ON AN IDENTITY INVOLVING BESSEL POLYNOMIALS

BY

F. M. RAGAB

1. Introduction

Bessel polynomials arise in the solution of the classical wave equation in spherical coordinates. They are defined by Krall and Frink [1] by the formula

$$(1) \qquad \gamma_n(x, a, b) = {}_2F_0(-n, a + n - 1; -xb).$$

Recently a number of papers have been written on these polynomials. Full references for these papers are given in Agarwal's paper [2], where a second definition is given on p. 414, namely

$$(2) \qquad \gamma_n(x, a, b) = \frac{1}{\Gamma(a + n - 1)} \int_0^\infty e^{-\lambda} \lambda^{a+n-2} (1 + \lambda x/b)^n \, d\lambda,$$

where $R(a + n - 1) \geqq 0$.

Also a divergent generating function was given by Brafman [3].

In §2 an identical relation between Bessel polynomials will be established, and in §3 an integral involving a modified Bessel function will be evaluated by means of this relation in terms of these polynomials. Some further identities for Bessel polynomials and Kummer functions are deduced in §4.

2. An identity involving Bessel polynomials

The formula to be established is

$$(3) \qquad \sum_{r=0}^{n} {}^nC_r(1 - a - 2k - n; r) \gamma_{k+n-r}(x, a + r, b)(b/x)^{-r} = \gamma_k(x, a, b),$$

where n, k, r are positive integers (or zero) and

$$(4) \qquad \begin{aligned} (\alpha; r) &= \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \cdots (\alpha + r - 1), \qquad r = 1, 2, 3, \cdots \\ (\alpha; 0) &= 1. \end{aligned}$$

To prove it, start with the two known relations for Bessel polynomials, namely, if k is any positive integer:

$$(5) \qquad \gamma_k(x, a, b) = \gamma_{k-1}(x, a + 1, b) + (x/b)(a + k - 1)\gamma_{k-1}(x, a + 2, b);$$

$$(6) \qquad \begin{aligned} &(2k + a - 1)\gamma_k(x, a, b) \\ &\qquad = k\gamma_{k-1}(x, a + 1, b) + (a + k - 1)\gamma_k(x, a + 1, b). \end{aligned}$$

Multiply (5) by $(2k + a - 1)$ and subtract (6); thus

Received June 14, 1957.

$$\gamma_k(x, a+1, b) + (1 - 2k - a)(b/x)^{-1}\gamma_{k-1}(x, a+2, b) = \gamma_{k-1}(x, a+1, b). \tag{7}$$

Here replace k by $k + 1$ and a by $a - 1$, and get

$$\gamma_{k+1}(x, a, b) + (-2k - a)(b/x)^{-1}\gamma_k(x, a+1, b) = \gamma_k(x, a, b), \tag{8}$$

which is formula (3) with $n = 1$.

Now assume (3) for a particular value of n, and apply (8) to each term on the left-hand side. This then becomes

$$\sum_{r=0}^{n} {}^nC_r(1 - a - 2k - n; r)(b/x)^{-r}$$
$$\times \left[\gamma_{k+n-r+1}(x, a+r, b) + (-2k - 2n + r - a)\left(\frac{b}{x}\right)^{-1}\gamma_{k+n-r}(x, a+r+1, b)\right].$$

But

$${}^nC_r(1 - a - 2k - n; r) + {}^nC_{r+1}(1 - a - 2k - n; r-1)(-2k - 2n + r - a) = {}^{n+1}C_r(1 - a - 2k - n - 1; r).$$

Therefore (3) holds with $n + 1$ in place of n. It holds, however, when $n = 1$; hence it holds for all positive integral values of n.

3. An integral involving a modified Bessel function

The integral formula

$$\int_0^\infty e^{-(a+b/x)\lambda}\lambda^{m-k-1}(1+\lambda)^{k-m}I_{2m}[2\sqrt{\{x^{-1}ab\lambda(1+\lambda)\}}]\,d\lambda$$
$$= \frac{\Gamma(2m-k)}{\Gamma(2m+1)}a^m\left(\frac{b}{x}\right)^{k-m}{}_1F_1(2m-k; 1+2m; a)\gamma_k(x, 1+2m-2k, b), \tag{9}$$

where $R(b) > |a|$, $R(2m - k) > 0$, $2m + 1$ is not a negative integer or zero, and k is any positive integer, will now be established.

To prove (9), assume $m > -\frac{1}{2}$, expand e^{-a} and I_{2m} in a series, and multiply. Then the integral becomes

$$\left(\frac{ab}{x}\right)^m \int_0^\infty e^{-b\lambda}\lambda^{2m-k-1}(1+\lambda)^k \times \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\frac{\{x^{-1}ab\lambda(1+\lambda)\}^{r-s}(a\lambda)^s}{\Gamma(2m+r-s+1)s!(r-s)!}\,d\lambda$$
$$= \left(\frac{ab}{x}\right)^m \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\frac{(ab/x)^{r-s}a^s}{\Gamma(2m+r-s+1)s!(r-s)!}\left(\frac{b}{x}\right)^{k-2m-r}$$
$$\cdot \int_0^\infty e^{-\lambda}\lambda^{2m-k+r-1}(1+\lambda x/b)^{k+r-s}\,d\lambda$$
$$= \left(\frac{ab}{x}\right)^m \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\frac{(ab/x)^{r-s}a^s}{\Gamma(2m+r-s+1)s!(r-s)!}\left(\frac{b}{x}\right)^{k-2m-r}$$
$$\cdot\Gamma(2m-k+r)\gamma_{k+r-s}(x, 1+2m-2k+s, b)$$

by (2).

Therefore the left-hand side of (9) is equal to

$$a^m \left(\frac{b}{x}\right)^{k-m} \sum_{r=0}^{\infty} \frac{\Gamma(2m-k+r)}{r!\Gamma(2m+1+r)} a^r$$
$$\times \sum_{s=0}^{r} {}^rC_s(-2m-r;s) \left(\frac{b}{x}\right)^{-s} \gamma_{k+r-s}(x, 1+2m-2k+s, b)$$
$$= \frac{\Gamma(2m-k)}{\Gamma(2m+1)} a^m \left(\frac{b}{x}\right)^{k-m} \sum_{r=0}^{\infty} \frac{(2m-k;r)}{r!(2m+1;r)} a^r \gamma_k(x, 1+2m-2k, b),$$

by (3). From this (9) follows after removing the restriction $m > -\frac{1}{2}$ by analytical continuation.

4. Further identities

In (9) assume $m > -\frac{1}{2}$, expand I_{2m}, and apply formula (2); the left-hand side of (9) then becomes

$$\left(\frac{ab}{x}\right)^m \sum_{r=0}^{\infty} \frac{\Gamma(2m-k+r}{r!\,\Gamma(2m+1+r)} \left(\frac{ab}{x}\right)^r \left(a+\frac{b}{x}\right)^{k-2m-r} \gamma_{k+r}(x, 1+2m-2k, ax+b).$$

Thus if $R(b) > |a|$, $R(2m-k) > 0$, and k is any positive integer,

$$(10) \quad {}_1F_1(2m-k; 2m+1; a)\gamma_k(x, 1+2m-2k, b) = \left(\frac{b}{ax+b}\right)^{2m-k}$$
$$\times \sum_{r=0}^{\infty} \frac{(2m-k;r)}{r!\,(2m+1;r)} \left(\frac{ab}{ax+b}\right)^r \gamma_{k+r}(x, 1+2m-2k, b+ax),$$

where the restriction $m > -\frac{1}{2}$ is now removed. In (3) take $k = 0$, and get

$$(11) \quad 1 = \sum_{r=0}^{n} {}^nC_r(1-a-n;r)(b/x)^{-r}\gamma_{n-r}(x, a+r, b),$$

which can be proved alternatively by considering the Cauchy product for a double series and then applying Gauss's theorem.

In (9), take $k = 0$, $x = 1$; thus if $R(b) > |a|$, $R(m) > 0$, and $2m + 1$ is not a negative integer or zero,

$$(12) \quad \int_0^{\infty} e^{-(a+b)\lambda} \lambda^{m-1}(1+\lambda)^{-m} I_{2m}[2\sqrt{\{ab\lambda(1+\lambda)\}}]\, d\lambda$$
$$= \left(\frac{a}{b}\right)^m \left\{\frac{1}{2m} + \frac{1}{1!}\frac{a}{2m+1} + \frac{1}{2!}\frac{a^2}{(2m+2)} + \frac{1}{3!}\frac{a^3}{2m+3} + \cdots\right\},$$

which is a new integral formula.

Again in (10) take $k = 0$, and get

$$(13) \quad \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{2m}{2m+r}\right)\left(\frac{ab}{ax+b}\right)^r \gamma_r(x, 1+2m, b+ax)$$
$$= \left(\frac{ax+b}{b}\right)^m \left\{1 + \frac{a}{1!}\frac{m}{m+\frac{1}{2}} + \frac{a^2}{2!}\frac{m}{m+1} + \frac{a^3}{3!}\frac{m}{m+\frac{3}{2}} + \cdots\right\},$$

where $R(b) > |a|$, $R(m) > 0$, and $2m + 1$ is not a negative integer or zero

Finally I may mention the following formulae:

$$\sum_{r=0}^{k} {}^{k}C_r(a + 2n + 1 - k; r)\, \gamma_{n+1-k}(x, a + r, b)(b/x)^{-r} = \gamma_{n+1}(x, a, b), \tag{14}$$

where k, n are any positive integers (or zero) such that $n + 1 - k \geqq 0$; and

$$\begin{aligned} n(1 - a - n)\gamma_{n-1}(x, a + 2, b) & \\ + (2 - a - b/x)\gamma_n(x, a, b) & \\ + (b^2/x^2)\gamma_{n+1}(x, a - 2, b) &= 0. \end{aligned} \tag{15}$$

References

1. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc., vol. 65 (1949), pp. 100–115.
2. R. P. Agarwal, *On Bessel polynomials*, Canadian J. Math., vol. 6 (1954), pp. 410–415.
3. F. Brafman, *A set of generating functions for Bessel polynomials*, Proc. Amer. Math. Soc., vol. 4 (1953), pp. 275–277.

The Institute for Advanced Study
Princeton, New Jersey
Ein Shams University
Abbassia, Cairo, Egypt