

EQUATIONS OF WIENER-HOPF TYPE

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1. Introduction

In this paper we shall be concerned with integral equations of the form

$$(1) \quad \int_0^{\infty} k(x-y)f(y) dy = f(x), \quad x > 0.$$

If $k(x)$ vanishes exponentially at $\pm\infty$, the technique of Wiener and Hopf (see, for instance, [5], §11.17) solves (1) more or less completely. However, we shall be concerned here with kernels which do not necessarily vanish exponentially at infinity; our only growth restriction on $k(x)$ will be $k(x) \in L_2(-\infty, \infty)$.

Equations similar to (1), with kernel $k(x)$ not vanishing exponentially at infinity, have been considered before. In a paper of Carlson and Heins [2], for instance, a nonhomogeneous analogue of (1) was investigated by introducing a small modification of the kernel. This amounted to replacing $k(x)$ by something like $e^{-\epsilon|x|}k(x)$ and applying the Wiener-Hopf technique to the modified equation. Presumably, then, the solution of the modified equation tends to a solution of the original as $\epsilon \rightarrow 0$. A similar device has been used by Carrier [3].

Using probabilistic methods, Spitzer [7] has obtained an elegant theory of (1) under the assumption that $k(x)$ is a probability density, i.e., that $k(x) \geq 0$ and $\int_{-\infty}^{\infty} k(x) dx = 1$.

Another approach has been used by Sparenberg [4]. Whereas in the Wiener-Hopf technique analytic functions which agree in a strip are combined, Sparenberg noted that a similar method, in which analytic functions which agree on a line are combined, could be used to solve (1) and nonhomogeneous analogues.

The method of the present paper is an extension of that of Sparenberg. Our main contribution is the following. Sparenberg assumed that the function $1 - \int_{-\infty}^{\infty} e^{izx}k(x) dx$ has no (real) zeros; we shall not make this assumption. In fact, it will be seen that it is the behavior of this function near its zeros that determines the nature of the solutions of (1).

2. General procedure

Let $f(x)$ be a solution of (1), and define $f(x)$ for negative values of x by the left side of (1). Set

$$F_+(z) = \int_0^{\infty} e^{izx}f(x) dx, \quad F_-(z) = \int_{-\infty}^0 e^{izx}f(x) dx.$$

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If we let $K(\xi)$ be the Fourier transform of $k(x)$,

$$K(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} k(x) dx,$$

(1) is equivalent, at least formally, to

$$(2) \quad F_-(\xi) = (K(\xi) - 1) F_+(\xi).$$

If $K(\xi) - 1$ can be written as the quotient of two analytic functions, say $K(\xi) - 1 = R_-(\xi)/R_+(\xi)$ where $R_+(z)$ and $R_-(z)$ are analytic in $\Re z > 0$ and $\Re z < 0$ respectively, then (2) becomes

$$(3) \quad F_-(\xi)/R_-(\xi) = F_+(\xi)/R_+(\xi).$$

Now if $f(x)$ does not grow too rapidly, $F_+(z)$ will be analytic in $\Re z > 0$ and $F_-(z)$ in $\Re z < 0$. Thus the left side of (3) can be extended to be analytic in $\Re z < 0$ and the right to be analytic in $\Re z > 0$, so we might be able to combine $F_-(z)/R_-(z)$ and $F_+(z)/R_+(z)$, giving a single analytic (but not necessarily entire) function $P(z)$. We shall then have

$$F_+(z) = R_+(z) P(z), \quad F_-(z) = R_-(z) P(z),$$

and so, for any $\gamma > 0$,

$$(4) \quad f(x) = \begin{cases} \frac{1}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} e^{-ixz} F_+(z) dz, & x > 0 \\ \frac{1}{2\pi} \int_{-i\gamma-\infty}^{-i\gamma+\infty} e^{-ixz} F_-(z) dz, & x < 0. \end{cases}$$

In this section we shall prove only one result, which gives a sufficient condition for a function on the real line to be the quotient of boundary values of analytic functions of a certain kind.

LEMMA. Let $\psi(\xi)$ be a (complex-valued) function on the real line satisfying

- (i) $\psi(\xi)$ is bounded and bounded away from zero,
- (ii) $\psi(\xi) - 1 \in L_2(-\infty, \infty)$ and $\lim_{\xi \rightarrow \pm\infty} \psi(\xi) = 1$,
- (iii) a continuous $\arg \psi(\xi)$ exists, satisfies a Lipschitz condition uniformly in $(-\infty, \infty)$, and $\Delta_{-\infty < \xi < \infty} \arg \psi(\xi) = 0$.

Then we can find functions $\psi_+(z)$ and $\psi_-(z)$ satisfying

- (iv) $\psi_+(z)$ and $\psi_-(z)$ are analytic for $\Re z > 0$ and $\Re z < 0$ respectively,
- (v) $\psi_+(z)$ and $\psi_-(z)$ are bounded and bounded away from zero in their respective half-planes,
- (vi) the limits $\psi_+(\xi) = \lim_{\varepsilon \rightarrow 0+} \psi_+(\xi + i\varepsilon)$ and $\psi_-(\xi) = \lim_{\varepsilon \rightarrow 0+} \psi_-(\xi - i\varepsilon)$ exist almost everywhere, and $\psi(\xi) = \psi_-(\xi)/\psi_+(\xi)$ almost everywhere.

Proof. By adding a suitable multiple of 2π to $\arg \psi(\xi)$, we may assume $\arg \psi(\pm\infty) = 0$. Setting $\log \psi(\xi) = \log |\psi(\xi)| + i \arg \psi(\xi)$ with this determination of the argument, it follows easily from (i) and (ii) that $\log \psi(\xi)$ is bounded and in $L_2(-\infty, \infty)$.

The integral

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \psi(\zeta)}{\zeta - z} d\zeta$$

represents an analytic function which we denote by $\chi_+(z)$ in $\Im z > 0$, and an analytic function which we denote by $\chi_-(z)$ in $\Im z < 0$. Since $\log \psi(\xi) \in L_2$, $\chi_+(\xi + i\varepsilon)$ and $\chi_-(\xi - i\varepsilon)$ converge almost everywhere as $\varepsilon \rightarrow 0+$ to functions $\chi_+(\xi)$ and $\chi_-(\xi)$ satisfying

$$\begin{aligned} \chi_+(\xi) &= \frac{1}{2\pi i} \text{pv} \int_{-\infty}^{\infty} \frac{\log \psi(\zeta)}{\zeta - \xi} d\zeta + \frac{1}{2} \log \psi(\xi), \\ \chi_-(\xi) &= \frac{1}{2\pi i} \text{pv} \int_{-\infty}^{\infty} \frac{\log \psi(\zeta)}{\zeta - \xi} d\zeta - \frac{1}{2} \log \psi(\xi). \end{aligned}$$

(See [5], §5.3.) If we set

$$\psi_+(z) = \exp(-\chi_+(z)), \quad \psi_-(z) = \exp(-\chi_-(z)),$$

(iv) and (vi) are immediate. As for (v), it suffices to prove that $\Re \chi_+(z)$ and $\Re \chi_-(z)$ are bounded in their respective half-planes. We have, for $\eta > 0$,

$$\Re \chi_+(\xi + i\eta) = \frac{\eta}{2\pi} \int_{-\infty}^{\infty} \frac{\log |\psi(\zeta)|}{\eta^2 + (\zeta - \xi)^2} d\zeta + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\zeta - \xi) \arg \psi(\zeta)}{\eta^2 + (\zeta - \xi)^2} d\zeta.$$

Since $\log |\psi(\xi)|$ is bounded, the first term is certainly bounded. As for the second term, denote by $\rho(\xi)$ the Hilbert transform of $\arg \psi(\xi)$,

$$\rho(\xi) = \frac{1}{\pi} \text{pv} \int_{-\infty}^{\infty} \frac{\arg \psi(\zeta)}{\zeta - \xi} d\zeta.$$

The second term is then ([5], §5.3)

$$(5) \quad \frac{\eta}{2\pi} \int_{-\infty}^{\infty} \frac{\rho(\zeta)}{\eta^2 + (\zeta - \xi)^2} d\zeta.$$

Since $\arg \psi(\xi)$ is in L_2 and satisfies a Lipschitz condition, $\rho(\xi)$ is bounded, so (5) is bounded. Thus $\Re \chi_+(\xi + i\eta)$ is bounded; a similar argument applies to $\Re \chi_-(\xi - i\eta)$. The lemma is therefore proved.

Before proceeding, we should state explicitly what we mean by a solution of (1). Our solutions $f(x)$ will not necessarily be such that the left side of (1) converges; rather we consider the integral as evaluated by the Abel method. More precisely: we call $f(x)$ ($-\infty < x < \infty$) a solution of (1) if, for each $\varepsilon > 0$, $e^{-\varepsilon|x|}f(x) \in L_2(-\infty, \infty)$ and

$$(6) \quad \text{l.i.m.}_{\varepsilon_1 \rightarrow 0+} e^{-\varepsilon_1|x|} \int_0^{\infty} k(x - y)e^{-\varepsilon_1 y} f(y) dy = e^{-\varepsilon_1|x|} f(x)$$

in $L_2(-\infty, \infty)$.

Of course, if $f(x)$ is a solution in this sense, and the left side of (1) converges for almost all x , then (1) is satisfied almost everywhere.

3. $K(\xi)$ smooth

In this section we assume, apart from $k(x) \in L_2(-\infty, \infty)$, that

- (a) $K(\xi)$ is continuous and $K(\pm\infty) = 0$,
- (b) $K(\xi) - 1$ has finitely many (real) zeros $\alpha_1, \alpha_2, \dots$,
- (c) for each j there is a (necessarily unique) integer m_j such that $(K(\xi) - 1)(\xi - \alpha_j)^{-m_j}$ is continuous and nonzero in a neighborhood of α_j ,
- (d) $\arg(K(\xi) - 1) \prod (\xi - \alpha_j)^{-m_j}$ satisfies a Lipschitz condition uniformly in $(-\infty, \infty)$.

We shall find all solutions of (1) such that

- (e) $F_+(z)$ is bounded in $\Im z > 0$, except possibly near the α_j , where $(z - \alpha_j)^{m_j} F_+(z)$ is bounded.

By (6) we have, for fixed $\varepsilon > 0$,

$$\begin{aligned} F_+(\xi + i\varepsilon) + F_-(\xi - i\varepsilon) &= \lim_{\varepsilon_1 \rightarrow 0} \int_{-\infty}^{\infty} e^{i\xi x} e^{-\varepsilon_1 |x|} dx \int_0^{\infty} k(x-y) e^{-\varepsilon_1 y} f(y) dy \\ (7) \qquad \qquad \qquad &= \lim_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{K(\zeta) F_+(\zeta + i\varepsilon_1)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta. \end{aligned}$$

Since

$$\lim_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{F_+(\zeta + i\varepsilon_1)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta = \lim_{\varepsilon_1 \rightarrow 0} F_+(\xi + i\varepsilon + i\varepsilon_1) = F_+(\xi + i\varepsilon),$$

(7) is the same as

$$(8) \qquad F_-(\xi - i\varepsilon) = \lim_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{(K(\zeta) - 1) F_+(\zeta + i\varepsilon_1)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta.$$

Now it follows from (e) above that $F_+(\zeta) = \lim_{\varepsilon \rightarrow 0} F_+(\zeta + i\varepsilon)$ exists almost everywhere, and from (a), (c), and (e) that $(K(\zeta) - 1) F_+(\zeta + i\varepsilon_1)$ is bounded as $\varepsilon_1 \rightarrow 0$ uniformly in ζ . Thus we can take the limit under the integral sign in (8), obtaining

$$(9) \qquad F_-(\xi - i\varepsilon) = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{(K(\zeta) - 1) F_+(\zeta)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta.$$

Since $(K(\zeta) - 1) F_+(\zeta)$ is bounded, the right side of (9) converges almost everywhere, as $\varepsilon \rightarrow 0+$, to $(K(\xi) - 1) F_+(\xi)$. (See, for example, [5], §1.17.) Thus $F_-(\xi) = \lim_{\varepsilon \rightarrow 0+} F_-(\xi - i\varepsilon)$ exists almost everywhere, and (2) holds almost everywhere.

Set $n = \sum m_j$. Then the function

$$\frac{1 - K(\xi)}{\prod (\xi - \alpha_j)^{m_j}} (\xi^2 + 1)^{n/2}$$

is continuous, nowhere zero, and tends to 1 at $\pm \infty$. If we denote the variation of its argument (as ξ goes from $-\infty$ to ∞) by $-2k\pi i$, then it is not hard to verify that the function

$$\psi(\xi) = \frac{1 - K(\xi)}{\prod(\xi - \alpha_j)^{m_j}} (\xi^2 + 1)^{n/2} \left(\frac{\xi - i}{\xi + i} \right)^k$$

satisfies the conditions of the lemma of §2. Thus we can find functions $\psi_+(z)$ and $\psi_-(z)$ as described in that lemma. Define

$$(10) \quad \begin{aligned} \phi_+(z) &= (z + i)^{n/2-k} \psi_+(z), & \Re z > 0 \\ \phi_-(z) &= (z - i)^{-n/2-k} \psi_-(z), & \Re z < 0. \end{aligned}$$

Then we have

$$\frac{\phi_-(\xi)}{\phi_+(\xi)} = \frac{1 - K(\xi)}{\prod(\xi - \alpha_j)^{m_j}},$$

and so from (2)

$$(11) \quad -\frac{F_-(\xi)}{\phi_-(\xi)} = \frac{F_+(\xi)}{\phi_+(\xi)} \prod(\xi - \alpha_j)^{m_j}.$$

Now $F_-(\xi - i\varepsilon)$ is bounded (for all ξ and ε) by (9), and $\phi_-(\xi - i\varepsilon)$ is bounded away from zero (for bounded ξ and ε). Thus

$$(12) \quad \lim_{\varepsilon \rightarrow 0} \int \left| \frac{F_-(\xi - i\varepsilon)}{\phi_-(\xi - i\varepsilon)} - \frac{F_-(\xi)}{\phi_-(\xi)} \right| d\xi = 0,$$

the integral being taken over any finite interval. Similarly (using (e))

$$(13) \quad \lim_{\varepsilon \rightarrow 0} \int \left| \frac{F_+(\xi + i\varepsilon)}{\phi_+(\xi + i\varepsilon)} \prod(\xi + i\varepsilon - \alpha_j)^{m_j} - \frac{F_+(\xi)}{\phi_+(\xi)} \prod(\xi - \alpha_j)^{m_j} \right| d\xi = 0,$$

again over any finite interval. We obtain from (11), (12), and (13),

$$\lim_{\varepsilon \rightarrow 0} \int \left| \frac{F_+(\xi + i\varepsilon)}{\phi_+(\xi + i\varepsilon)} \prod(\xi + i\varepsilon - \alpha_j)^{m_j} + \frac{F_-(\xi - i\varepsilon)}{\phi_-(\xi - i\varepsilon)} \right| d\xi = 0.$$

It now follows from a theorem of Carleman ([2], Theorem II, p. 40) that the functions

$$-\frac{F_-(z)}{\phi_-(z)} \quad \text{and} \quad \frac{F_+(z)}{\phi_+(z)} \prod(z - \alpha_j)^{m_j}$$

are analytic continuations of each other, and so together represent an entire function $P(z)$. Moreover from the boundedness of $F_-(z)$, (e), (10), and the fact that $\psi_+(z)$ and $\psi_-(z)$ are bounded away from zero, we see that $P(z) = O(|z|^{n/2+k})$ at infinity. Thus $P(z)$ must be a polynomial of degree at most $\frac{1}{2}n + k$. But since

$$\int_{-\infty}^{\infty} |\phi_-(\xi - i)P(\xi - i)|^2 d\xi = \int_{-\infty}^{\infty} |F_-(\xi - i)|^2 d\xi < \infty$$

and $1/\phi_-(z) = O(|z|^{n/2+k})$ near infinity, we see that the degree of $P(z)$ must be less than $[\frac{1}{2}n + k]$.

Thus

$$(14) \quad F_+(z) = \frac{P(z)\phi_+(z)}{\prod(z - \alpha_j)^{m_j}}, \quad F_-(z) = -P(z)\phi_-(z),$$

and $f(x)$ is given by (4).

We now show that (14) and (4) give a solution of (1) (in the sense (6)) for any polynomial $P(z)$ of degree less than $[\frac{1}{2}n + k]$.

Note first that by (14) and (10), $F_+(\xi + i\varepsilon)$ and $F_-(\xi - i\varepsilon)$ are in L_2 for any $\varepsilon > 0$, that is, $e^{-\varepsilon|x|}f(x) \in L_2$ for any $\varepsilon > 0$. Taking Fourier transforms of both sides of (6), and noting that $\text{l.i.m.}_{\varepsilon_1 \rightarrow 0} F_+(\xi + i\varepsilon_1 + i\varepsilon) = F_+(\xi + i\varepsilon)$, we see that it suffices to prove

$$(15) \quad F_-(\xi - i\varepsilon) = \text{l.i.m.}_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{(K(\zeta) - 1)F_+(\zeta + i\varepsilon_1)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta.$$

Since

$$F_-(\xi - i\varepsilon) = \text{l.i.m.}_{\varepsilon_1 \rightarrow 0} F_-(\xi - i\varepsilon - i\varepsilon_1) = \text{l.i.m.}_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{F_-(\zeta - i\varepsilon)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta,$$

(15) is equivalent to

$$(16) \quad \text{l.i.m.}_{\varepsilon_1 \rightarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{(K(\zeta) - 1)F_+(\zeta + i\varepsilon_1) - F_-(\zeta - i\varepsilon_1)}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta = 0.$$

Since each of the functions $(K(\zeta) - 1)F_+(\zeta + i\varepsilon_1)$ and $F_-(\zeta - i\varepsilon_1)$ is dominated by an L_2 function independent of ε_1 , and since the difference of the functions approaches zero almost everywhere, this difference approaches zero in L_2 mean. This fact and Young's inequality give (16).

4. $K(\xi)$ very smooth

Assume now, in addition to the assumptions of §3, that $K(\xi)$ is $2m_j + 1$ times continuously differentiable near each α_j . In this case we can get a good idea of the behavior of the solutions of (1) at infinity.

We see that $\log \psi(\xi)$ is $m_j + 1$ times continuously differentiable near α_j ; and so $\chi_+(z)$, and therefore also $\phi_+(z)$, is m_j times continuously differentiable near α_j . Hence we can find a polynomial $Q(z)$, of degree at most $n - 1$, such that

$$\frac{P(z)\phi_+(z) - Q(z)}{\prod(z - \alpha_j)^{m_j}}$$

is bounded near each α_j .

We have

$$\frac{1}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} \frac{e^{-izx}Q(z)}{\prod(z - \alpha_j)^{m_j}} dz = \sum_j Q_j(x)e^{-i\alpha_j x},$$

where $Q_j(x)$ is a polynomial of degree less than m_j . Thus if we move the line of integration in (4) down to the real axis we obtain, for $x > 0$,

$$f(x) = \sum_j Q_j(x)e^{-i\alpha_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} \frac{P(\xi)\phi_+(\xi) - Q(\xi)}{\prod(\xi - \alpha_j)^{m_j}} d\xi,$$

the integral representing a function in L_2 .

If $\int_{-\infty}^{\infty} |x^{m_j-1}k(x)| dx < \infty$ for each j , the integral in (1) converges absolutely for all y , and (1) is satisfied in the usual sense.

5. $K(\xi)$ has a corner

We can treat certain cases in which $K(\xi) - 1$ is not smooth near its zeros; for instance, $K(\xi) - 1$ might have a ‘‘corner’’ at one or more of these zeros. In order not to obscure the modifications necessary in this situation, we shall treat only a fairly special case. In addition to (a) of §3 we shall assume $K(\xi)$ is real, $K(\xi) - 1$ is zero for $\xi = 0$ and nowhere else, and

$$(K(\xi) - 1)/|\xi|^\sigma \tag{\sigma > 0}$$

is bounded and bounded away from zero near $\xi = 0$. (Actually, only in case $\sigma = 1$ are we justified in saying that $K(\xi)$ has a corner at $\xi = 0$.)

We shall find all solutions of (1) satisfying: $F_+(z)$ is bounded in $\Im z > 0$ except possibly near $z = 0$; there is a $\delta < 1$ such that $|z|^{\sigma+\delta}F_+(z)$ is bounded near $z = 0$.

Again, $F_+(\xi) = \lim_{\varepsilon \rightarrow 0+} F_+(\xi + i\varepsilon)$ exists almost everywhere, and we obtain (8). Since $(K(\zeta) - 1)F_+(\zeta + i\varepsilon_1)$ is bounded for ζ bounded away from zero, and for ζ near zero is $O(|\zeta|^{-\delta})$ uniformly in ε_1 , we obtain (9). Also, since

$$\frac{(K(\zeta) - 1)F_+(\zeta)}{1 + \zeta^2} \in L_1(-\infty, \infty),$$

we obtain (2) from (9) by letting $\varepsilon \rightarrow 0$ ([5], §1.17).

The function

$$\psi(\xi) = \frac{1 - K(\xi)}{|\xi|^\sigma} (\xi^2 + 1)^{\sigma/2}$$

satisfies the conditions of the lemma of §2, and we can find the corresponding functions $\psi_+(z)$ and $\psi_-(z)$. Set

$$\phi_+(z) = (z + i)^{\sigma/2} z^{-\sigma/2} \psi_+(z), \tag{\Im z > 0}$$

$$\phi_-(z) = (z - i)^{-\sigma/2} z^{\sigma/2} \psi_-(z), \tag{\Im z < 0}$$

where the many-valued functions are determined by $i = e^{\pi i/2}$ and $-i = e^{-\pi i/2}$. Then $\phi_-(\xi)/\phi_+(\xi) = 1 - K(\xi)$, so from (2)

$$(17) \quad -[F_-(\xi)/\phi_-(\xi)] = F_+(\xi)/\phi_+(\xi).$$

Now I claim $F_-(z) \leqq A(1 + |z|^{-\delta})$ in $\mathcal{G}_z < 0$ for some constant A . (A will always represent a constant, but the constant may change with each use of A .) From (9) and the fact that $(K(\zeta) - 1)F_+(\zeta) \leqq A(1 + |\zeta|^{-\delta})$ we obtain

$$|F_-(\xi - i\varepsilon)| \leqq A + A\varepsilon \int_{-\infty}^{\infty} \frac{|\zeta|^{-\delta}}{\varepsilon^2 + (\zeta - \xi)^2} d\zeta.$$

Now

$$\varepsilon \int_{-\infty}^{\infty} \cdots d\zeta = \varepsilon \int_{|\zeta| \leqq \varepsilon} \cdots d\zeta + \varepsilon \int_{|\zeta| \geqq \varepsilon} \cdots d\zeta \leqq A\varepsilon^{-\delta},$$

and

$$\varepsilon \int_{-\infty}^{\infty} \cdots d\zeta = \varepsilon \int_{|\zeta| \leqq |\xi|/2} \cdots d\zeta + \varepsilon \int_{|\zeta| \geqq |\xi|/2} \cdots d\zeta \leqq A\varepsilon |\xi|^{-1-\delta} + A |\xi|^{-\delta}.$$

Thus $|F_-(\xi - i\varepsilon)| \leqq A + A \min(\varepsilon^{-\delta}, (\varepsilon |\xi|^{-1} + 1) |\xi|^{-\delta})$, from which we easily derive the desired inequality.

Since $F_+(z)$ and $F_-(z)$ are bounded except near $z = 0$, we may use (17) and apply Carleman's theorem (across any interval not including zero) and see that $-F_-(z)/\phi_-(z)$ and $F_+(z)/\phi_+(z)$ are analytic continuations of each other, and together represent a function $P(z)$ which is analytic everywhere except possibly at $z = 0$. Using our bounds, we easily see that $P(z)$ is bounded at infinity and $O(|z|^{-\sigma/2-\delta})$ near zero. Thus, if we set

$$r = \begin{cases} \frac{1}{2}\sigma & \frac{1}{2}\sigma \text{ an integer,} \\ \lceil \frac{1}{2}\sigma \rceil + 1 & \frac{1}{2}\sigma \text{ not an integer,} \end{cases}$$

$P(z)$ has a pole at $z = 0$ of order at most r . Now if $P(z)$ approached a non-zero constant at infinity, we would have $\int_{-\infty}^{\infty} |F_+(\xi + i)|^2 d\xi = \infty$, which is false; therefore $P(\infty) = 0$. It follows that $P(z) = z^{-1}Q(z^{-1})$, where Q is a polynomial of degree less than r , and

$$(18) \quad F_+(z) = z^{-1}Q(z^{-1})\phi_+(z), \quad F_-(z) = -z^{-1}Q(z^{-1})\phi_-(z).$$

To prove that (18) gives a solution of (1) for any Q of degree less than r , note first that $F_+(\xi + i\varepsilon)$ and $F_-(\xi - i\varepsilon)$ are in L_2 for any $\varepsilon > 0$, so $f(x)$ given by (4) is such that $e^{-\varepsilon|x|}f(x) \in L_2(-\infty, \infty)$. Again, it suffices to prove (16). But since $(K(\zeta) - 1)F_+(\zeta + i\varepsilon_1) - F_-(\zeta - i\varepsilon_1)$ tends to zero in $L_1(-1, 1)$ and in $L_2(1, \infty)$ and $L_2(-\infty, -1)$, (16) follows upon application of two versions of Young's inequality.

Thus we have obtained r linearly independent solutions of (1). Note that if σ is an even integer, we have a special case of the situation of §3, the solutions of course being the same.

6. An example

We consider here the equation

$$(19) \quad \frac{1}{\pi} \int_0^{\infty} \frac{f(y)}{1 + (x - y)^2} dy = \lambda f(x), \quad 0 < \lambda \leqq 1.$$

Here $K(\xi) = \lambda^{-1}e^{-|\xi|}$. If $\lambda < 1$ the methods of §3 and §4 apply, and we obtain a solution (unique up to a multiplicative constant) which has the form

$$f(x) = A\lambda^{ix} + B\lambda^{-ix} + g(x) \quad (x > 0)$$

where A and B are constants and $g(x) \in L_2(0, \infty)$. We have been unable to obtain $F_+(z)$ explicitly in terms of known functions.

In case $\lambda = 1$, the method of §5 applies with $\sigma = 1$. (This example was also considered by Spitzer [7].) We have, for $\Re z > 0$,

$$\begin{aligned} \chi_+(iz) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log [(1 - e^{-|\xi|}) |\xi|^{-1} (\xi^2 + 1)^{1/2}]}{\xi - iz} d\xi \\ (20) \qquad &= \frac{1}{2} \log \left(1 + \frac{1}{z} \right) + \frac{z}{\pi} \int_0^{\infty} \frac{\log (1 - e^{-\xi})}{\xi^2 + z^2} d\xi. \end{aligned}$$

Let

$$I(\alpha) = \frac{z}{\pi} \int_0^{\infty} \frac{\log (1 - e^{-\alpha\xi})}{\xi^2 + z^2} d\xi \quad (\alpha > 0).$$

Then

$$I'(\alpha) = \frac{z}{\pi} \int_0^{\infty} \frac{\xi}{e^{\alpha\xi} - 1} \frac{d\xi}{\xi^2 + z^2} = \frac{z}{2\pi} \log \frac{\alpha z}{2\pi} - \frac{1}{2\alpha} - \frac{z}{2\pi} \frac{\Gamma'}{\Gamma} \left(\frac{\alpha z}{2\pi} \right).$$

(See, for instance, [6], §12.32.) Therefore

$$I(\alpha) = \left(\frac{\alpha z}{2\pi} - \frac{1}{2} \right) \log \frac{\alpha z}{2\pi} - \log \Gamma \left(\frac{\alpha z}{2\pi} \right) + C(z),$$

where $C(z)$ is independent of α . Since, clearly, $\lim_{\alpha \rightarrow \infty} I(\alpha) = 0$, Stirling's theorem gives $C(z) = \frac{1}{2} \log 2\pi$, so

$$I(1) = \left(\frac{z}{2\pi} - \frac{1}{2} \right) \log \frac{z}{2\pi} - \frac{z}{2\pi} + \frac{1}{2} \log 2\pi - \log \Gamma \left(\frac{z}{2\pi} \right),$$

and, by (20),

$$\chi_+(iz) = \frac{1}{2} \log (1 + z) + \left(\frac{z}{2\pi} - 1 \right) \log \frac{z}{2\pi} - \frac{z}{2\pi} - \log \Gamma \left(\frac{z}{2\pi} \right).$$

Hence from (18), suppressing an irrelevant multiplicative constant,

$$F_+(iz) = z^{-3/2} (z + 1)^{1/2} \exp (-\chi_+(iz)) = z^{-3/2} (z/2\pi)^{1-z/2\pi} e^{z/2\pi} \Gamma(z/2\pi).$$

Consequently, again suppressing a multiplicative constant,

$$f(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{2\pi xz} \left(\frac{e}{z} \right)^{z+1/2} \Gamma(z) dz, \quad x > 0.$$

It is not hard to see that $f(x) \sim Ax^{1/2}$ at infinity, so the integral in (19) converges absolutely.

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