A CLASS OF TWO-POINT BOUNDARY PROBLEMS

BY
WILLIAM T. REID[1]

1. Introduction

For a two-point boundary problem, which in vector form may be written

$$
\begin{aligned}
(1.1)\qquad & y' = A(x)y + \lambda B(x)y, \qquad a \leqq x \leqq b,\\
& My(a) + Ny(b) = 0,
\end{aligned}
$$

Bliss [1] introduced the concept of "self-adjointness under a nonsingular transformation $z = T(x)y$," and considered in detail a special class of such problems, termed "definitely self-adjoint." This class of problems includes the so-called accessory boundary problem for a nonsingular simple integral problem of the calculus of variations involving no differential equations as restraints, but includes the accessory problem for a variational problem of Lagrange or Bolza type only in case very strong normality conditions hold. In 1938 Bliss [2] gave a new definition of definite self-adjointness with the involved normality assumption considerably weaker than in the original definition, so that the class of problems definitely self-adjoint according to this modified definition does include the accessory system for a Lagrange or Bolza type problem which is normal, but not necessarily normal on the interval of the minimizing arc.

In both the original and modified definitions of Bliss the definiteness property of the system is possessed by the matrix $S(x) = T^*(x)B(x)$. Subsequently Reid [7] considered a boundary problem (1.1) satisfying the conditions of Bliss [2] aside from the definiteness condition, and with this hypothesis replaced by a suitable condition of definiteness on the functional $\int_a^b y^*T^*(x)\,[y' - A(x)y]dx$; extension of the results to systems with complex coefficients was also discussed by Reid [7; Section 12]. At about the same time, E. Hölder [5] treated a real system (1.1) with $B(x)$ of constant rank on $a \leqq x \leqq b$, and satisfying the hypotheses of Bliss [2], with the exception of the normality condition; through the consideration of a related canonical system of twice the dimension of (1.1), Hölder reduced the determination of normal solutions for this problem to the solution of a pair of adjoint vector integral equations, and thus obtained results on the existence of normal proper values and associated expansions in terms of the normal proper solutions of (1.1).

The initial activity of the author in the direction of the present paper was an analysis of the various conditions satisfied by systems (1.1) which are self-

Received September 5, 1957.

[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

adjoint and definite in the sense of either Bliss [2] or Reid [7]. Results of this nature were presented to the American Mathematical Society on two occasions (see Reid [8, 9]), although in neither instance were the results submitted for publication. The present paper presents extensions and modernization of this earlier work of the author. Instead of dealing with a system (1.1) in which the differential equation is solved for the derivative vector function $y'(x)$, we shall consider a system of the more general form

$$\begin{aligned} A_1(x)y' + A_0(x)y &= \lambda B(x)y, \qquad a \leqq x \leqq b, \\ My(a) + Ny(b) &= 0, \end{aligned} \tag{1.2}$$

with $A_1(x)$ nonsingular on $ab: a \leqq x \leqq b$, since for systems (1.2) the results have certain niceties of character not present for the restricted case (1.1). In terms of a nonsingular transformation

$$z(x) = T(x)y(x), \qquad a \leqq x \leqq b, \tag{1.3}$$

the conditions for (1.2) corresponding to those of Bliss [2] are as follows:

(i) (1.2) is equivalent to its adjoint under (1.3);

(ii) $S(x) \equiv T^*(x)B(x)$ is hermitian on ab;

(iii) $S(x)$ is nonnegative definite on ab;

(iv) if $A_1 y' + A_0 y = 0$, $My(a) + Ny(b) = 0$, and $By \equiv 0$, then $y \equiv 0$ on ab.

For (1.2) the conditions corresponding to those of Reid [7] are the above conditions (i), (ii), (iv), while (iii) is replaced by the condition that

$$\int_a^b y^*T^* [A_1 y' + A_0 y]dx > 0$$

for arbitrary $y(x)$ satisfying $My(a) + Ny(b) = 0$, $By \not\equiv 0$, and for which there is an associated vector function $g(x)$ such that $A_1 y' + A_0 y = Bg$.

Section 2 is strictly prefatory to the subsequent study of the paper, and Section 3 is concerned with the basic theorems on the equivalence of two general systems of the form (1.2) under a nonsingular linear transformation. Section 4 is an analysis of condition (i) for problems (1.2), while Section 5 deals with systems (1.2) that satisfy (i) and (ii). In particular, the results of Section 5 culminate in Theorem 5.3, to the effect that whenever (1.2) satisfies condition (i) with a transformation (1.3), then there is a second transformation $z = T_1(x)y$ with which (1.2) satisfies both conditions (i) and (ii), the matrix $A_1^*T_1$ is skew-hermitian on ab, and the corresponding equivalent system $T_1^*[A_1 y' + A_0 y] = \lambda T_1^*By$, $My(a) + Ny(b) = 0$ is self-adjoint in the classical Lagrange sense; moreover, if (1.2) satisfies with (1.3) conditions (i) and (ii), then there is a real constant k_1 such that $T_1^*B \equiv k_1 T^*B$ on ab.

Section 6 is devoted to an analysis of the normality condition (iv) for problems (1.2). The principal result of this section is Theorem 6.2, which shows that if (1.2) satisfies (i) with the transformation (1.3), but condition (iv) does not hold, then for any associated transformation $z = T_1(x)$ satisfying the conditions of Theorem 5.3 described above there is a second system of the form (1.2) which is "equivalent" to the original system, and such that the second system with the transformation $z = T_1(x)y$ satisfies conditions (i), (ii) and (iv).

Finally, for systems (1.2) satisfying conditions (i), (ii) with a transformation (1.3) there is discussed in Section 7 a condition of definiteness that includes as special cases the above condition (iii) and the modification of this condition used by Reid [7]. In particular, a problem that is definite in the sense of Section 7 and satisfies the normality condition (iv) is equivalent to an integral equation of the type considered by Zimmerberg [15]; also, the equivalent integral equation for such a problem is a special case of symmetrizable transformations considered by Reid [10] and Zaanen [12, 13, 14]. Known results for such definite and normal problems, together with the result of Theorem 6.2, provide immediate results for a problem (1.2) that is definite but not normal. In particular, this method of treating abnormal problems seems decidedly simpler than the procedure employed by Hölder [5].

Matrix notation is used throughout the present paper. The symbol E is used for the $n \times n$ identity matrix, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. Matrices of one column are termed vectors, and the inner product z^*y of two n-dimensional vectors y, z is denoted by (y, z). The symbol C' will signify the class of matrices of arbitrary dimensions with elements which are continuously differentiable functions of the real variable x on the considered interval ab. Moreover, if $y(x)$, $z(x)$ are n-dimensional vector functions such that $(y(x), z(x))$ is integrable on ab, we shall write $\langle y, z\rangle$ for the integral $\int_a^b (y(x), z(x))\, dx$.

2. Adjoint boundary problems

In the following it will be assumed that the elements of the $n \times n$ matrices $A_0(x)$, $A_1(x)$, $B(x)$ are complex-valued functions of the real variable x on a given compact interval $ab{:}a \leqq x \leqq b$, with $A_0(x)$, $B(x)$ continuous, $B(x) \not\equiv 0$, and $A_1(x)$ nonsingular and of class C' on this interval. The elements of the $n \times n$ matrices M, N are supposed to be complex-valued, with the $n \times 2n$ matrix $\| M\ N \|$ of rank n. For brevity, the considered two-point boundary problem (1.2) is written as

$$(2.1) \qquad L[y] = \lambda B(x)y, \qquad s[y] \equiv My(a) + Ny(b) = 0,$$

where $L[y]$ denotes the formal vector differential operator $A_1(x)y' + A_0(x)y$. If a complex number λ is such that (2.1) has nonidentically vanishing solutions $y(x)$, then λ is termed a *proper value* of (2.1), and any such $y(x)$ is a corresponding *proper solution*; the dimension $i(\lambda)$ of the linear space of all solutions $y(x)$ of (2.1) for a proper value λ is called the *index* of λ as a proper value.

If $L^\star[z]$ denotes the formal Lagrange adjoint differential operator

$$[-A_1^* z]' + A_0^* z = -A_1^* z' + [A_0^* - A_1^{*\prime}]z,$$

then

$$(L[y], z) - (y, L^\star[z]) \equiv (A_1 y, z)' \qquad \text{for} \quad y, z \in C', \tag{2.2}$$

and consequently a vector function $z(x) \in C'$ is such that

$$\langle L[y], z\rangle - \langle y, L^\star[z]\rangle = 0, \tag{2.3}$$

for all $y(x) \in C'$ satisfying $s[y] = 0$, if and only if $t[z] \equiv P^* z(a) + Q^* z(b) = 0$, where P, Q are $n \times n$ matrices with the $n \times 2n$ matrix $\| P^* \; Q^* \|$ of rank n and

$$MA_1^{-1}(a)P - NA_1^{-1}(b)Q = 0. \tag{2.4}$$

Moreover, if M_1, N_1, P_1, Q_1 are $n \times n$ matrices such that

$$\begin{Vmatrix} P_1 & P \\ Q_1 & Q \end{Vmatrix} \cdot \begin{Vmatrix} M & N \\ M_1 & N_1 \end{Vmatrix} = \begin{Vmatrix} -A_1(a) & 0 \\ 0 & A_1(b) \end{Vmatrix}, \tag{2.5}$$

then the linear forms $s[y]$, $t[z]$, $s_1[y] \equiv M_1 y(a) + N_1 y(b)$, and $t_1[z] \equiv P_1^* z(a) + Q_1^* z(b)$ satisfy the algebraic identity

$$(s[y], t_1[z]) + (s_1[y], t[z]) \equiv (A_1(b)y(b), z(b)) - (A_1(a)y(a), z(a)). \tag{2.6}$$

For a given boundary problem (2.1) the corresponding adjoint boundary problem is

$$L^\star[z] = \lambda B^*(x)z, \qquad t[z] \equiv P^* z(a) + Q^* z(b) = 0. \tag{2.7}$$

If the index of a proper value λ for (2.7) is denoted by $i^*(\lambda)$, then relations (2.2), (2.6) imply the following result.

LEMMA 2.1. *A constant λ_0 is a proper value for* (2.1) *if and only if $\bar\lambda_0$ is a proper value for* (2.7); *moreover, $i(\lambda_0) = i^*(\bar\lambda_0)$.*

For brevity, we shall write $L[y; \lambda]$ for $L[y] - \lambda B(x)y$, and $L^\star[z; \lambda]$ for $L^\star[z] - \lambda B^*(x)z$. If $\lambda = \lambda_0$ is not a proper value for (2.1), then for arbitrary continuous vector functions $g(x)$ the nonhomogeneous differential system

$$L[y; \lambda_0] = g(x), \qquad s[y] = 0, \tag{2.8}$$

has a unique solution given by

$$y(x) = \int_a^b G(x, t; \lambda_0) g(t)\, dt, \qquad a \leqq x \leqq b, \tag{2.9}$$

where $G(x, t; \lambda_0)$ is the Green's matrix for the incompatible homogeneous system

$$L[y; \lambda_0] = 0, \qquad s[y] = 0; \tag{2.10}$$

for an explicit determination of the Green's matrix the reader is referred to Bliss [1; Section 5]. The following lemma is a ready consequence of Lemma 2.1 and the fact that (2.3) holds for arbitrary $y(x), z(x) \in C'$ satisfying $s[y] = 0$, $t[z] = 0$.

Lemma 2.2. *If* $\lambda = \lambda_0$ *is not a proper value for* (2.1), *and* $G(x, t; \lambda_0)$ *is the Green's matrix for* (2.10), *then* $G^*(t, x; \lambda_0)$ *is the Green's matrix for* (2.7) *with* $\lambda = \bar{\lambda}_0$.

In view of the above definitions and discussion the following lemma is immediate.

Lemma 2.3. *If* $U(x)$, $V(x)$ *are* $n \times n$ *matrices which are nonsingular and of class* C' *on* ab, *while* c_0, c_1 *are constants with* $c_1 \neq 0$, *then for the boundary problem*

$$L_0[y; \lambda] \equiv U(x)L[Vy; c_0 + c_1\lambda] = 0, \qquad s_0[y] \equiv s[Vy] = 0, \tag{2.11}$$

the adjoint is given by

$$L_0^\star[z; \lambda] \equiv V^*(x)L^\star[U^*z; \bar{c}_0 + \bar{c}_1\lambda] = 0, \qquad t_0[z] \equiv t[U^*z] = 0. \tag{2.12}$$

A value $\lambda = \lambda_0$ *is a proper value for* (2.11) *of index* k *if and only if* $\lambda = c_0 + c_1\lambda_0$ *is a proper value for* (2.1) *of index* k; *moreover, if* $\lambda = \lambda_0$ *is not a proper value for* (2.11), *then the Green's matrix* $G_0(x, t; \lambda_0)$ *of* (2.11) *for* $\lambda = \lambda_0$ *is given by* $V^{-1}(x)G(x, t; c_0 + c_1\lambda_0)U^{-1}(t)$, *where* $G(x, t; \lambda)$ *denotes the Green's matrix for* (2.1).

3. Equivalent boundary problems

Consider a second boundary problem

$$\begin{aligned} L^0[u; \lambda] &\equiv A_1^0(x)u' + A_0^0(x)u - \lambda B^0(x)u = 0, \\ s^0[u] &\equiv M^0u(a) + N^0u(b) = 0, \end{aligned} \tag{3.1}$$

of the general form (2.1), with the coefficient matrices $A_0^0(x)$, $A_1^0(x)$, $B^0(x)$, M^0, N^0 satisfying the same conditions as prescribed above for the corresponding matrices of (2.1). Following the terminology of Bliss [1, 2], the system (2.1) is said to be equivalent to (3.1) under the transformation

$$u(x) = H(x)y(x), \qquad a \leqq x \leqq b, \tag{3.2}$$

provided $H(x)$ is a nonsingular matrix of class C' on ab such that for arbitrary values λ a vector function $y(x)$ satisfies the differential equation or boundary condition of (2.1) if and only if the corresponding $u(x)$ of (3.2) satisfies the respective differential equation or boundary condition of (3.1). If (2.1) is equivalent to (3.1) under the transformation (3.2), then clearly a constant λ is a proper value for (2.1) of index k if and only if λ is a proper value for (3.1) of the same index. Now the condition that $L[y; \lambda] = 0$ if and only if $L^0[Hy; \lambda] = 0$ is readily seen to be equivalent to the identity

$$L^0[Hy; \lambda] \equiv A_1^0(x)H(x)A_1^{-1}(x)L[y; \lambda], \qquad a \leqq x \leqq b, \tag{3.3}$$

for all values λ and arbitrary $y(x) \in C'$, while the condition that $s[y] = 0$ if and only if $s^0[Hy] = 0$ is equivalent to the existence of a nonsingular constant matrix K such that

$$M^0H(a) = KM, \qquad N^0H(b) = KN. \tag{3.4}$$

In view of these remarks, and Theorem 4.1 of Reid [6], we have the following result.

THEOREM 3.1. *The boundary problem* (2.1) *is equivalent to* (3.1) *under the transformation* (3.2) *if and only if* $H(x)$ *is a nonsingular matrix of class* C' *satisfying*

$$A_1^0H' + A_0^0H - A_1^0HA_1^{-1}A_0 = 0, \qquad B^0H - A_1^0HA_1^{-1}B = 0, \qquad a \leqq x \leqq b, \tag{3.5}$$

$$M^0H(a)A_1^{-1}(a)P - N^0H(b)A_1^{-1}(b)Q = 0, \tag{3.6}$$

where P, Q *are* $n \times n$ *matrices with the* $n \times 2n$ *matrix* $\| P^* \; Q^* \|$ *of rank* n *and satisfying* (2.4). *Moreover, the general solution* $H(x)$ *of the matrix differential equation of* (3.5) *is* $H(x) = U(x)CY^{-1}(x)$, *where* $U(x)$, $Y(x)$ *are nonsingular solutions of the respective matrix differential equations*

$$L[Y; 0] \equiv A_1 Y' + A_0 Y = 0, \qquad L^0[U; 0] \equiv A_1^0U' + A_0^0U = 0,$$

and C *is an arbitrary* $n \times n$ *constant matrix; in particular, if* $H(x)$ *is a solution of this matrix differential equation that is nonsingular for some value* x_0 *on* ab, *then* $H(x)$ *is nonsingular throughout* ab.

The following theorem is an immediate consequence of the linearity in H of the conditions (3.5), (3.6), and the fact that (2.1) is equivalent to (3.1) under (3.2) if and only if (3.1) is equivalent to (2.1) under the transformation $y(x) = H^{-1}(x)u(x)$.

THEOREM 3.2. *If* (2.1) *is equivalent to* (3.1) *under each of the transformations* $u = H_\alpha(x)y$, $(\alpha = 1, 2)$, *and* c_1, c_2 *are constants such that*

$$H(x) = c_1 H_1(x) + c_2 H_2(x)$$

is nonsingular for some value x_0 *on* ab, *then* $H(x)$ *is nonsingular throughout* ab, *and* (2.1) *is equivalent to* (3.1) *under the transformation* $u = H(x)y$. *Moreover, if* (2.1) *is equivalent to* (3.1) *under each of the transformations* $u = H_\beta(x)y$, $(\beta = 1, 2, 3)$, *then* (2.1) *is equivalent to* (3.1) *under the transformation*

$$u = H_3(x)H_2^{-1}(x)H_1(x)y.$$

THEOREM 3.3. *If* (2.1) *is equivalent to* (3.1) *under the transformation* (3.2), *and* λ *is not a proper value of these differential systems, then the Green's matrices* $G(x, t; \lambda)$ *and* $G^0(x, t; \lambda)$ *of* (2.1) *and* (3.1), *respectively, satisfy the relation*

$$H(x)G(x, t; \lambda)A_1(t) \equiv G^0(x, t; \lambda)A_1^0(t)H(t), \qquad a \leqq x, t \leqq b, \quad x \neq t. \tag{3.7}$$

The relation (3.7) is a ready consequence of the identity (3.3) and the definitive solvability property of the Green's matrix.

4. Problems (2.1) which are equivalent to their adjoints

As a special case of Theorem 3.1 we have the following result on the equivalence of (2.1) to its adjoint (2.7) under a transformation

$$z(x) = T(x)y(x), \qquad a \leqq x \leqq b. \tag{4.1}$$

THEOREM 4.1. *The boundary problem* (2.1) *is equivalent to its adjoint* (2.7) *under the transformation* (4.1) *if and only if* $T(x)$ *is a nonsingular matrix of class* C' *satisfying*

$$(A_1^*T)' - A_1^*TA_1^{-1}A_0 - A_0^*T = 0, \qquad A_1^*TA_1^{-1}B + B^*T = 0, \qquad a \leqq x \leqq b, \tag{4.2}$$

$$P^*T(a)A_1^{-1}(a)P - Q^*T(b)A_1^{-1}(b)Q = 0, \tag{4.3}$$

where P, Q *are* $n \times n$ *matrices with the* $n \times 2n$ *matrix* $\| P^* \; Q^* \|$ *of rank* n *and satisfying* (2.4). *Moreover, the general solution of the matrix differential equation of* (4.2) *is* $T(x) = A_1^{*-1}(x)Y^{*-1}(x)CY^{-1}(x)$, *where* $Y(x)$ *is a nonsingular solution of* $L[Y; 0] = 0$.

Now if $T = T_0(x)$ is a nonsingular matrix of class C' satisfying (4.2), (4.3), it may be verified readily that $T = A_1^{*-1}(x)T_0^*(x)A_1(x)$ is also a nonsingular matrix of class C' satisfying (4.2), (4.3), so that by Theorem 3.2 we have the following result.

COROLLARY. *If* (2.1) *is equivalent to* (2.7) *under* (4.1), *then* (2.1) *is also equivalent to* (2.7) *under the transformation*

$$z(x) = T_1(x)y(x), \qquad a \leqq x \leqq b, \tag{4.4}$$

where $T_1(x)$ *is any matrix of the form*

$$T_1(x) = c_1 T(x) + c_2 A_1^{*-1}(x)T^*(x)A_1(x), \tag{4.5}$$

with c_1, c_2 *constants such that* $T_1(x)$ *is nonsingular for some* x_0 *on* ab.

It is to be remarked that (4.3) holds if and only if

$$MT^{-1}(a)A_1^{*-1}(a)M^* - NT^{-1}(b)A_1^{*-1}(b)N^* = 0, \tag{4.3$'$}$$

a form of the condition that has been used frequently for problems of the special form (1.1) (see, for example, Bliss [1,2], or Reid [6]).

As an immediate consequence of the definition of equivalence of a problem (2.1) and its adjoint, one has the following result.

THEOREM 4.2. *If* $U(x)$, $V(x)$ *are* $n \times n$ *nonsingular matrices of class* C' *on* ab, *while* c_0, c_1 *are real constants with* $c_1 \neq 0$, *then* (2.1) *is equivalent to its adjoint* (2.7) *under the transformation* (4.1) *if and only if* (2.11) *is equivalent to its adjoint* (2.12) *under the transformation* $z = U^{*-1}(x)T(x)V(x)y$.

Now if (2.1) is equivalent to (3.1) under the transformation (3.2), it follows from the identity (3.3) that (3.1) is of the form (2.11) with

$$U = A_1^0 H A_1^{-1}, \qquad V = H^{-1}, \qquad y = u,$$

and hence the following result is a corollary to the above theorem.

COROLLARY. *If* (2.1) *is equivalent to* (3.1) *under the transformation* (3.2), *and* (2.1) *is equivalent to its adjoint under* (4.1), *then* (3.1) *is equivalent to its adjoint under the transformation*

$$v = A_1^{0*-1}(x)H^{*-1}(x)A_1^*(x)T(x)H^{-1}(x)u.$$

THEOREM 4.3. *If* (2.1) *is equivalent to its adjoint* (2.7) *under the transformation* (4.1), *then*

(i) λ_0 *is a proper value for* (2.1) *if and only if* λ_0 *is a proper value for* (2.7), *and* $\bar{\lambda}_0$ *is a proper value for* (2.1); *moreover*, $i(\lambda_0) = i^*(\lambda_0) = i(\bar{\lambda}_0)$;

(ii) *if* λ *is not a proper value for* (2.1), *then*

$$T(x)G(x, t; \lambda)A_1(t) + G^*(t, x; \bar{\lambda})A_1^*(t)T(t) \equiv 0, \qquad a \leqq x, t \leqq b, \quad x \neq t;$$

(iii) *if* $y_1(x)$, $y_2(x)$ *are proper solutions of* (2.1) *corresponding to respective proper values* λ_1, λ_2 *with* $\lambda_1 \neq \bar{\lambda}_2$, *then*

$$\langle T^*By_1, y_2\rangle = 0, \qquad \langle T^*L[y_1], y_2\rangle = 0.$$

Conclusion (i) follows from Lemma 2.1 and the definition of equivalence of (2.1) with (2.7), while (ii) is a direct consequence of Lemma 2.2 and Theorem 3.3. In order to establish (iii), it is to be noted that

$$\lambda_1(T^*By_1, y_2) = (T^*L[y_1], y_2) = (L[y_1], Ty_2);$$

as $s[y_1] = 0$, $t[Ty_2] = 0$, $L^\star[Ty_2] = \lambda_2 B^*Ty_2$, and (2.3) holds for arbitrary $y(x)$, $z(x) \in C'$ satisfying $s[y] = 0$, $t[z] = 0$, it then follows that

$$\begin{aligned}\lambda_1\langle T^*By_1, y_2\rangle = \langle L[y_1], Ty_2\rangle &= \langle y_1, L^\star[Ty_2]\rangle \\ &= \bar{\lambda}_2\langle y_1, B^*Ty_2\rangle = \bar{\lambda}_2\langle T^*By_1, y_2\rangle,\end{aligned}$$

and $\langle T^*By_1, y_2\rangle = 0$ whenever $\lambda_1 \neq \bar{\lambda}_2$. The final statement of (iii) then follows from the relation $\langle T^*L[y_1], y_2\rangle = \lambda_1\langle T^*By_1, y_2\rangle$.

5. Symmetrizable boundary problems

A boundary problem (2.1) is said to be *symmetrizable under a transformation* (4.1) if (2.1) is equivalent to its adjoint (2.7) under this transformation, and the associated matrix $S(x) = T^*(x)B(x)$ is hermitian on ab. If (2.1) is symmetrizable under (4.1), then it follows from Theorem 4.1 and its corollary that (2.1) is also symmetrizable under the transformation (4.4) with

$$T_1(x) = A_1^{*-1}(x)T^*(x)A_1(x),$$

and $T_1^*(x)B(x) = -T^*(x)B(x)$ on ab. Now whenever (2.1) is equivalent to its adjoint under (4.1), the second relation of (4.2) implies that

$$T^*B - B^*T \equiv (T^* + A_1^* TA_1^{-1})B.$$

As $T + A_1^{*-1} T^*A_1$ is of constant rank on ab in view of Theorem 4.1 and its corollary, and $B(x) \not\equiv 0$ on ab, if (2.1) is symmetrizable under (4.1) then $T + A_1^{*-1} T^*A_1$ is singular throughout ab, and the constant rank of this matrix does not exceed $n - q$, where q is the maximum rank of $B(x)$ on this interval. Moreover, the above relation implies that (2.1) is symmetrizable under (4.1) whenever (2.1) is equivalent to its adjoint (2.7) under this transformation and A_1^*T is skew-hermitian, (i.e., $A_1^*T = -(A_1^*T)^*$), on ab.

THEOREM 5.1. *If* (2.1) *is symmetrizable under a transformation* (4.1), *then*

(i) *for* λ *not a proper value of* (2.1) *the matrix*

$$K_1(x, t; \lambda) = S(x)G(x, t; \lambda)B(t)$$

is such that $K_1(x, t; \lambda) = K_1^*(t, x; \bar{\lambda})$ *for* $a \leqq x, t \leqq b, \quad x \neq t$;

(ii) *if* Λ *denotes the linear space of vector functions* $y(x) \in C'$ *satisfying* $s[y] = 0$, $L[y] = B(x)g(x)$ *with* $g(x)$ *continuous on* ab, *then for arbitrary real constants* c_1, c_2 *the functional* $L[y; c_1, c_2; T] \equiv T^*(x)(c_1 L[y] + c_2 B(x)y)$ *is hermitian on* Λ *in the sense that*

$$\langle L[y_1 ; c_1, c_2 ; T], y_2\rangle = \langle y_1, L[y_2 ; c_1, c_2 ; T]\rangle \qquad \text{for} \quad y_1, y_2 \in \Lambda. \tag{5.1}$$

Whenever (2.1) is symmetrizable under (4.1), it follows from the second relation of (4.2) and the hermitian character of $S = T^*B$ that

$$A_1^*TA_1^{-1}B = -B^*T = -S;$$

consequently for λ not a proper value of (2.1) it follows from (ii) of Theorem 4.3 that

$$\begin{aligned} K_1(x, t; \lambda) &= B^*(x)T(x)G(x, t; \lambda)B(t) \\ &= -B^*(x)G^*(t, x; \bar{\lambda})A_1^*(t)T(t)A_1^{-1}(t)B(t) \\ &= B^*(x)G^*(t, x; \bar{\lambda})S(t) = K_1^*(t, x; \bar{\lambda}) \end{aligned}$$

for $a \leqq x, t \leqq b, x \neq t$, thus establishing conclusion (i). Now whenever (2.1) is equivalent to (2.7) under (4.1), relation (3.3) implies that

$$L^\star[Ty] = -A_1^*TA_1^{-1}L[y]$$

for arbitrary $y(x) \in C'$. If (2.1) is symmetrizable under (4.1) and $y_2 \in \Lambda$ with $L[y_2] = Bg_2$, then from the second relation of (4.2) and the hermitian character of $S(x)$ we conclude that

$$L^\star[Ty_2] = -A_1^*TA_1^{-1}Bg_2 = B^*Tg_2 = T^*Bg_2 = T^*L[y_2].$$

Consequently, if $y_1, y_2 \in \Lambda$, then

$$\begin{aligned}\langle L[y_1; c_1, c_2; T], y_2\rangle &= c_1\langle L[y_1], Ty_2\rangle + c_2\langle Sy_1, y_2\rangle \\ &= c_1\langle y_1, L^\star[Ty_2]\rangle + c_2\langle y_1, Sy_2\rangle \\ &= c_1\langle y_1, T^*L[y_2]\rangle + c_2\langle y_1, Sy_2\rangle \\ &= \langle y_1, L[y_2; c_1, c_2; T]\rangle,\end{aligned}$$

thus establishing conclusion (ii).

An important instance of symmetrizable problems is the case of problems (2.1) that are *self-adjoint* in the classical Lagrange sense; that is,

$$L[y;\lambda] \equiv L^\star[y;\lambda]$$

for all complex values λ and arbitrary $y(x) \in C'$, while $s[y] = 0$ if and only if $t[y] = 0$. In terms of the coefficient matrices of (2.1) the conditions of self-adjointness are

$$A_1 = -A_1^*, \qquad B = B^*, \qquad A_1' = A_0 - A_0^*, \qquad a \leqq x \leqq b, \tag{5.2}$$

$$MA_1^{*-1}(a)M^* - NA_1^{*-1}(b)N^* = 0. \tag{5.3}$$

Whenever we refer to self-adjoint problems we shall mean self-adjointness in this classical sense. The following two theorems present the basic relations between the class of symmetrizable problems (2.1) and self-adjoint problems.

THEOREM 5.2. *If* (2.1) *is symmetrizable under a transformation* (4.1) *for which the matrix* $A_1^*(x)T(x)$ *is skew-hermitian on* ab, *then the boundary problem* $T^*(x)L[y;\lambda] = 0$, $s[y] = 0$ *is self-adjoint. Moreover, if there exist nonsingular matrices* $U(x)$, $V(x) \in C'$ *such that for some real constants* c_0, $c_1 \neq 0$ *the problem* (2.11) *is self-adjoint, then* $T(x) = U^*(x)V^{-1}(x)$ *is such that* $A_1^*(x)T(x)$ *is skew-hermitian on* ab, *and* (2.1) *is symmetrizable under the corresponding transformation* (4.1).

If (2.1) is symmetrizable under (4.1) with A_1^*T skew-hermitian on ab, then the relation $A_1^*T + T^*A_1 = 0$, together with (4.2) and (4.3′), imply conditions (5.2), (5.3) for the boundary problem $T^*(x)L[y;\lambda] = 0$, $s[y] = 0$; that is, with $U(x) = T^*(x)$, $V(x) = E$, $c_0 = 0$, $c_1 = 1$ the problem (2.11) is self-adjoint.

On the other hand, if $U(x)$, $V(x)$ are nonsingular matrices of class C' such that (2.11) with real constants c_0, $c_1 \neq 0$ is self-adjoint, the conditions (5.2), (5.3) for this problem (2.11) are equivalent to

$$UA_1V = -V^*A_1^*U^*, \qquad UBV = V^*B^*U^*, \tag{5.4}$$

$$(UA_1V)' = U(A_1V' + A_0V) - (V^{*\prime}A_1^* + V^*A_0^*)U^*, \tag{5.5}$$

$$MV(a)U^{*-1}(a)A_1^{*-1}(a)M^* - NV(b)U^{*-1}(b)A_1^{*-1}(b)N^* = 0. \tag{5.6}$$

Now the matrix $T(x) = U^*(x)V^{-1}(x)$ is nonsingular and of class C' on ab, and conditions (5.4) imply that A_1^*T is skew-hermitian and T^*B is hermitian on ab; in particular, the second relation of (4.2) holds for this $T(x)$. In turn, the differential equation (5.5) and the skew-hermitian character of A_1^*T imply that $T(x)$ satisfies the differential equation of (4.2). Finally, (5.6) is the condition (4.3′) for $T = U^*V^{-1}$, thus completing the proof that (2.1) is symmetrizable under (4.1) with $T(x) = U^*(x)V^{-1}(x)$.

THEOREM 5.3. *If* (2.1) *is equivalent to its adjoint* (2.7) *under the transformation* (4.1), *then there exists a nonsingular* $T_1(x)$ *of the form* (4.5) *with* $A_1^*T_1$ *skew-hermitian on* ab, *and such that* (2.1) *is symmetrizable under the corresponding transformation* (4.4), *and the boundary problem*

$$T_1^*(x)L[y;\lambda] = 0, \qquad s[y] = 0, \tag{5.7}$$

is self-adjoint; *moreover, if* (2.1) *is symmetrizable under* (4.1), *then for each such* $T_1(x)$ *there is a corresponding nonzero real constant* k_1 *satisfying* $T_1^*B \equiv k_1\, T^*B$ *on* ab.

If (2.1) is equivalent to (2.7) under (4.1), then the matrix $T_1(x)$ of (4.5) is such that $T_1(a)$ is nonsingular for

$$c_1 = (1 + i \tan \theta)/2, \qquad c_2 = (-1 + i \tan \theta)/2,$$

with $\theta \neq (2k + 1)\pi/2$, $(k = 0, \pm 1, \cdots)$, and $e^{i2\theta}$ not a proper value of the matrix $T^*(a)A_1(a)T^{-1}(a)A_1^{*-1}(a)$; by the Corollary to Theorem 4.1 we have that (2.1) is equivalent to (2.7) under the corresponding transformation (4.4). For this $T_1(x)$ the matrix $A_1^*\, T$ is equal to

$$\tfrac{1}{2} \sec \theta(e^{i\theta}\, A_1^*\, T - e^{-i\theta}\, T^*A_1),$$

and is therefore skew-hermitian on ab, so that by the comments of the first paragraph of this section the problem (2.1) is symmetrizable under the corresponding transformation (4.4); moreover, in view of Theorem 5.2 the corresponding problem (5.7) is self-adjoint.

Now if (2.1) is symmetrizable under (4.1), and $T_1(x)$ is any matrix of the form (4.5) with T_1 nonsingular and $A_1^*T_1$ skew-hermitian on ab, then T^*B and T_1^*B are hermitian on ab. Moreover, by the second relation of (4.2) we have

$$T_1^*B = \bar{c}_1\, T^*B + \bar{c}_2\, A_1^*TA_1^{-1}B = \bar{c}_1\, T^*B - \bar{c}_2\, B^*T = (\bar{c}_1 - \bar{c}_2)T^*B,$$

and since $B \not\equiv 0$ on ab, it follows that $\bar{c}_1 - \bar{c}_2$ is a nonzero real constant k_1.

In view of Theorem 5.3, for a system (2.1) that is equivalent to its adjoint under a transformation (4.1), the general results on the distribution and properties of proper values and solutions are the same as such results for a problem (2.1) that is self-adjoint. It is to be noted that without further hypotheses these results are essentially limited to those of Theorem 4.3. Indeed, for an arbitrary boundary problem (2.1) and its adjoint (2.7) let

$w(x) = (w_\beta(x))$, $(\beta = 1, \cdots, 2n)$, with

$$w_\alpha(x) = y_\alpha(x), \qquad w_{n+\alpha}(x) = z_\alpha(x), \qquad\qquad (\alpha = 1, \cdots, n),$$

and consider the corresponding boundary problem in $w(x)$ defined by

$$\begin{aligned} L^\star[z; \lambda] &= 0, \qquad t[z] = 0, \\ L[y; \lambda] &= 0, \qquad s[y] = 0. \end{aligned} \tag{5.8}$$

If for arbitrary $n \times n$ matrices M_1, M_2 we denote by $(M_1; M_2)$ the $2n \times 2n$ matrix

$$\begin{Vmatrix} 0 & M_1 \\ M_2 & 0 \end{Vmatrix},$$

then (5.8) is of the form

$$(-A_1^*; A_1)w' + (A_0^* - A_1^{*\prime}; A_0)w = \lambda(B^*; B)w,$$

$$(P^*; M)w(a) + (Q^*; N)w(b) = 0,$$

and it may be verified readily that (5.8) is self-adjoint. Clearly λ is a proper value for (5.8) if and only if λ is a proper value for either (2.1) or (2.7), and the index of λ as a proper value of (5.8) is equal to $i(\lambda) + i^*(\lambda)$.

It is to be remarked that if the coefficient matrices of (2.1) are all real-valued, and (2.1) is equivalent to its adjoint (2.7) under a transformation (4.1) with $T(x)$ real-valued on ab, then in general the matrix $T_1(x)$ of Theorem 5.3 may not be chosen real-valued. For typographical simplicity in the presentation of an example to illustrate this possibility, let E, J, K denote the 2×2 constant matrices

$$E = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \qquad K = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

and for arbitrary 2×2 matrices M_1, M_2, let $M_1 \dotplus M_2$ denote the direct sum 4×4 matrix

$$\begin{Vmatrix} M_1 & 0 \\ 0 & M_2 \end{Vmatrix}.$$

In order that the boundary problem

$$y' = \lambda(0 \dotplus J)y, \qquad (E \dotplus E)y(a) + (K \dotplus (-E))y(b) = 0 \tag{5.9}$$

be equivalent to its adjoint under a transformation (4.1), it follows from condition (4.2) that the 4×4 matrix $T(x)$ must be a constant matrix of the form $R_1 \dotplus R_2$ with R_2 of the form $e_1 E + e_2 J$, and condition (4.3′) implies that R_1 is of the form $e_3 E + e_4 K$. A corresponding $T_1(x)$ satisfying the conditions of Theorem 5.3 is then a nonsingular matrix of the form

$$T_1(x) = i(d_3 E + d_4 K) \dotplus (id_1 E + d_2 J)$$

with d_1, d_2, d_3, d_4 real constants, and clearly there is no such $T_1(x)$ that is real-valued. In particular, (5.9) is symmetrizable under (4.1) with

$$T(x) = E \dotplus J,$$

and for the determination of $T_1(x)$ as in the proof of Theorem 5.3 the choice $\theta = \pi/4$ leads to the corresponding skew-hermitian matrix $T_1(x) = iE \dotplus J$.

6. Normality and abnormality for problems (2.1)

For the problem (2.1) let $\Lambda_{0,0}$ denote the linear space of vector functions $y(x)$ which are solutions of $L[y] = 0$ and satisfy $s[y] = 0$, $B(x)y(x) \equiv 0$ on ab. The symbol $\Lambda_{0,0}^{\star}$ will signify the corresponding linear space for the adjoint problem (2.7), that is, $\Lambda_{0,0}^{\star}$ is the totality of vector functions $z(x)$ satisfying $L^{\star}[z] = 0$, $t[z] = 0$, $B^*(x)z(x) \equiv 0$ on ab. If $\Lambda_{0,0}$ is zero-dimensional, the problem (2.1) is said to be *normal*, or to have *abnormality of order zero*, whereas if dim $\Lambda_{0,0} = r > 0$, the problem (2.1) is said to be *abnormal*, with *order of abnormality* r, and a vector function $y(x)$ of $\Lambda_{0,0}$ with $y(x) \not\equiv 0$ on ab is termed an *abnormal solution* of (2.1). If dim $\Lambda_{0,0} = r > 0$, then clearly all complex numbers λ are proper values for (2.1) with $i(\lambda) \geqq r$; in case $i(\lambda) > r$ the integer $i_n(\lambda) = i(\lambda) - r$ is termed the *normal index* of λ as a proper value for (2.1), and a proper solution $y(x)$ with $B(x)y(x) \not\equiv 0$ is called a *normal proper solution.* Correspondingly, if dim $\Lambda_{0,0}^{\star} = r^* > 0$, a proper value λ of (2.7) with $i^*(\lambda) > r^*$ is said to have normal index $i_n^*(\lambda) = i^*(\lambda) - r^*$.

Let Λ_0 denote the linear space of vector functions $y(x)$ satisfying $L[y] = 0$, $B(x)y(x) \equiv 0$ on ab; similarly, $\Lambda_0^{\star}$ will signify the linear space of vector functions $z(x)$ satisfying $L^{\star}[z] = 0$, $B^*(x)z(x) \equiv 0$ on ab. As $\Lambda_0 \supset \Lambda_{0,0}$, if Λ_0 is zero-dimensional, then $\Delta_{0,0}$ is zero-dimensional also, and (2.1) is normal. Now (2.4) implies that a vector function $y(x)$ satisfies $s[y] = 0$ if and only if there is a constant vector ξ such that $y(a) = A_1^{-1}(a)P\xi$, $y(b) = -A_1^{-1}(b)Q\,\xi$. Consequently, if dim $\Lambda_0 = p > 0$, and η is an $n \times p$ matrix whose column vectors form a basis for Λ_0, then dim $\Lambda_{0,0} = r \geqq 0$ is equivalent to the condition that the $(n + p) \times 2n$ matrix

$$\begin{Vmatrix} P^* & Q^* \\ \eta^*(a)A_1^*(a) & -\eta^*(b)A_1^*(b) \end{Vmatrix}$$

is of rank $n + p - r$.

If Λ_1 denotes the linear space of vector functions $y(x)$ satisfying $L[y] = B(x)g(x)$ with some continuous vector function $g(x)$, then from (2.2) we have that $z^*(x)A_1(x)y(x)$ is constant on ab whenever $y(x) \in \Lambda_1$, $z(x) \in \Lambda_0^{\star}$, and consequently

$$(6.1)\qquad z^*(a)A_1(a)y(a) - z^*(b)A_1(b)y(b) = 0, \qquad \text{for} \quad y(x) \in \Lambda_1, \quad z(x) \in \Lambda_0^{\star}.$$

In particular, the boundary condition (6.1) holds for $y(x)$ any proper solution of (2.1).

It is to be emphasized that in general dim $\Lambda_{0,0} \neq$ dim $\Lambda^{\star}_{0,0}$, and dim $\Lambda_0 \neq$ dim $\Lambda^{\star}_0$. For example, for the particular system (2.1) given by

$$
\begin{aligned}
y_1' &= \lambda y_1, & y_1(0) + y_1(1) &= 0, \\
y_2' &= \lambda x y_1, & y_2(0) - y_2(1) &= 0,
\end{aligned}
\tag{6.2}
$$

we have dim $\Lambda_{0,0} =$ dim $\Lambda_0 = 1$, and a basis for $\Lambda_{0,0}$ is provided by the vector $\eta(x) \equiv (\eta_\alpha(x))$, $(\alpha = 1, 2)$, with $\eta_1(x) \equiv 0$, $\eta_2(x) \equiv 1$; indeed, for (6.2) each complex number λ is a proper value with $i(\lambda) = 1$, $i_n(\lambda) = 0$. On the other hand, the system adjoint to (6.2) is

$$
\begin{aligned}
-z_1' &= \lambda(z_1 + x z_2), & z_1(0) + z_1(1) &= 0, \\
-z_2' &= 0, & z_2(0) - z_2(1) &= 0,
\end{aligned}
$$

with dim $\Lambda^{\star}_0 = 0$ and $i^*(\lambda) = 1 = i^*_n(\lambda)$ for all complex λ.

If (2.1) is equivalent to its adjoint (2.7) under a transformation (4.1), however, then dim $\Lambda_0 =$ dim $\Lambda^{\star}_0$, and dim $\Lambda_{0,0} =$ dim $\Lambda^{\star}_{0,0}$; indeed, for such a problem it follows from the second relation of (4.2) that $y(x)$ belongs to Λ_0 or $\Lambda_{0,0}$ if and only if $z(x) = T(x)y(x)$ belongs to the respective $\Lambda^{\star}_0$ or $\Delta^{\star}_{0,0}$. In particular, if (2.1) is equivalent to (2.7) under (4.1), and dim $\Delta_{0,0} =$ dim $\Delta^{\star}_{0,0} = r > 0$, then for $\eta(x)$ an $n \times r$ matrix whose column vectors form a basis for $\Lambda_{0,0}$, the matrix $\zeta(x) = T(x)\eta(x)$ is such that its column vectors form a basis for $A^{\star}_{0,0}$, and the $(n + r) \times 2n$ matrix

$$
\begin{Vmatrix} M & N \\ \zeta^*(a)A_1(a) & -\zeta^*(b)A_1(b) \end{Vmatrix}
$$

is of rank n. Moreover, if σ is an $n \times r$ matrix such that

$$
\zeta^*(a)A_1(a) = \sigma^* M, \qquad -\zeta^*(b)A_1(b) = \sigma^* N,
$$

and τ is an $n \times (n - r)$ matrix of rank $n - r$ such that $\sigma^*\tau = 0$, then the boundary conditions $s[y] = 0$ are equivalent to

$$
\begin{aligned}
&\tau^* s[y] = 0, \\
&\sigma^* s[y] \equiv \zeta^*(a)A_1(a)y(a) - \zeta^*(b)A_1(b)y(b) = 0.
\end{aligned}
$$

Consequently, if θ, ϕ are $n \times r$ matrices such that the $r \times r$ matrix $\theta^*\eta(a) + \phi^*\eta(b)$ is nonsingular, the boundary problem

$$
\begin{aligned}
L[y; \lambda] &= 0, \\
\tau^* s[y] &= 0, \\
\theta^* y(a) + \phi^* y(b) &= 0,
\end{aligned}
\tag{6.3}
$$

is a normal problem. Moreover, since (6.1) implies that $\sigma^* s[y] = 0$ for any proper solution of (2.1), the problem (6.3) is "equivalent" to (2.1) in the sense that if $y(x)$ is a proper solution of (6.3) for a value λ, then $y(x)$ is a normal solution of (2.1) for this value λ, while if $y(x)$ is a solution of (2.1) there is a unique r-dimensional constant vector ρ such that $y(x) + \eta(x)\rho$ is a solution of (6.3). Clearly λ is a proper value for (6.3) of index k if and only if λ is a proper value for (2.1) with normal index $i_n(\lambda)$ equal to k.

The central result of this section is given in the following theorem.

THEOREM 6.1. *If* (2.1) *is an abnormal problem that is equivalent to its adjoint* (2.7) *under a transformation* (4.1) *for which* $A_1^*(x)T(x)$ *is skew-hermitian on* ab, *then there exists an equivalent normal problem* (6.3) *that is equivalent to its adjoint under the same transformation* (4.1).

As (2.1) is equivalent to (2.7) under (4.1), it follows from (4.3′) that the coefficient matrices of the adjoint boundary conditions may be chosen as $P = T^{*-1}(a)M^*$, $Q = T^{*-1}(b)N^*$. In the following argument it will be understood that P, Q are so chosen, and that M_1, N_1, P_1, Q_1 are $n \times n$ matrices satisfying (2.5); in particular, we have

$$-M_1 A_1^{-1}(a)T^{*-1}(a)M^* + N_1 A_1^{-1}(b)T^{*-1}(b)N^* = E. \tag{6.4}$$

As above, $\eta(x)$ will denote an $n \times r$ matrix whose column vectors form a basis for $\Lambda_{0,0}$, and σ will signify an $n \times r$ matrix such that

$$\eta(a) = T^{-1}(a)A_1^{*-1}(a)M^*\sigma, \qquad \eta(b) = -T^{-1}(b)A_1^{*-1}(b)N^*\sigma,$$

while τ is an $n \times (n - r)$ matrix of rank $n - r$ such that $\sigma^*\tau = 0$. For R the $n \times n$ matrix defined as

$$R = -\tfrac{1}{2}[M_1 T^{-1}(a)A_1^{*-1}(a)M_1^* - N_1 T^{-1}(b)A_1^{*-1}(b)N_1^*], \tag{6.5}$$

it will be shown that whenever $A_1^*(x)T(x)$ is skew-hermitian on ab, the boundary problem

$$\begin{aligned} L[y; \lambda] &= 0, \\ \tau^*s[y] &= 0, \\ \sigma^*(s_1[y] - Rs[y]) &= 0, \end{aligned} \tag{6.6}$$

is a normal problem equivalent to (2.1), and (6.6) is equivalent to its adjoint under the same transformation (4.1). Indeed, relation (6.4) and the skew-hermitian character of $T^{-1}(x)A_1^{*-1}(x)$ imply that the $r \times r$ matrix

$$\begin{aligned} \sigma^*(s_1[\eta] - Rs[\eta]) &= \sigma^*s_1[\eta] \\ &= \sigma^*(M_1 T^{-1}(a)A_1^{*-1}(a)M^* - N_1 T^{-1}(b)A_1^{*-1}(b)N^*)\sigma \end{aligned}$$

is the nonsingular matrix $\sigma^*\sigma$, thus establishing the normality of (6.6). Moreover, the skew-hermitian character of $A_1^*(x)T(x)$ implies that the constant matrix R of (6.5) is skew-hermitian, and by direct computation it may be verified that the coefficient matrices of the boundary conditions of (6.6) satisfy (4.3′) with the given $T(x)$, thus proving that (6.6) is equivalent to its adjoint under the same transformation (4.1).

The above proof of Theorem 6.1 is a direct generalization of the method that has been used in the case of the accessory minimum problem for abnormal problems of Bolza type to determine an equivalent normal problem of similar type (see Hestenes [4; Section 3], and Bliss [3; Section 81]).

Finally, as a corollary to the combined results of Theorems 5.3 and 6.1 we have the following theorem.

THEOREM 6.2. *If* (2.1) *is an abnormal problem which is equivalent to its adjoint under a transformation* (4.1), *then for an associated transformation* (4.4) *with matrix* $T_1(x)$ *satisfying the conditions of Theorem* 5.3 *there is an equivalent normal problem*

$$L[y; \lambda] = 0, \qquad s_0[y] = 0, \tag{6.7}$$

that is symmetrizable under (4.4), *and the system*

$$T_1^*(x)L[y; \lambda] = 0, \qquad s_0[y] = 0, \tag{6.8}$$

is self-adjoint; moreover, if the original problem (2.1) *is symmetrizable under* (4.1), *then for each such* $T_1(x)$ *there is a nonzero real constant* k_1 *such that*

$$T_1^*(x)B(x) \equiv k_1\, T^*(x)B(x)$$

on ab.

7. Definite boundary problems

If (2.1) is symmetrizable under a transformation (4.1), and Λ denotes the linear space of vector functions satisfying $s[y] = 0$, $L[y] = B(x)g(x)$ with $g(x)$ continuous on ab, then for arbitrary real constants c_1, c_2 the formal operator $L[y; c_1, c_2; T) \equiv T^*(x)(c_1 L[y] + c_2 B(x)y)$ is hermitian on Λ in the sense of (5.1); in particular,

$$I[y; c_1, c_2; T] = \langle L[y; c_1, c_2; T], y\rangle \tag{7.1}$$

is real-valued for $y \in \Lambda$. We shall say that the problem (2.1) is *definite* $[c_1, c_2; T]$ whenever (2.1) is symmetrizable under (4.1), and for the corresponding $T(x)$ and suitable real constants c_1, c_2 the functional (7.1) is positive for all $y(x) \in \Lambda$ with $B(x)y(x) \not\equiv 0$ on ab. It is to be noted that this condition of definiteness is satisfied vacuously in case $B(x)y(x) \equiv 0$ for all $y(x) \in \Lambda$; an example of such a problem is the self-adjoint system

$$\begin{aligned} -y_2' &= \lambda y_1, & y_1(0) &= 0, \\ y_1' &= 0, & y_2(1) &= 0. \end{aligned}$$

For symmetrizable problems (2.1) the condition of definiteness corresponding to that considered by Bliss [1, 2] for problems with real coefficients is the nonnegative definiteness of the hermitian matrix $S(x) = T^*(x)B(x)$ on ab; such systems clearly satisfy the above condition of definiteness $[c_1, c_2; T]$ with $c_1 = 0$, $c_2 = 1$. If definiteness $[c_1, c_2; T]$ holds with $c_1 \neq 0$, then there is no essential additional restriction in supposing that the problem is definite $[1, 0; T]$, as such is attainable through replacing λ by $\lambda - c_2/c_1$ in (2.1), and the possible substitution of $-T$ or $A_1^{*-1}T^*A_1$ for T. In this connection it is to be noted that if (2.1) is symmetrizable under (4.1), then by the second relation of (4.2) we have

$$L[y; c_1, c_2; -T] = -L[y; c_1, c_2; T] = L[y; c_1, c_2; A_1^{*-1}T^*A_1]$$

for $y(x) \in \Lambda$. A problem that is normal and definite $[0, 1; T]$ may be treated by methods similar to those of Bliss [2], while a problem that is normal and definite $[1, 0; T]$ may be handled by the corresponding method of Reid [7]; in this connection the reader is referred to the comments of Reid [7; Section 13].

Now if (2.1) is definite $[c_1, c_2; T]$ and normal, we have $\langle Sy, y\rangle \neq 0$ for all proper solutions of this problem, in view of the relation

$$\langle T^*L[y], y\rangle = \lambda\langle Sy, y\rangle$$

for a proper solution $y(x)$ of (2.1) corresponding to a proper value λ. From conclusion (iii) of Theorem 4.3 it then follows that for such a problem (2.1) all proper values are real, and therefore at most denumerably infinite in number as they are the zeros of an entire function

$$\Delta(\lambda) = \det [MY(a; \lambda) + NY(b; \lambda)],$$

where $Y(x; \lambda)$ is a fundamental matrix of solutions of $L[y; \lambda] = 0$ with elements that are entire functions of λ for fixed x on ab. Moreover, by well-known methods, (see, for example, Bliss [1, 2]), it may be established that for such a problem (2.1) the index of each proper value is equal to its multiplicity as a zero of $\Delta(\lambda)$. If (2.1) is definite $[c_1, c_2; T]$ and normal, then for λ_0 a real number which is not a proper value it follows from conclusion (i) of Theorem 5.1 that the kernel $K(x, t) = G(x, t; \lambda_0)B(t)$ of the equivalent integral equation

$$y(x) = (\lambda - \lambda_0) \int_a^b G(x, t; \lambda_0)B(t)y(t)\, dt, \qquad a \leqq x \leqq b, \tag{7.2}$$

is such that the corresponding matrix $K_1(x, t) = S(x)K(x, t)$ satisfies

$$K_1(x, t) \equiv K_1^*(t, x), \qquad a \leqq x, t \leqq b, \quad x \neq t.$$

For $\mathcal{H}$ the Hilbert space of vector functions of integrable square norm on ab, the theory of the integral equation (7.2) is a special case of linear transformations on $\mathcal{H}$ to $\mathcal{H}$ which are completely continuous and fully symmetrizable in the sense of Reid [10]; in this connection, see also Zaanen [12, 13, 14]. Indeed, if (2.1) is normal and definite $[0, 1; T]$, then for any real λ_0 which is not a proper value the kernel $K(x, t) = G(x, t; \lambda_0)B(t)$ of (7.2) and the hermitian matrix $S(x) = T^*(x)B(x)$ are such that the transformations $\mathcal{K}: k = \mathcal{K}g$ and $\mathcal{S}: s = \mathcal{S}g$ on $\mathcal{H}$ to $\mathcal{H}$ with respective functional values

$$k(x) = \int_a^b K(x, t)g(t)\, dt, \qquad s(x) = S(x)g(x), \qquad a \leqq x \leqq b, \tag{7.3}$$

are such that $\mathcal{K}$ is completely continuous, fully symmetrizable by each of the symmetric transformations $\mathcal{S}\mathcal{K}^p$, $(p = 0, 1, \cdots)$, and $\mathcal{S}\mathcal{K}^2$ is nonnegative symmetric; in particular, if $S(x)$ is a nonnegative hermitian matrix on ab, then $\mathcal{S}$ is a nonnegative symmetric transformation.

If (2.1) is normal and definite $[1, 0; T]$, then $\lambda = 0$ is not a proper value of (2.1), and for $\lambda_0 = 0$ the kernel $K(x, t) = G(x, t; 0)B(t)$ and $S(x) = T^*(x)B(x)$ are such that for the corresponding transformations with respective functional values (7.3) we have that $\mathcal{K}$ is completely continuous and fully symmetrizable by each of the symmetric transformations $\mathcal{S}\mathcal{K}^p$, $(p = 0, 1, \cdots)$, while $\mathcal{S}\mathcal{K}$ is nonnegative symmetric.

For each of the normal definite problems described above, the general results of Reid [10] or Zaanen [12, 13, 14] provide for the respective integral equations (7.2) results on the existence and extremizing properties of proper values, integral expansions of Hilbert type, and convergence in the mean of associated Fourier series expansions; by a slight additional argument one may obtain results on the pointwise convergence of the Fourier series expansions. In this connection it is to be commented that the integral equations (7.2) equivalent to normal definite problems (2.1) involve transformations which belong to a class of fully symmetrizable, completely continuous transformations for which it is shown in Section 7 of Reid [10] that the spectral theory may be developed without the use of the general solvability theorems for completely continuous transformations. Moreover, the results of Reid or Zaanen are still applicable to (7.2) whenever the above described condition of definiteness is weakened to the point of requiring (2.1) to be normal and symmetrizable under a transformation (4.1), while for some real λ_0 not a proper value the kernel $K(x, t) = G(x, t; \lambda_0)B(t)$ and $S(x) = T^*(x)B(x)$ are such that for the transformations specified by (7.3) there is a nonnegative integer q for which $\mathcal{S}\mathcal{K}^q$ is a nonnegative symmetric transformation. For a direct discussion of certain vector integral equations which include the above equation (7.2) whenever (2.1) is normal and definite $[0, 1; T]$ or $[1, 0; T]$, the reader is referred to Wilkins [11] and Zimmerberg [15].

Now suppose that the problem (2.1) is definite $[c_1, c_2; T]$, but is abnormal. Let $T_1(x)$ be a matrix of the form (4.5) such that on ab the matrix $A_1^*T_1$ is skew-hermitian and T_1 is nonsingular, and denote by Λ^0 the linear class Λ for a normal problem (6.7) that is equivalent to (2.1) and symmetrizable under the corresponding transformation (4.4); that is, Λ^0 consists of all vector functions $y(x)$ satisfying $s_0[y] = 0$, $L[y] = B(x)g(x)$ with $g(x)$ continuous on ab. If $y(x) \in \Lambda^0$ then $y(x) \in \Lambda$ in view of (6.1), and by a direct computation it follows readily that $I[y; c_1, c_2; T] = I[y; c_1/k_1, c_2/k_1; T_1]$, where k_1 is the nonzero real constant such that $T_1^*B \equiv k_1 T^*B$ on ab. Moreover, if $y(x) \in \Lambda$, then there is an abnormal solution $y_0(x)$ of (2.1) such that $y_1(x) = y(x) + y_0(x) \in \Lambda^0$, and it may be verified directly that

$$I[y_1; c_1/k_1, c_2/k_1; T_1] = I[y; c_1, c_2; T].$$

Consequently, the condition that the abnormal problem (2.1) be definite $[c_1, c_2; T]$ is equivalent to the condition that the corresponding normal problem (6.7) be definite $[c_1/k_1, c_2/k_1; T_1]$. Application of the above described analysis to the normal definite problem (6.7) provides for the original abnormal problem results on the existence and extremizing properties of normal proper values, integral expansions of Hilbert type, and convergence in the mean properties of associated Fourier expansions in terms of the normal proper solutions of (2.1).

For a real boundary problem (2.1) with $B(x)$ of constant rank on ab, and which in the terminology of the present paper is symmetrizable under a transformation (4.1) with $T(x)$ real and $T^*(x)B(x)$ nonnegative definite on ab, E. Hölder [5] has obtained results on the existence and extremizing properties of normal proper values, together with associated expansion theorems. His method of treatment involves the consideration of a related canonical system of $2n$ linear differential equations and boundary conditions, for which the determination of normal solutions is equivalent to the solution of a pair of adjoint n-dimensional vector integral equations. As noted above, the results of Section 6 on the existence of equivalent normal problems lead to results for abnormal problems under conditions of definiteness much more general than that treated by Hölder. Moreover, it is felt that even for the particular problem considered by Hölder the method of the present paper is considerably simpler than that which he employed.

Bibliography

1. G. A. Bliss, *A boundary value problem for a system of ordinary linear differential equations of the first order*, Trans. Amer. Math. Soc., vol. 28 (1926), pp. 561–584.
2. ———, *Definitely self-adjoint boundary value problems*, Trans. Amer. Math. Soc., vol. 44 (1938), pp. 413–428.
3. ———, *Lectures on the calculus of variations*, University of Chicago Press, 1946.
4. M. R. Hestenes, *Sufficient conditions for the problem of Bolza in the calculus of variations*, Trans. Amer. Math. Soc., vol. 36 (1934), pp. 793–818.
5. E. Hölder, *Einordnung besonderer Eigenwertprobleme in die Eigenwerttheorie kanonischer Differentialgleichungssyteme*, Math. Ann., vol. 119 (1943), pp. 21–66.
6. W. T. Reid, *Some remarks on linear differential systems*, Bull. Amer. Math. Soc., vol. 45 (1939), pp. 414–419.
7. ———, *A new class of self-adjoint boundary value problems*, Trans. Amer. Math. Soc., vol. 52 (1942), pp. 381–425.
8. ———, *Definitely self-adjoint differential systems*, Abstract no. 222, Bull. Amer. Math. Soc., vol. 51 (1945), p. 881.
9. ———, *Self-adjoint boundary value problems*, Abstract no. 535, Bull. Amer. Math. Soc., vol. 55 (1949), p. 1063.
10. ———, *Symmetrizable completely continuous linear transformations in Hilbert space*, Duke Math. J., vol. 18 (1951), pp. 41–56.
11. J. E. Wilkins, Jr., *Definitely self-conjugate adjoint integral equations*, Duke Math. J., vol. 11 (1944), pp. 155–166.
12. A. C. Zaanen, *Ueber vollstetige symmetrische und symmetrisierbare Operatoren*, Nieuw Arch. Wisk. (2), vol. 22 (1947), pp. 57–80.

13. ———, *On the theory of linear integral equations*, I, II; III; IV, IVa; V; VI; VII; VIII; VIIIa, Nederl. Akad. Wetensch. Proc., vol. 49 (1946), pp. 194–212; pp. 292–301; pp. 409–423; pp. 571–585; pp. 608–621; vol. 50 (1947), pp. 357–368; pp. 465–473; pp. 612–617 [= Indag. Math., vol. 8 (1946), pp. 91–109; pp. 161–170; pp. 264–278; pp. 352–366; pp. 367–380; vol. 9 (1947), pp. 215–226; pp. 271–279; pp. 320–325].

14. ———, *Normalisable transformations in Hilbert space and systems of linear integral equations*, Acta Math., vol. 83 (1950), pp. 197–248.

15. H. J. Zimmerberg, *Definite integral systems*, Duke Math. J., vol. 15 (1948), pp. 371–388.

Northwestern University
Evanston, Illinois