how very elementary considerations of duality theory can simplify and unify many known results on quasi-Frobenius rings, and give new characterizations for these rings.[1]

2. Modules with perfect duality

By a ring I will always understand an associative ring A having a unit element 1; all A-modules are supposed to be unitary. The elementary theory of duality [2, §4] associates to each left (right) A-module E its dual E^*, which is a right (left) A-module; further, to every submodule M of E (resp. E^*) is associated its *orthogonal* M^0, which is a submodule of E^* (resp. E); one has the trivial relations:

(i) $M \subset N$ implies $N^0 \subset M^0$, $\quad (M + N)^0 = M^0 \cap N^0$;

(ii) $M \subset M^{00}$, $\quad M^0 = M^{000}$.

In addition, the theory defines

(iii) a natural homomorphism $E \rightarrow E^{**}$;

(iv) a natural isomorphism $(E/M)^* \rightarrow M^0$;

(v) a natural monomorphism $E^*/M^0 \rightarrow M^*$,

M being an arbitrary submodule of E; moreover,

(vi) if E is a direct sum $M_1 + \cdots + M_n$, E^* is naturally identified to the direct sum $M_1^* + \cdots + M_n^*$ (M_i^* being identified by (iv) to the orthogonal of $\sum_{j \neq i} M_j$).

Finally, if A_s (resp. A_d) is A considered as left (resp. right) A-module,

(vii) $(A_s)^* = A_d$, $\quad (A_d)^* = A_s$,

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[1] *Added in proof.* After this paper was written, Professor A. Rosenberg kindly drew my attention to the following paper which I had overlooked:

K. Morita and H. Tachikawa, *Character modules, submodules of a free module, and quasi-Frobenius rings*, Math. Zeit., vol. 65 (1956), pp. 414–428.

In that paper, the authors study the quasi-Frobenius rings from the point of view of duality of A-modules, and prove a slightly weaker version of result (3.4) below (they assume that the duals of simple A-modules are simple), by essentially the same arguments as mine.

and for a left ideal $\mathfrak{l}$ (resp. a right ideal $\mathfrak{r}$) of A, $\mathfrak{l}^0$ (resp. $\mathfrak{r}^0$) is the right annihilator $r(\mathfrak{l})$ (resp. the left annihilator $l(\mathfrak{r})$).

In the "best" cases, e.g. for finite-dimensional vector spaces, we have in addition:

(A) $E \to E^{**}$ is an isomorphism;

(B) $E^*/M^0 \to M^*$ is an isomorphism for any submodule $M \subset E$;

(C) $M^{00} = M$ for any submodule $M \subset E$;

(D) $(M \cap N)^0 = M^0 + N^0$ for any pair of submodules, M, N of E.

We are therefore led to evaluate the "perfection" of modules with respect to duality theory by the extent to which these four properties are satisfied. This leads to a variety of problems, of which we only mention the two following ones:

(I) Let us say that a given module E has *perfect duality* if properties (A), (B), (C), and (D) hold for *E and for its dual* E^*. Find all modules with perfect duality.

(II) Let $\mathcal{C}$ be a *category* of A-modules, such that every submodule and every quotient module of a module of $\mathcal{C}$ is still in $\mathcal{C}$, and in addition the dual of any module of $\mathcal{C}$ belongs to $\mathcal{C}$. Find all categories $\mathcal{C}$ consisting of modules with perfect duality.

We will only be concerned with a special case of problem (II). But before we go into the details, it may be worthwhile to observe that there are logical connections between properties (A), (B), (C), and (D), and also (when (A) holds) between these properties and the corresponding ones for E^*, which we write (B*), (C*), and (D*). To begin with:

(2.1) *Condition* (B) *implies condition* (D).

This has been proved by Ikeda-Nakayama [13, p. 16], when $E = A_s$, and their proof extends at once to the general case: if $x' \in (M \cap N)^0$, the mapping $x + y \to \langle x, x' \rangle$ is a well defined linear form on $M + N$ ($x \in M$, $y \in N$); hence by assumption there is $y' \in E^*$ such that $\langle x + y, y' \rangle = \langle x, x' \rangle$ for $x \in M$ and $y \in N$; if we take $x = 0$, we obtain $y' \in N^0$, and $y = 0$ yields $x' - y' \in M^0$; hence $x' \in M^0 + N^0$, Q.E.D.

Condition (D) implies neither (B) nor (C), even when (A) also holds: a trivial counterexample is provided by taking $E = A_s$, A being a commutative domain of integrity other than a field (for instance a principal ideal ring). I do not know if (C) implies (D) or not.

In the theory of quasi-Frobenius rings, examples turn up showing that (C) does not imply (B) (even when (D) also holds) [11, p. 48]. Ikeda-Nakayama [13, p. 16] have shown that for $E = A_s$, (B) implies (C*) when A is noetherian. But it is possible to give examples of commutative rings where (B) holds but not (C): it is not hard to see that we obtain such a ring by taking an infinite product (= "complete direct sum") of fields.

Properties (B) and (C) (hence also (D)) may hold without (A) being valid, as the example of infinite-dimensional vector spaces shows.

A well-known [4, Theorem 4.5] and fairly obvious result (by (i) and (ii)) is:

(2.2) *If both conditions* (C) *and* (C*) *are satisfied, so is* (D) (*and* (D*)).

We have seen above that (B) and (B*) may be both satisfied, but neither (C) nor (C*). The major unsolved problem in this connection is whether *both* conditions (C) and (C*) (under the assumption that (A) holds) imply perfect duality.

(2.3) (a) *If* (B*) *and* (C) *are satisfied, then* (A) *holds for every quotient module of* E.

(b) *If* (A) *holds for every quotient module of* E, *then* (C) *is satisfied*; *if in addition* (C*) *is satisfied, then so is* (B*).

(c) *If* (A) *holds for every quotient module of* E *and every quotient module of* E^*, *then* E *has perfect duality.*

(a) follows at once from the definitions and from (iv). Conversely it is readily verified that the natural homomorphism of E/M into $(E/M)^{**}$ is the natural homomorphism $E/M \to E/M^{00}$; in case (b), the assumption thus implies $M^{00} = M$ and $(M^0)^* = E/M$; if in addition (C*) holds, then any submodule of E^* has the form M^0, and (B*) is satisfied. The conclusion of (c) is then obvious.

The converse of (2.3.c) follows from the definitions, and in addition (A) holds for all submodules of E and E^*; but this last condition by itself implies none of the others, as the example $E = A_s$, A a principal ideal ring, shows again.

Finally, let us consider the following condition (see [7, p. 455] and [3, p. 14]):

(N) *For any pair of submodules* M, N *of* E *such that* $N \subset M$ *and* $N \neq M$ *the dual* $(M/N)^*$ *is not reduced to* 0.

Then

(2.4) *Conditions* (B) *and* (N) *imply* (C).

Let M be a submodule of E. Its dual is then identified to E^*/M^0 by (B); so is the dual of M^{00}. But in E^*/M^0, considered as the dual of M^{00}, the orthogonal submodule of M is $M^0/M^0 = \{0\}$; hence the dual of M^{00}/M is reduced to 0; condition (N) then implies that $M^{00} = M$.

3. Finitely generated modules with perfect duality

For any ring A, there are always A-modules which do not have perfect duality, for instance the direct sum of infinitely many copies of A_s. This leads us to consider the category $\mathcal{C}_f(A)$ of *finitely generated* A-modules (left or right); to be a category in the sense introduced in §2, one has of course to suppose that A is both *left and right noetherian.* Then any module of $\mathcal{C}_f(A)$ is isomorphic to a quotient A_s^n/R (or A_d^n/R) and therefore its dual is isomorphic to the submodule R^0 of A_d^n (resp. A_s^n) and hence belongs also to $\mathfrak{C}_f(A)$. The main result of this section is then:

(3.1) *In order that all the modules in $\mathfrak{C}_f(A)$ have perfect duality, it is necessary and sufficient that A be a quasi-Frobenius ring.*

The necessity is of course obvious, by applying (C) to A_s and A_d. Most of the sufficiency part of the theorem is known: that conditions (C) and (C*) hold is substantially the Hall-Nakayama theorem ([4, Theorem 5.2] and [9, Theorem 12]); and (B) has been proved for the case $E = A_s$ by Ikeda and Nakayama ([6], [13]). We will not use any of these theorems, but give a very elementary proof which needs no structure theory, and which starts from very much weaker assumptions than the properties (C) and (C*) for $E = A_s$.

In what follows, we suppose (unless the contrary is explicitly stated) that A is a left and right artinian ring (i.e. satisfies both minimal conditions). We consider the following conditions:

($\mathrm{c_s}$) Every minimal left ideal in A satisfies $\mathfrak{l}^{00} = \mathfrak{l}$.

($\mathrm{c_d}$) Every minimal right ideal in A satisfies $\mathfrak{r}^{00} = \mathfrak{r}$.

($\mathrm{m_s}$) The dual of any simple left A-module has length $\leqq 1$.

($\mathrm{m_d}$) The dual of any simple right A-module has length $\leqq 1$.

We first observe that

(3.2) *Condition* ($\mathrm{c_d}$) *implies condition* ($\mathrm{m_s}$).

Indeed any simple left A-module is isomorphic to a quotient $A_s/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal left ideal; its dual is therefore isomorphic to $\mathfrak{m}^0$. If $\mathfrak{m}^0$ had a length $\geqq 2$, it would properly contain a minimal right ideal $\mathfrak{r}$; hence $\mathfrak{r}^0 \supset \mathfrak{m}^{00} \supset \mathfrak{m}$, and therefore $\mathfrak{r}^0 = A_s$ or $\mathfrak{r}^0 = \mathfrak{m}$; but as $\mathfrak{r}^{00} = \mathfrak{r}$ by assumption, both conclusions contradict the definition of $\mathfrak{r}$; hence $\mathfrak{m}^0$ has length $\leqq 1$.

In what follows, we will denote by $\lambda(E)$ the length of a (left or right) A-module E.

(3.3) *Let A be an arbitrary left artinian ring, and suppose the dual of any simple left A-module has length $\leqq k$. Then for any left A-module E of finite length, $\lambda(E^*) \leqq k\cdot\lambda(E)$.*

The proof is by induction on $n = \lambda(E)$, the result being true by assumption for $n = 1$. Let M be a nontrivial submodule of E; then $\lambda(M) < n$ and $\lambda(E/M) < n$; hence by (iv) and the inductive assumption, $\lambda(M^0) \leqq k\lambda(E/M)$, and by (v) and the inductive assumption, $\lambda(E^*/M^0) \leqq \lambda(M^*) \leqq k\cdot\lambda(M)$; hence $\lambda(E^*) = \lambda(M^0) + \lambda(E^*/M^0) \leqq k(\lambda(E/M) + \lambda(M)) = k\cdot\lambda(E)$.

Our strengthened version of (3.1) is then the following one:

(3.4) *Consider (for a left and right artinian ring A) the three conditions* ($\mathrm{m_s}$), ($\mathrm{m_d}$), *and*

(e) *the modules A_s and A_d have the same length.*

Then any two of these conditions imply the third and imply that the modules in $\mathcal{C}_f(A)$ have perfect duality.

If (m_s) and (m_d) are both satisfied, then it follows from (3.3) that

$$\lambda(E^*) \leqq \lambda(E)$$

for any (left or right) module E in $\mathcal{C}_f(A)$; in particular $\lambda(A_s) = \lambda(A_d)$, i.e. condition (e) holds.

Suppose then that (m_s) and (e) are satisfied, and consider an arbitrary left A-module E of finite length; it is isomorphic to a quotient A_s^n/R. By (3.3) and (m_s) we have $\lambda(R^0) \leqq \lambda(A_s^n/R)$ and $\lambda(A_d^n/R^0) \leqq \lambda(R^*) \leqq \lambda(R)$; however $\lambda(A_d^n) = \lambda(R^0) + \lambda(A_d^n/R^0) \leqq \lambda(A_s^n/R) + \lambda(R) = \lambda(A_s^n) = \lambda(A_d^n)$, hence

$$\lambda(E^*) = \lambda(R^0) = \lambda(A_s^n/R) = \lambda(E).$$

If we apply this to any quotient module E/M of E, we get

$$\lambda(M^0) = \lambda(E/M) = \lambda(E) - \lambda(M),$$

and if we apply the same result to M, we have by (v) $\lambda(E^*/M^0) \leqq \lambda(M^*) = \lambda(M)$, and as $\lambda(E^*) = \lambda(E)$, this implies $\lambda(E^*/M^0) = \lambda(M^*) = \lambda(M)$; hence $M^* = E^*/M^0$, i.e. condition (B) is satisfied. On the other hand the equality $\lambda(E^*) = \lambda(E)$ shows that condition (N) of §2 is also satisfied; hence by (2.4) condition (C) holds. In particular (c_s) is satisfied, hence also (m_d) by (3.2). But we can now exchange A_s and A_d in the previous argument; hence $\lambda(E^{**}) = \lambda(E^*) = \lambda(E)$; furthermore, (C) applied to the submodule $\{0\}$ of E yields $(E^*)^0 = \{0\}$, and $(E^*)^0$ is the kernel of the natural mapping $E \to E^{**}$; this mapping is thus injective, and the relation $\lambda(E^{**}) = \lambda(E)$ shows that $E^{**} = E$, which ends the proof.

As a corollary, we have immediately:

(3.5) *For any module E in $\mathcal{C}_f(A)$, where A is a quasi-Frobenius ring, the centralizers of E and E^* are inverse isomorphic rings.*

We have only to observe that as $E^{**} = E$, the transposition $u \to {}^t u$ is a bijection of the centralizer of E onto the centralizer of E^*. This generalizes a result of Nakayama for simple A-modules [8, pp. 620–621].

4. Study of conditions (m_s) and (N)

Using now much more structure theory and previous results of Nakayama and Ikeda, we turn to the study of rings in which (m_s) *alone* is satisfied. Our goal is to prove that

(4.1) *An algebra of finite rank which satisfies* (m_s) *is a quasi-Frobenius ring.*

We will reach this by a series of intermediate results, some of which have independent interest (and in which we do not suppose that A is an algebra). We denote by $\mathfrak{N}$ the radical of A, by $\mathfrak{R}$ (resp. $\mathfrak{L}$) the right (resp. left) socle of

A, which, as is well known, is the left annihilator $l(\mathfrak{N})$ (resp. the right annihilator $r(\mathfrak{N})$).

(4.2) *The following properties are equivalent:*

(a) *Condition* (N) *holds for any left A-module of finite length.*

(b) *The dual of any simple left A-module is not reduced to* 0.

(c) $l(\mathfrak{L}) = \mathfrak{N}$.

(d) *For any simple left A-module M, there is a minimal left ideal in A isomorphic to M.*

It is obvious that (a) implies (b). Conversely, if (b) is satisfied and E is any left A-module of finite length, M a maximal submodule of E, then the dual of E/M is not reduced to 0, hence $M^0 \neq \{0\}$ by (iv), and a fortiori $E^* \neq \{0\}$. To prove that (c) implies (b), suppose (b) is not verified. Consider the semisimple ring $\bar{A} = A/\mathfrak{N}$, which is the direct sum of minimal two-sided ideals $\bar{\mathfrak{a}}_k$ $(1 \leqq k \leqq r)$; denote by $\mathfrak{a}_k$ the inverse image of $\bar{\mathfrak{a}}_k$ in A; as a left A-module, $\bar{\mathfrak{a}}_k = \mathfrak{a}_k/\mathfrak{N}$ is a direct sum of isomorphic simple A-modules, and any simple left A-module is isomorphic to a simple submodule of one of the $\bar{\mathfrak{a}}_k$. Our assumption implies by (vi) that the dual of one $\bar{\mathfrak{a}}_k$ is $\{0\}$; hence (again by (vi)), the dual of $A_s/\mathfrak{N}$ is the direct sum of the $\bar{\mathfrak{a}}_h^*$ for $h \neq k$, and therefore $\bar{\mathfrak{a}}_k$ is orthogonal to $(A_s/\mathfrak{N})^*$; however, by (iv), $(A_s/\mathfrak{N})^*$ is identified to $\mathfrak{L} = r(\mathfrak{N})$, and hence $\mathfrak{a}_k$ is contained in the left annihilator $l(\mathfrak{L})$; in other words (c) does not hold.

Conversely, suppose (b) is satisfied; then none of the $\bar{\mathfrak{a}}_k^*$ is reduced to 0, and their direct sum is $\mathfrak{L}$; furthermore, if $\mathfrak{b}_k = \sum_{h \neq k} \mathfrak{a}_h$, $\bar{\mathfrak{a}}_k^*$ is identified (by (vi)) to the right annihilator $r(\mathfrak{b}_k) \subset \mathfrak{L}$. Now suppose $l(\mathfrak{L}) \neq \mathfrak{N}$; as $l(\mathfrak{L})$ is a two-sided ideal, it would contain one of the $\mathfrak{a}_k$, and hence $\bar{\mathfrak{a}}_k^*$ would be in the right annihilator of both $\mathfrak{a}_k$ and $\mathfrak{b}_k$; however, as $A = \mathfrak{a}_k + \mathfrak{b}_k$, this implies $\bar{\mathfrak{a}}_k^* = \{0\}$, contrary to assumption.

It is almost immediate that (d) implies (b), for the annihilator $\mathfrak{l}^0$ of a minimal left ideal $\mathfrak{l}$ cannot be equal to A; hence (by (v)) the dual of $\mathfrak{l}$ is not reduced to 0, which proves (b). Conversely, there is in $\mathfrak{a}_k$ an idempotent u_k such that its class $\bar{u}_k$ mod $\mathfrak{N}$ is the unit element of the simple ring $\bar{\mathfrak{a}}_k$, and $1 = \sum_{k=1}^r u_k$; moreover, $u_k \mathfrak{L}$ is the foot of the left socle $\mathfrak{L}$ consisting of the sum of all minimal left ideals isomorphic to a simple submodule of $\bar{\mathfrak{a}}_k$; condition (c) implies that $u_k \mathfrak{L} \neq \{0\}$ for any k, and therefore (c) implies (d).

We will denote by $(\mathrm{N_s})$ any one of the four equivalent conditions of (4.2), by $(\mathrm{N_d})$ the corresponding condition for right A-modules.

(4.3) *The condition $\mathfrak{N} \subset \mathfrak{L}$ implies* $(\mathrm{N_s})$.

Indeed, if condition $(\mathrm{N_s})$ were not satisfied, the right annihilator of one of the $\mathfrak{a}_k$ would contain $\mathfrak{L}$, hence also $\mathfrak{N}$. However, we have $\mathfrak{a}_k \supset Au_k$, and therefore $r(\mathfrak{a}_k) \subset (1 - u_k)A$; hence $\mathfrak{N}$ would be contained in $(1 - u_k)A$; but the right ideal $u_k A$ contains a minimal right ideal, and this brings a contradiction.

Observe that one of Nakayama's examples [11, p. 49] shows that $\mathfrak{R} \subset \mathfrak{L}$ is not a necessary condition for ($\mathrm{N_s}$). On the other hand, it is easy to form examples in which only one of the two conditions ($\mathrm{N_s}$), ($\mathrm{N_d}$) is satisfied: such is the algebra A over a field K, having a basis consisting of the unit 1 and two elements u, v, with $u^2 = u$, $uv = v$, $v^2 = vu = 0$ as multiplication table; we have here $Kv = \mathfrak{N} = \mathfrak{R}$, and $\mathfrak{L} = Ku + Kv$; ($\mathrm{N_s}$) is satisfied, but $r(\mathfrak{R}) = \mathfrak{L} \neq \mathfrak{R}$.

(4.4) *Condition* ($\mathrm{m_s}$) *implies* $\mathfrak{L} \subset \mathfrak{R}$, *and both conditions* ($\mathrm{N_s}$) *and* ($\mathrm{N_d}$).

Decompose A into a direct sum of indecomposable right ideals $e_{ki} A$ ($1 \leqq k \leqq r$, $1 \leqq i \leqq f(k)$ for each k), the classes $\bar{e}_{ki}$ mod $\mathfrak{N}$ forming a system of primitive orthogonal idempotents in $\bar{A}$, and the $e_{ki} A$ which correspond to the same value of k being isomorphic to each other. It is clear that $l(e_{ki}\mathfrak{L})$ contains the maximal left ideal $A(1 - e_{ki}) + \mathfrak{N}$; by condition ($\mathrm{m_s}$),

$$r(l(e_{ki}\mathfrak{L})) \supset e_{ki}\mathfrak{L}$$

has length $\leqq 1$; hence $e_{ki}\mathfrak{L}$ is 0 or a minimal right ideal; therefore

$$\mathfrak{L} = \textstyle\sum_{i,k} e_{ki}\mathfrak{L}$$

is a sum of minimal right ideals; in other words $\mathfrak{L} \subset \mathfrak{R}$. This already implies ($\mathrm{N_d}$) by (4.3). Now consider a maximal right ideal $\mathfrak{n}$ in A; by (iv) and ($\mathrm{N_d}$), the left ideal $\mathfrak{n}^0$ is not reduced to 0, hence contains a minimal left ideal $\mathfrak{l}$. As $\mathfrak{l}^0$ cannot be equal to A and contains $\mathfrak{n}$, $\mathfrak{l}^0 = \mathfrak{n}$; conditions ($\mathrm{m_s}$) and (v) then imply that the dual $\mathfrak{l}^*$ is isomorphic to $A_d/\mathfrak{n}$. However, there are r nonisomorphic simple right A-modules $A_d/\mathfrak{n}$; hence there are also r nonisomorphic minimal left ideals in A, and by (4.2.d), this proves condition ($\mathrm{N_s}$).

Conditions ($\mathrm{N_s}$) and ($\mathrm{N_d}$) show that none of the feet $u_k\mathfrak{L}$, $\mathfrak{R}u_k$ of the two socles is reduced to 0; moreover it follows from the proof of (4.4) that each $u_k\mathfrak{L}$ is contained in one of the feet of $\mathfrak{R}$, which we will denote by $\mathfrak{R}u_{\pi(k)}$, π being a permutation of the integers $\leqq r$; this implies of course $u_k\mathfrak{L} = \mathfrak{L}u_{\pi(k)}$.

Proof of (4.1). Denote by d_k the dimension (over the field of scalars) of the simple A-modules $\bar{A}\bar{e}_{k1}$ and $\bar{e}_{k1}\bar{A}$. As each simple right A-module $e_{ki}\mathfrak{L}$ is contained in $\mathfrak{R}u_{\pi(k)}$, it is isomorphic to $\bar{e}_{\pi(k),1}\bar{A}$; hence the dimension of $u_k\mathfrak{L}$ is $f(k)\, d_{\pi(k)}$. On the other hand, $u_k\mathfrak{L} = \mathfrak{L}u_{\pi(k)}$ is the direct sum of the left submodules $\mathfrak{L}e_{\pi(k),j}$ ($1 \leqq j \leqq f(\pi(k))$); these modules are isomorphic (since the $Ae_{\pi(k),j}$ are); hence none is 0, and each contains therefore a simple left A-module isomorphic to $\bar{A}\bar{e}_{k1}$; computing the dimensions of these modules, we obtain the inequality

$$f(\pi(k))\, d_k \leqq f(k)\, d_{\pi(k)},$$

or equivalently

$$d_k/f(k) \leqq d_{\pi(k)}/f(\pi(k)).$$

As π is a permutation, we must have equality. This implies that each $\mathfrak{L}e_{ki}$ is a minimal left ideal; however, it follows from a theorem of Nakayama

[11, p. 45, Theorem 1, where left and right are exchanged] that an algebra having that property is quasi-Frobeniusean.

I have not been able to decide whether, when A is not an algebra, condition (m_s) implies equality of the two socles, nor whether, when we assume both (m_s) and $\mathfrak{L} = \mathfrak{R}$, condition ($c_d$) follows. Examples of Ikeda [5] show that a ring may verify (c_d) without being a quasi-Frobenius ring.

5. A criterion for Frobenius rings

Following Nakayama [9, p. 8], we define for a simple (left or right) A-module M the *colength* of M as the dimension of M considered as a vector space over its centralizer (which is a sfield); then, for any A-module E of finite length, we define the *colength* $\lambda'(E)$ as the sum of the colengths of the quotients in a Jordan-Hölder sequence of E; it is then clear that for every submodule M of E, $\lambda'(E) = \lambda'(E/M) + \lambda'(M)$, and that $\lambda'(M) = \lambda'(E)$ if and only if $M = E$. With the help of these formal properties, similar to those of the length, we can prove the following counterpart of criterion (3.4):

(5.1) *Consider the three conditions:*
(m_s') *For any simple left A-module M, $\lambda'(M^*) \leqq \lambda'(M)$.*
(m_d') *For any simple right A-module M, $\lambda'(M^*) \leqq \lambda'(M)$.*
(e') *The modules A_s and A_d have the same colength.*
Then any two of these conditions imply the third, and imply that A is a Frobenius ring and that $\lambda'(E^*) = \lambda'(E)$ for any A-module of finite length.

The first part of the proof consists in replacing λ by λ' in the argument of (3.4), up to the point where condition (C) is proved for any left A-module of finite length. In particular (c_s) holds, and then it follows from results of Ikeda [5][2] that $\mathfrak{L} = \mathfrak{R}$; the dual of the minimal left A-module $\bar{A}\bar{e}_{ki}$ is then identified with $e_{ki}\mathfrak{R}$, and hence is semisimple and sum of simple A-modules isomorphic to $\bar{e}_{\pi(k),1}\bar{A}$; but the colength of $\bar{A}\bar{e}_{k1}$, equal to that of $\bar{e}_{k1}\bar{A}$, is $f(k)$; as the relation $\lambda'(E^*) = \lambda'(E)$ is valid for any left A-module of finite length, we have $f(\pi(k)) \leqq f(k)$ by applying that relation to $\bar{A}\bar{e}_{ki}$. As π is a permutation, this again implies that $f(\pi(k)) = f(k)$ and that each $e_{ki}\mathfrak{R}$ is a minimal right ideal; in other words (m_s) holds; as (m_d) also holds, A is a quasi-Frobenius ring, and the relation $f(\pi(k)) = f(k)$ shows that it is a Frobenius ring. As (m_d') is then satisfied, the relation $\lambda'(E^*) = \lambda'(E)$ holds for any right A-module of finite length by the first part of the proof.

Similarly,

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[2] Actually Ikeda states the result without proof, but it is very easy to supply a simple proof: if $\mathfrak{r}$ is a minimal right ideal, $\mathfrak{r}^0$ must be a maximal left ideal: otherwise, it would be strictly contained in a maximal left ideal $\mathfrak{m}$, and by (3.2) and (4.4), $\mathfrak{m}^0$ is a minimal right ideal contained in $\mathfrak{r}^{00} = \mathfrak{r}$, hence equal to $\mathfrak{r}$; as $\mathfrak{m}^{00} = \mathfrak{m}$, we reach a contradiction. Now any maximal left ideal in A contains one of the $\mathfrak{b}_k$, hence $\mathfrak{r} = \mathfrak{r}^{00}$ is contained in one of the feet $u_k\mathfrak{L}$ of $\mathfrak{L}$, i.e. $\mathfrak{R} \subset \mathfrak{L}$.

(5.2) *Let A be an algebra of finite rank over a field K. Consider the two conditions:*

(m_s'') *For any simple left A-module M,* $\dim_K (M^*) \leqq \dim_K (M)$.

(m_d'') *For any simple right A-module M,* $\dim_K (M^*) \leqq \dim_K (M)$.

These two conditions are equivalent, and they imply that A is a Frobenius algebra and that $\dim_K (E^*) = \dim_K E$ *for any A-module of finite length.*

We suppress the proof, which is even simpler than that of (5.1), owing to (4.1).

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