A PROPERTY OF BROWNIAN MOTION PATHS[1]

BY

H. F. TROTTER

1. Introduction

Let $x(t, \omega)$ be a separable one-dimensional Brownian motion process with $x(0, \omega) = 0$. We suppose that, if necessary, a set of sample points of measure zero has been discarded so that all the sample functions of the process are continuous. Then for every t and ω a measure $\mu(\cdot\,, t, \omega)$ is defined by taking $\mu(A, t, \omega)$ to be the Lebesgue measure of the set $\{\tau : 0 \leqq \tau \leqq t, x(\tau, \omega) \in A\}$. In this paper we prove the following result.

THEOREM. *With probability one, $\mu(\cdot\,, t, \omega)$ has a continuous density function. That is, for almost all ω, there exists a function $f(x, t, \omega)$ which is continuous in x and t such that*

$$\mu(A, t, \omega) = \int_A f(x, t, \omega)\, dx$$

for every Borel set A.

The proof occupies Sections 2 and 3. Explicit bounds for the moduli of continuity of f are given by (2.1) and (2.3).

This theorem represents a partial extension of results of P. Lévy connected with the notion of "mesure du voisinage" (of the set of zeros of a Brownian motion sample function) introduced and investigated by him [5, 6]. Let $F(x, \omega) = \mu([-\infty, x], t, \omega)$. Then one of Lévy's theorems may be paraphrased as follows: *For any fixed ξ, $F'(\xi)$ exists with probability one, and is equal to $(\pi/2)^{1/2}$ times the "measure du voisinage" of the set*

$$A_\xi = \{\tau : \tau \leqq t, x(\tau, \omega) = \xi\}.$$

Our result is stronger on the one hand, in that we show that with probability one, $F'(\xi)$ exists for all ξ simultaneously and is continuous. On the other hand, we do not show any connection between $F'(\xi)$ and the set A_ξ.

We have stated the theorem for the case of Brownian motion, but it can be extended to a very general class of one-dimensional diffusion processes. Let $x(t, \omega)$ be a process such that the infinitesimal generator of the associated semigroup [1] has the form [2]

$$\Omega = \frac{d}{dm}\frac{d}{dx},$$

where m is an arbitrary strictly increasing function. Then $\mu(\cdot\,, t, \omega)$ can be defined as before, and the conclusion is that for almost all ω there is a con-

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tinuous $f(x, t, \omega)$ such that

$$\mu(A, t, \omega) = \int_A f(x, t, \omega)\, dm(x).$$

This generalization is an immediate consequence of results of Itô and McKean (not yet published), which show that all such diffusion processes can be derived from Brownian motion by suitable modification of the time-scale.

I am indebted to H. P. McKean for very helpful discussion, and particularly for pointing out a serious error in my original version of Lemma 1.

2. Proof of the theorem

For any $r > 0$, define $f_r(x, t, \omega)$ to be the step-function having the value $r^{-1}\mu([ir, (i + 1)r), t, \omega)$ on the interval $[ir, (i + 1)r)$ for every integer i. Thus f_r is an approximate density function for μ. The function f_r will have discontinuities at the integral multiples of r; define $\delta_r(t, \omega)$ to be the maximum of these discontinuities, so that $| f_r(x, t, \omega) - f_r(x', t, \omega) | \leqq \delta_r(t, \omega)$ for all x, x' with $| x - x' | \leqq r$.

LEMMA 1. *For any T, there exists a constant K such that if r is sufficiently small and $r^{-1/2}d$ is bounded from zero,*

$$\Pr \{\delta_r(t, \omega) \geqq d\} \leqq Kr^{-1} \exp(-r^{-1/2}d)$$

for all $t \leqq T$.

The proof of this lemma is rather lengthy and will be given in Section 3.

Let T be fixed, and define A_n to be the set of all integral multiples of 2^{-2n} between 0 and $T + 1$. Take $r_n = 2^{-n}$ and $d_n = 3n2^{-n/2}$. Define

$$G_n = \bigcup_{i=n}^{\infty} \bigcup_{t \in A_i} \{\omega : \delta_{r_i}(t, \omega) \geqq d_i\}.$$

Provided that n is so large that r_n is sufficiently small for Lemma 1 to apply,

$$\begin{aligned}\Pr \{G_n\} &\leqq \textstyle\sum_{i=n}^{\infty} (T + 1)2^{2i} \Pr \{\omega : \delta_{r_i}(t, \omega) \geqq d_i\} \\ &\leqq K(T + 1)\textstyle\sum_{i=n}^{\infty} 2^{3i}e^{-3i} \\ &= K(T + 1)\textstyle\sum_{i=n}^{\infty} (2/e)^{3i},\end{aligned}$$

and since this last is a convergent series, it follows that

$$\lim_{n \to \infty} \Pr \{G_n\} = 0.$$

We shall now write $f_n(x, t, \omega)$ for $f_{r_n}(x, t, \omega)$. For every x and n, $f_n(x, t, \omega)$ is obviously a monotone function of t. Furthermore,

$$| f_n(x, t', \omega) - f_n(x, t, \omega) | \leqq r_n^{-1} | t' - t |,$$

because of the step-function nature of f_n and the fact that

$$\int | f_n(x, t', \omega) - f_n(x, t, \omega) | \, dx = | t' - t | .$$

Hence if t' and t'' are two adjacent members of A_n and $t' \leqq t \leqq t''$,

$$f_n(x, t, \omega) - f_n(x, t', \omega) \leqq r_n^{-1}(t'' - t') = 2^{-n}.$$

Consequently, unless $\omega \in G_n$,

$$\delta_{r_n}(t, \omega) \leqq \delta_{r_n}(t', \omega) + 2 \cdot 2^{-n} \leqq d_n + 2^{-n+1}$$

for every t between 0 and T, since every such t lies between two adjacent members of A_n.

Now observe that f_n can be obtained from f_{n+1} by averaging over pairs of adjacent "steps", so that

$$\max_x | f_{n+1}(x, t, \omega) - f_n(x, t, \omega) | \leqq \tfrac{1}{2}\delta_{r_{n+1}}(t, \omega).$$

Then unless $\omega \in G_n$, for any m, p with $m \geqq n$,

$$\begin{aligned} | f_{m+p}(x, t, \omega) - f_m(x, t, \omega) | &\leqq \sum_{i=m}^{m+p-1} | f_{i+1}(x, t, \omega) - f_i(x, t, \omega) | \\ &< \sum_{i=m+1}^{\infty} d_i + 2^{-i+1} \\ &\leqq Km2^{-m/2}. \end{aligned}$$

(Here, and subsequently, we use K to denote a sufficiently large positive constant, depending in general on ω and T, whose particular value is immaterial.) Hence, if $\omega \notin G_n$, *the sequence* $\{f_n(x, t, \omega)\}$ *converges uniformly to a limit, which we shall call* $f(x, t, \omega)$. *Furthermore*

$$| f(x, t, \omega) - f_m(x, t, \omega) | \leqq Km2^{-m/2} \leqq Kr_m^{1/2} | \log r_m |$$

for every $m \geqq n$. It is clear that the measures having $f_n(x, t, \omega)$ as density functions converge at least weakly to $\mu(\cdot, t, \omega)$, and consequently $f(x, t, \omega)$ *is a density function for* $\mu(\cdot, t, \omega)$. The continuity of f remains to be established.

Take an arbitrary pair of numbers x and x', and define m by the requirement

$$\tfrac{1}{2} | x - x' | < r_m \leqq | x - x' | .$$

Then if $\omega \notin G_n$ and $| x - x' |$ is so small that $m \geqq n$,

$$\begin{aligned} | f(x, t, \omega) - f(x', t, \omega) | &\leqq | f(x, t, \omega) - f_m(x, t, \omega) | \\ &\quad + | f_m(x, t, \omega) - f_m(x', t, \omega) | \\ &\quad + | f_m(x', t, \omega) - f(x', t, \omega) | \\ &\leqq Kr_m^{1/2} | \log r_m | \\ &\leqq K | x - x' |^{1/2} | \log | x - x' || . \end{aligned} \tag{2.1}$$

This assertion involves the assumption that $| x - x' | \leqq r_n$, but it is now clear that for each $\omega \notin G_n$, a K may be found sufficiently large that (2.1) holds for all x, x'.

Turning now to the modulus of continuity with respect to t, we have, for $\omega \notin G_n$ and any $m \geqq n$,

$$
\begin{aligned}
|f(x, t, \omega) - f(x, t', \omega)| &\leqq |f(x, t, \omega) - f_m(x, t, \omega)| \\
&\quad + |f_m(x, t, \omega) - f_m(x, t', \omega)| \\
&\quad + |f_m(x, t', \omega) - f(x, t', \omega)| \\
&\leqq K r_m^{1/2} |\log r_m| + K r_m^{-1} |t - t'|.
\end{aligned} \tag{2.2}
$$

This expression is approximately minimized by choosing m so that r_m is as near $|t - t'|^{2/3} |\log |t - t'||^{-2/3}$ as possible. This m will be greater than n if $|t - t'|$ is sufficiently small. Substituting in (2.2) gives

$$
|f(x, t, \omega) - f(x, t', \omega)| \leqq K |t - t'|^{1/3} |\log |t - t'||^{2/3}. \tag{2.3}
$$

As before, by choosing K sufficiently large (2.3) can be made to hold for all t, t' between 0 and T without the restriction that $|t - t'|$ be small. Note that if f satisfies both (2.1) and (2.3), it is continuous in x and t jointly.

The results (2.1) and (2.3) hold for all $\omega \notin \bigcap_n G_n$, that is, for almost all ω, since the sets G_n form a monotone sequence and $\lim_{n\to\infty} \Pr\{G_n\} = 0$. Hence $f(x, t, \omega)$ is continuous with probability one, for $t \leqq T$. Since the set of sample points for which $f(x, t, \omega)$ is defined and continuous for all t is the (countable) intersection of the sets on which $f(x, t, \omega)$ is defined and continuous for $t \leqq n$, the proof of the theorem is complete, except for Lemma 1.

3. Proof of Lemma 1

Throughout this section we shall write $x_\xi(t, \omega)$ for a standard Brownian motion which starts with $x = \xi$ at $t = 0$. If ξ is a multiple of r, the discontinuity of $f_r(x, t, \omega)$ at ξ will have the same probability distribution as r^{-1} times the absolute value of

$$
R(\xi, t, \omega) = \int_0^t V(x_\xi(\tau, \omega))\, d\tau, \tag{3.1}
$$

where

$$
\begin{aligned}
V(x) &= 1 \quad \text{on} \quad (-r, 0) \\
&= -1 \quad \text{on} \quad (0, r) \\
&= 0 \quad \text{elsewhere.}
\end{aligned}
$$

We shall investigate the distribution of R by examining its Laplace transform

$$
m(\xi, u, t) = E\{\exp(-uR(\xi, t, \omega))\}. \tag{3.2}
$$

To avoid interruption of the main argument at a later stage we prove two preliminary lemmas.

LEMMA 2. *For $u > 0$ and $\xi \geqq 0$, $m(\xi, u, t)$ is an increasing function of t.*

Proof. There is no loss of generality in taking $u = 1$. Since

$$\exp(-R(\xi, t, \omega)) = 1 + \int_0^t - V(x_\xi(\tau, \omega)) \exp(-R(\xi, \tau, \omega))\, d\tau,$$

$$m(\xi, 1, t) = 1 + \int_0^t E\{-V(x_\xi(\tau, \omega)) \exp(-R(\xi, \tau, \omega))\, d\tau,$$

by taking expectations of both sides and using Fubini's theorem. Hence it will be sufficient to prove $E\{-V(x_\xi(t, \omega)) \exp(-R(\xi, t, \omega))\} \geqq 0$.

If $x_\xi(\tau, \omega) = 0$ for some $\tau \leqq t$, define $t^*(\omega)$ to be the greatest such τ not exceeding t, and otherwise define $t^*(\omega) = t$. Let

$$R_1(\omega) = \int_0^{t^*} V(x_\xi(\tau, \omega))\, d\tau \quad \text{and} \quad R_2(\omega) = \int_{t^*}^{t} V(x_\xi(\tau, \omega))\, d\tau,$$

so that $R(\xi, t, \omega) = R_1 + R_2$.

From considerations of symmetry, the conditional distribution of $| R_2 |$ given $t^* = T$ and $x(t) = X$ depends only on T and $| X |$, if $T < t$. Also, R_2 has the same sign as $-x(t)$, so that

$$E\{-V(x_\xi(t, \omega)) \exp(-R_2(\omega)) \mid x(t) = X, t^* = T\}$$
$$+ E\{-V(x_\xi(t, \omega)) \exp(-R_2(\omega)) \mid x(t) = -X, t^* = T\} \geqq 0.$$

Since the conditional distribution of $x_\xi(t, \omega)$ given $t^* = T$ is symmetric when $T < t$, it follows that

$$E\{-V(x_\xi(t, \omega)) \exp(-R_2(\omega)) \mid t^* = T\} \geqq 0 \tag{3.3}$$

when $T < t$. On the other hand, $t^* = t$ implies that the sample function does not reach the origin before time t, and since $\xi \geqq 0$, $-V(x(t))$ must then be nonnegative, and $E\{-V(x(t)) \exp(-R(\omega)) \mid t^* = t\} \geqq 0$. When conditioned on $t^* = T < t$, R_1 is independent of $x(t)$ and R_2, so that

$$E\{-V(x_\xi(t, \omega)) \exp(-R(\xi, t, \omega)) \mid t^* = T\}$$
$$= E\{-V(x_\xi(t, \omega)) \exp(-R_2(\omega)) \mid t^* = T\} \cdot E\{\exp(-R_1(\omega)) \mid t^* = T\} \geqq 0,$$

and integrating over the distribution of t^* gives the desired result.

LEMMA 3. *For $u > 0$ and $\xi \geqq 0$,*

$$E\{\sinh(-uR(\xi, t, \omega))\} \geqq 0.$$

Proof. As in Lemma 2, there is no loss of generality in taking $u = 1$. Define $t^*(\omega)$ to be the least value of τ for which $x_\xi(\tau, \omega) = 0$, or t, whichever is smaller. Define R_1 and R_2 as in Lemma 2, using this new value of t^*. When conditioned by t^*, R_1 and R_2 are independent, and consequently

$$E\{\sinh(-R(\xi, t, \omega)) \mid t^* = T\}$$
$$= -E\{\sinh R_1(\omega) \cosh R_2(\omega) \mid t^* = T\}$$
$$-E\{\cosh R_1(\omega) \mid t^* = T\} \cdot E\{\sinh R_2(\omega) \mid t^* = T\}.$$

By symmetry, $E\{\sinh R_2(\omega) \mid t^* = T\} = 0$, and since $\xi \geqq 0$,

$$\sinh R_1(\omega) \cosh R_2(\omega) \leqq 0$$

for all ω. Integrating over the distribution of t^* completes the proof.

We now consider the Laplace transform

$$F(\xi, u, s) = \int_0^\infty e^{-st} m(\xi, u, t)\, dt, \tag{3.4}$$

which we can calculate explicitly, using a method due to Kac [3, 4]. Let $\bar{V}(x) = V(x) + 1$, and define $\bar{F}(\xi, u, s)$ by (3.1), (3.2), and (3.4), using $\bar{V}$ in place of V. The function $\bar{V}$ is nonnegative, and we may apply the results of [3]. It follows that for $s > 0$,

$$\bar{F}(0, u, s) = \int_{-\infty}^\infty \psi(x)\, dx,$$

where $\psi(x)$ is the unique[2] function such that

$$-\tfrac{1}{2}\psi'' + \{s + u\bar{V}(x)\}\psi = 0 \tag{3.5}$$

except at the origin and the discontinuities of $\bar{V}$, and

(i) $\psi \to 0$ at infinity,
(ii) ψ' is continuous except at the origin,
(iii) $\psi'(-0) - \psi'(+0) = 2$.

Thus ψ is the Green's function with singularity at $x = 0$ for the equation (3.5). Writing $\psi(x, \xi)$ for the Green's function with singularity at $x = \xi$, we have

$$\bar{F}(\xi, u, s) = \int_{-\infty}^\infty \psi(x, \xi)\, dx.$$

This, however, says that $\bar{F}$ is the result of transforming the constant function 1 by the kernel $\psi(x, \xi)$, and hence is the solution of the inhomogeneous equation

$$-\tfrac{1}{2}\bar{F}'' + \{s + u\bar{V}(x)\}\bar{F} = 1$$

[2] I am indebted to the referee for pointing out that the uniqueness proof indicated in [3] tacitly assumes that V is continuous and (3.5) holds everywhere except at the origin. The gap is easily filled. Formula (4.7) of [3] implies that $\psi'(x)$ is *continuous* (for $x \neq 0$) and not merely bounded. With this additional condition the integration by parts called for in the uniqueness proof is legitimate, and the proof is valid. Note that there is a misprint in the footnote on page 9 of [3]; in the next to the last line "by φ'" should read "by φ".

which has a continuous first derivative and is bounded at infinity. Since $u\bar{V}(x) = uV(x) + u$, $\bar{F}(\xi, u, s) = F(\xi, u, s + u)$. Replacing $s + u$ by s in the equation above gives $F(\xi, u, s)$ as the solution of

$$-\tfrac{1}{2}F'' + \{s + uV(x)\}F = 1 \tag{3.6}$$

which has continuous first derivative and is bounded at infinity.

The function $s + uV(x)$ is piecewise constant, and the solution of (3.6) is made up piecewise of linear combinations of exponentials and constants. These combinations must be matched at $-r$, 0, and r so that F and F' are continuous.

It will be convenient to describe F in terms of the functions

$$\begin{aligned} C(s, u, x) &= \cosh (2s + 2u)^{1/2}x \\ S(s, u, x) &= \sinh (2s + 2u)^{1/2}x/x(2s + 2u)^{1/2} \\ Q(s, u, x) &= (\cosh (2s + 2u)^{1/2}x - 1)/x^2(s + u) \end{aligned}$$

and $\bar{C}(s, u, x) = C(s, -u, x)$, $\bar{S}(s, u, x) = S(s, -u, x)$, and $\bar{Q}(s, u, x) = Q(s, -u, x)$. Note that since the hyperbolic cosine is even and the hyperbolic sine odd, these functions are even functions of x and single-valued analytic functions of all three arguments, in spite of the appearance of square roots in the definitions. Of course, C, S, and Q are defined to have the value 1 when $(s + u)x = 0$.

Now letting $F(0) = q$ and $F'(0) = p$ we have

$$F(x) = qC(x) + pxS(x) - x^2Q(x) \qquad \text{on} \quad (-r, 0), \tag{3.7}$$

and

$$F(x) = q\bar{C}(x) + px\bar{S}(x) - x^2\bar{Q}(x) \qquad \text{on} \quad (0, \mathrm{r}), \tag{3.8}$$

as may be checked by substitution in (3.6). The condition that F be bounded at infinity implies that

$$\begin{aligned} F(x) &= s^{-1} + c \exp (-(2s)^{1/2}x), & x &\geqq r \\ &= s^{-1} + c' \exp ((2s)^{1/2}x), & x &\leqq -r \end{aligned}$$

for some constants c, c'. Hence

$$F'(r) = -(2s)^{1/2}\{F(r) - s^{-1}\},$$

and there is a similar matching condition at $-r$. Applying these conditions to (3.7) and (3.8) yields two equations for p and q, which after collecting coefficients become

$$\begin{aligned} (\bar{C} + (2s)^{1/2}r\bar{S})p + ((2s)^{1/2}\bar{C} + 2(s - u)r\bar{S})q &= (2/s)^{1/2} + 2r\bar{S} + (2s)^{1/2}r^2\bar{Q}, \\ -(C + (2s)^{1/2}rS)p + ((2s)^{1/2}C + 2(s + u)rS)q &= (2/s)^{1/2} + 2rS + (2s)^{1/2}r^2Q, \end{aligned}$$

where S, C, etc., stand for $S(s, u, r)$, $C(s, u, r)$, etc.

The determinant of this system of equations is

$$D = 2(2s)^{1/2}(C\bar{C} + (2s)^{1/2}r(C\bar{S} + \bar{C}S) + 2sr^2S\bar{S}) - 2ur(C\bar{S} - \bar{C}S), \tag{3.9}$$

and the solution is given by

$$\begin{aligned} Dp = {} & 2u(2s)^{-1/2}\{2r(S + \bar{S}) + 4(2s)^{1/2}r^2S\bar{S} + 2sr^3(Q\bar{S} + \bar{Q}S)\} \\ & + 2(C - \bar{C}) + 2(2s)^{1/2}r(S - \bar{S} + C\bar{S} - \bar{C}S) \\ & + 2sr^2(C\bar{Q} - \bar{C}Q) + 2s(2s)^{1/2}r^3(S\bar{Q} - \bar{S}Q), \\ Dq = {} & 2s^{-1/2}(C + \bar{C}) + 2r(C\bar{S} + \bar{C}S + S + \bar{S}) \\ & + (2s)^{1/2}r^2(C\bar{Q} + \bar{C}Q + 4S\bar{S}) + 2sr^3(S\bar{Q} + \bar{S}Q). \end{aligned}$$

These formulas show that for each fixed x and u, the F defined by (3.6) is an analytic function of s in the right half-plane, except at the zeros of D. Since $m(\xi, u, t)$ is nonnegative and (3.4) holds for $s > u$, it follows from a well-known theorem on the Laplace transform [7, p. 58] that (3.4) remains true for s greater than or equal to the largest real zero of D.

We shall be interested in the asymptotic behaviour of F as $u \to \infty$ and $r \to 0$ while u^2r^3 remains bounded. Suppose $s \leqq u$. Then $(s \pm u)r^2 \to 0$ and $C = 1 + (s + u)r^2/2 + o(ur^2)$, $S = 1 + (s + u)r^2/6 + o(ur^2)$, etc. Using these approximations we get

$$D = 2(2s)^{1/2} - \tfrac{4}{3}\, u^2r^3 + o(u^2r^3).$$

We shall now fix $u^2r^3 = 1$. Then it is clear that if r is sufficiently small, D has no zeros for $u \geqq s \geqq 1$. On the other hand, since C, S, $\bar{C}$, and $\bar{S}$ are all positive for $s \geqq u$, formula (3.9) shows that $D > 0$ for $s \geqq u$. Hence, if we take F to be defined by (3.6), *the relation* (3.4) *is valid for* $s \geqq 1$, *provided* $u^2r^3 = 1$ *and* r *is sufficiently small.*

Now suppose that s remains fixed as $r \to 0$. Then we obtain

$$q = s^{-1}(1 + O(u^2r^3)) = O(1)$$

and $p = 2urs^{-1}(1 + O(u^2r^3)) = O(ur) = O(r^{-1/2})$. It follows that

$$\begin{aligned} F(0, u, s) - s^{-1} &= O(1), \\ F(r, u, s) - s^{-1} &= q\bar{C} + pr\bar{S} - r^2\bar{Q} = O(1), \\ F(x, u, s) - s^{-1} &= \exp(-(2s)^{1/2}x)\cdot O(1), \qquad x \geqq r. \end{aligned}$$

Now

$$\begin{aligned} F(x, u, s) - s^{-1} &= \int_0^\infty e^{-st}(m(x, u, t) - 1)\, dt \\ &\geqq \int_T^\infty e^{-st}(m(x, u, t) - 1)\, dt \\ &\geqq s^{-1}e^{-sT}(m(x, u, T) - 1), \end{aligned}$$

provided x is positive, since then m is monotone by Lemma 2. From this, by setting $s = 2^{1/2}$, we obtain

$$m(x, u, t) - 1 \leqq K \exp(-2x)$$

for $x \geqq 0$, $t \leqq T$, and $u = r^{-3/2}$.

Lemma 3 now gives

$$\begin{aligned} & E\{\cosh r^{-3/2}R(\xi, t, \omega) - 1\} \\ & \qquad = m(\xi, t, \omega) - 1 - E\{\sinh(-r^{-3/2}R(\xi, t, \omega))\} \\ & \qquad \leqq m(\xi, t, \omega) - 1 \\ & \qquad \leqq K \exp(-2\,|\,\xi\,|\,). \end{aligned} \tag{3.10}$$

This has been proved under the assumption $\xi \geqq 0$, but since (3.10) is symmetric with respect to ξ, it holds in general.

Denote the discontinuity of $f(x, t, \omega)$ at $\xi = nr$ by Δ_n. Then

$$\begin{aligned} \Pr\{\Delta_n \geqq d\} &= \Pr\{r^{-1}\,|\,R(\xi, t, \omega)\,|\, \geqq d\} \\ &= \Pr\{r^{-3/2}\,|\,R(\xi, t, \omega)\,|\, \geqq r^{-1/2}d\} \\ &\leqq (\cosh r^{-1/2}d - 1)^{-1}E\{\cosh r^{-3/2}R(\xi, t, \omega)\} \\ &\leqq K \exp(-r^{-1/2}d) \exp(-r\,|\,n\,|\,), \end{aligned}$$

provided r is sufficiently small for (3.10) to be valid and $r^{-1/2}d$ is bounded from zero. Finally,

$$\begin{aligned} \Pr\{\delta_r(t, \omega) \geqq d\} &\leqq \sum_{n=-\infty}^{\infty} \Pr\{\Delta_n \geqq d\} \\ &\leqq Kr^{-1} \exp(-r^{-1/2}d). \end{aligned}$$

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Princeton University
Princeton, New Jersey