CLASSES OF PERIODIC SEQUENCES[1]

BY

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1. Introduction

In certain psychological experiments connected with the learning of periodic sequences of symbols,[2] it is reasonable to identify two sequences if we can get one from the other by beginning at a different point, by permuting the symbols, or by a combination of these operations. For example, if there are two symbols, say 0 and 1, and if the period is 3, the following sequences are equivalent:

$$(011),\quad (101),\quad (110),\quad (100),\quad (010),\quad (001).$$

Also, (000) and (111) are equivalent. Thus, the eight possible sequences fall into two equivalence classes. It is of interest to determine how many classes there are for a given period n. Since a sequence of period n also has period kn, where k is any integer, there will be duplications as we run through all periods. For example, the class $\{(000), (111)\}$ will already have been counted for $n = 1$ and $n = 2$. We should therefore determine, for each n, the number of classes of sequences which have period n but no smaller period; that is, the number $F(n)$ of classes[3] with *primitive period n*. In our example, $F(1) = 1$, $F(2) = 1$, $F(3) = 1$, $F(4) = 2$, and so forth. If $F^*(n)$ denotes the total number of classes with period n, whether primitive or not, then

$$F^*(n) = \sum_{d \mid n} F(d), \tag{1}$$

where the summation is over all (positive) divisors d of n. This follows from the fact that every class of period n has a primitive period d which divides n.

2. Formulation

Let A denote the set of all periodic sequences

$$a = (\cdots, a_{-1}, a_0, a_1, \cdots),$$

where the a_j may take any of the q values $1, 2, \cdots, q$. Let Q denote the symmetric group on these q symbols, its elements being denoted generically by π, the identity element by e. Let T be the infinite cyclic group generated by the element τ. We can make Q and T act on A by the following rules:

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[2] E. H. Galanter and M. Kochen, *The acquisition and utilization of information in problem solving and thinking*, Lab Memo, University of Pennsylvania, 1957, and E. H. Galanter and W. A. S. Smith, *Some experiments on thought*, Lab Memo, University of Pennsylvania, 1957.

[3] It is easily verified that equivalent sequences have the same periods, so it makes sense to speak of a class with period n.

$$\pi(a) = (\cdots, \pi(a_1), \pi(a_0), \pi(a_1), \cdots), \tag{2}$$

$$\tau(a) = (\cdots, b_{-1}, b_0, b_1, \cdots), \tag{3}$$

where $b_j = a_{j+1}$. It is not difficult to see that π and τ^n commute: $\pi\tau^n(a) = \tau^n\pi(a)$ for all $a \in A$. Furthermore, Q and T have only the identity element in common, that is, $\pi(a) = \tau^n(a)$ for all $a \in A$ implies $\pi = \tau^n = e$. Therefore the group G of permutations of A generated by Q and T is their direct product, and every element of G has a unique representation in the form $\pi\tau^n$.

For each $a \in A$, there is a largest subgroup H_a of G under which a is invariant. That is,

$$\eta \in H_a \leftrightarrow \eta(a) = a. \tag{4}$$

If $H = H_a$ for some $a \in A$, we shall say that *a pertains to H*, and we shall call H a *special* subgroup of G. For each subgroup H of G, we define A_H as the set of all $a \in A$ which pertain to H, and we write $f(H)$ for the number of elements in A_H. Clearly, $f(H) > 0$ if and only if H is special. Also, if H is special, it contains τ^n for some $n > 0$, so every a which pertains to (or is merely invariant under) H must have period n. Since there are at most q^n such elements, $f(H)$ is finite. Also, H is of finite index $j(H)$ in G.

Let $k(H')$ denote the number of elements in A which are invariant under the special subgroup H'. Every such element belongs to a unique A_H for some $H \supset H'$. Conversely, if $a \in A_H$ for some $H \supset H'$, then a is invariant under H'. Hence

$$k(H') = \sum_{H \supset H'} f(H). \tag{5}$$

This is the first of two fundamental equations that will lead to a solution of our problem, the evaluation of $F(n)$.

To derive the second equation, we digress for a moment to count the number of elements in a class $C = C_a$ containing a given element $a \in A_H$. C, which is the orbit of a under G, consists of the set of all $\gamma(a)$ for $\gamma \in G$. Let $\beta_1, \beta_2, \cdots, \beta_j$ be a system of left coset representatives for H in G, so that

$$G = \beta_1 H \cup \beta_2 H \cup \cdots \cup \beta_j H,$$

where $j = j(H)$. If $\gamma \in G$, $\gamma = \beta_i h$ for some i and for some $h \in H$, so $\gamma(a) = \beta_i h(a) = \beta_i(a)$. These j elements are all distinct, for if $\beta_i(a) = \beta_{i'}(a)$, we have $\beta_{i'}^{-1}\beta_i(a) = a$, $\beta_{i'}^{-1}\beta_i \in H$, $\beta_i \in \beta_{i'} H$, and finally $i = i'$. Hence $C = \{\beta_1(a), \cdots, \beta_j(a)\}$, and $|C|$, the number of elements in C, is equal to $j(H_a)$.

Now, to determine $F(n)$, we can count each element a of primitive period n with frequency $1/|C_a|$. Hence

$$F(n) = \sum_a^{(n)} \frac{1}{|C_a|} = \sum_a^{(n)} \frac{1}{j(H_a)} = \sum_H^{(n)} \frac{1}{j(H)} \sum_{a \in A_H} 1,$$

$$F(n) = \sum_H^{(n)} \frac{f(H)}{j(H)}, \tag{6}$$

where the subscripts (n) indicate that we sum over those elements or subgroups with primitive period n. Thus, sufficient information about $f(H)$ and $j(H)$ will yield a solution to our problem. Once we have analyzed the structure of the special subgroups, $j(H)$ will be easy to compute, as will $k(H')$ in equation (5). A further analysis of the conditions under which $H \supset H'$ will enable us to combine (5) and (6) and to solve for $F(n)$. The method sketched above is fairly general and may be applied to many combinatorial problems.

3. The special subgroups

Let $H = H_a$ be a special subgroup of G, and let $Q_a = Q \cap H_a$. Define ρ as the set of all $i \in \{1, 2, \cdots, q\}$ such that $\pi(i) = i$ for all $\pi \in Q_a$, ρ_1 as the set of all i which appear in a. Clearly $\rho_1 \subset \rho$. If $\theta \in Q$ leaves ρ pointwise fixed, it does so to ρ_1, and therefore $\theta(a) = a$, $\theta \in Q_a$. Hence

$$Q_a = Q(\rho) \equiv \{\theta \in Q \mid \theta(i) = i \text{ for all } i \in \rho\}.$$

If the cardinality of ρ_1, say $r(\rho_1)$, is q or $q - 1$, then $Q_a = \{e\}$ and $\rho = \{1, \cdots, q\}$. If $r(\rho_1) \leqq q - 2$, then $\rho = \rho_1$. For suppose that $i \in \rho - \rho_1$. There exists an $i' \notin \rho_1$, $i' \neq i$. Let θ be the transposition (ii'). Then $\theta(a) = a$, so $\theta \in Q_a = Q(\rho)$, and $i' = \theta(i) = i$. This contradiction shows that $\rho \subset \rho_1$, and since $\rho_1 \subset \rho$, our assertion is proved. The possible values of $r(\rho)$ are $1, 2, \cdots, q - 2, q$, and the correspondence between ρ and $Q(\rho)$ is one-to-one. Obviously $Q(\rho)$ is isomorphic to the symmetric group on $q - r(\rho)$ letters.

Now we shall prove that Q_a is a direct factor in H. If $\beta \in H$, then $\beta = \lambda\tau^m$ uniquely, where $\lambda \in Q$. The set of all λ which arise in this way is a group $L \supset Q_a = Q(\rho)$. If $\lambda \in L$, $\theta \in Q(\rho)$, then $\theta' = \lambda^{-1}\theta\lambda \in Q$. But $\lambda^{-1}\theta\lambda = (\beta\tau^{-m})^{-1}\theta(\beta\tau^{-m}) = \beta^{-1}\theta\beta \in H$. Hence $\theta' \in Q \cap H = Q(\rho)$, and $Q(\rho)$ is normal in L. Next we show that every $\lambda \in L$ leaves ρ setwise invariant. For if not, there exist an $i \in \rho$ and $i' \notin \rho$ such that $\lambda(i') = i$ for some $\lambda \in L$. For each $\theta' \in Q(\rho)$, there is a $\theta \in Q(\rho)$ for which $\theta' = \lambda^{-1}\theta\lambda$, and

$$\theta'(i') = \lambda^{-1}\theta\lambda(i') = \lambda^{-1}\theta(i) = \lambda^{-1}(i) = i'.$$

Thus, every element of $Q(\rho)$ leaves i' fixed, contradicting the fact that $i' \notin \rho$, and our assertion is proved. Now the set of all elements in Q which leave ρ invariant as a set is precisely $Q(\rho) \times Q(\rho^*)$, where ρ^* is the complement of ρ if $r(\rho) \neq 1$, and $\rho^* = \{1, \cdots, q\}$ if $r(\rho) = 1$. Therefore $L \subset Q(\rho) \times Q(\rho^*)$, and every $\lambda \in L$ is representable uniquely as $\lambda = \theta\pi$, $\theta \in Q(\rho)$, $\pi \in Q(\rho^*)$. It follows that every $\beta \in H$ is representable uniquely as $\beta = \theta\pi\tau^m$, $\theta \in Q(\rho)$, $\pi \in Q(\rho^*)$. Let E be the subgroup of H consisting of all β for which $\theta = e$, that is, $\beta = \pi\tau^m$ with $\pi \in Q(\rho^*)$. Then $Q(\rho)$ commutes with E, the two together generate H, and $Q(\rho) \cap E = \{e\}$. (If $\theta = \pi\tau^m$, with $\theta \in Q(\rho)$, $\pi\tau^m \in E$, then $\theta = \theta \cdot e = e \cdot \pi\tau^m$, so $\theta = e$ by the uniqueness of the representation.) This proves that $H = Q(\rho) \times E$, as was asserted.

We shall now show that E is cyclic. Let M be the set of integers m for which there is a $\pi \in Q(\rho^*)$ with $\pi\tau^m \in E$. It is easy to see that M is a subgroup of the integers, therefore cyclic, with generator $\delta \geqq 0$. Since $\tau^n \in E$ for some minimal positive n, $\delta > 0$ and $\delta \mid n$. Let $\pi \in Q(\rho^*)$ be associated with $\delta(\pi\tau^\delta \in E)$, and let $\pi'\tau^m$ be an arbitrary element of E, $\pi' \in Q(\rho^*)$ and $m \in M$. Then $m = k\delta$, and $(\pi'\tau^m)(\pi\tau^\delta)^{-k} = \pi'\pi^{-k}$ belongs to E and to Q, hence to $H \cap Q = Q(\rho)$. But $Q(\rho) \cap E = \{e\}$, so $\pi' = \pi^k$ and $\pi'\tau^m = (\pi\tau^\delta)^k$. This proves that E is cyclic, with generator $\pi\tau^\delta$. We remark that if the order of π is $d = d(\pi)$, then $n = d\delta$. For $(\pi\tau^\delta)^d = \tau^{\delta d} \in H \cap T$, which is the cyclic group generated by τ^n. Hence $n \mid \delta d$. On the other hand,

$$\tau^{-n}(\pi\tau^\delta)^{n/\delta} = \pi^{n/\delta} \in E \cap Q = \{e\}.$$

Therefore n/δ is a multiple of d, so $\delta d \mid n$. Since δd and n are positive integers which divide each other, they are equal.

To summarize, every special subgroup H is of the form $Q(\rho) \times H_\pi^\delta$, where H_π^δ is the cyclic subgroup generated by $\pi\tau^\delta$, $\pi \in Q(\rho^*)$, and $\delta d(\pi) = n$ is the least positive integer for which $\tau^n \in H$. Furthermore, ρ, π, δ are uniquely determined by H. We may characterize n as the primitive period of any $a \in A_H$.

4. Development of the fundamental equations

From the results of the last section, we may write equation (6) in the form

$$F(n) = \sum_{\substack{\rho,\pi,\delta \\ \delta d(\pi)=n}} \frac{f(Q(\rho) \times H_\pi^\delta)}{j(Q(\rho) \times H_\pi^\delta)}. \tag{7}$$

To compute the index $j = (G:H)$, we observe that

$$H_e^n \subset H_e^1 \subset Q \times H_e^1 = G, \qquad H_e^n \subset H_\pi^\delta \subset Q(\rho) \times H_\pi^\delta \subset G.$$

Hence

$$(G:H_e^n) = (G:H_e^1)\,(H_e^1:H_e^n) = q!\,n.$$

Also

$$(G:H_e^n) = (G:H)\,(H:H_\pi^\delta)\,(H_\pi^\delta:H_e^n) = (G:H)\,(q-r)!\,d(\pi),$$

since $d(\pi)$ is the least positive integer t for which $(\pi\tau^\delta)^t = \tau^n$. (Here $r = r(\rho)$, of course.) Combining these results, we get

$$j(H) = (G:H) = \frac{q!\,n}{(q-r)!\,d(\pi)},$$

$$q!\,nF(n) = \sum_{\substack{\rho,\pi,\delta \\ \delta d(\pi)=n}} (q-r)!\,d(\pi)f_{\rho,\pi}(\delta), \tag{8}$$

where we have defined

$$f_{\rho,\pi}(\delta) = f(Q(\rho) \times H_\pi^\delta). \tag{9}$$

Let us now introduce the formal Dirichlet series

$$\mathfrak{F}(s) = \mathfrak{F}_q(s) = \sum_{n=1}^{\infty} \frac{nF(n)}{n^s}, \tag{10}$$

$$Z_{\rho,\pi}(s) = \sum_{\delta=1}^{\infty} \frac{f_{\rho,\pi}(\delta)}{\delta^s}. \tag{11}$$

No use will be made of convergence for these series, and we could equally well permit the summations to run over some fixed, but arbitrary, range $1, \cdots, N$. The results obtained would be valid for $n \leqq N$. Since N is arbitrary, they would hold for all n. However, it is more convenient to work with the full formal series in order to avoid clumsy circumlocutions.

We multiply (8) by n^{-s} and sum for all $n \geqq 1$, to get

$$q!\,\mathfrak{F}(s) = \sum_{n=1}^{\infty} \sum_{\substack{\rho,\pi,\delta \\ \delta d(\pi)=n}} (q-r)!\,d(\pi)\frac{f_{\rho,\pi}(\delta)}{n^s} = \sum_{\rho,\pi} (q-r)!\,d(\pi) \sum_{\delta=1}^{\infty} \frac{f_{\rho,\pi}(\delta)}{d(\pi)^s \cdot \delta^s},$$

$$q!\,\mathfrak{F}(s) = \sum_{\rho} (q-r)! \sum_{\pi \in Q(\rho^*)} d(\pi)^{1-s} Z_{\rho,\pi}(s). \tag{12}$$

Now if $r(\rho) = r(\rho')$, there is a permutation of the symbols which carries ρ into ρ' and $\pi \in Q(\rho^*)$ into $\pi' \in Q(\rho'^*)$. Since such a permutation does not change the number of elements pertaining to $Q(\rho) \times H_\pi^\delta$, we have $Z_{\rho',\pi'}(s) = Z_{\rho,\pi}(s)$. The inner sum therefore depends only on r. The number of ρ for which $r(\rho) = r$ is clearly $\binom{q}{r} = q!/(q-r)!\,r!$ Hence, for each $r = 1, 2, \cdots, q-2, q$ we may select a normalized $\rho = \{1, 2, \cdots, r\}$. $Q(\rho^*)$ is then the symmetric group on $\{1, 2, \cdots, r\}$, which we shall denote by S_r. Also, $Z_{\rho,\pi}$ will be written as $Z_{r,\pi}$ if ρ is normalized. Performing the indicated reductions, we obtain

$$\mathfrak{F}(s) = \sum_r \frac{1}{r!} \sum_{\pi \in S_r} d(\pi)^{1-s} Z_{r,\pi}(s). \tag{13}$$

If we adopt the convention that $Z_{q-1,\pi}(s) = 0$, we may regard the summation as running from $r = 1$ to $r = q$.

Now we turn to equation (5) and evaluate $k(H)$ directly. If a is to be invariant under $H = Q(\rho) \times H_\pi^\delta$, then a_i ($i = 1, 2, \cdots, \delta$) can be chosen arbitrarily from ρ, and all others are determined by

$$a_{i+m\delta} = (\tau^{m\delta}(a))_i = (\pi^{-m}(a))_i = \pi^{-m}(a_i),$$

for all i and m. Hence

$$k(H) = r^\delta,$$

and equation (5) becomes

$$r'^{\delta'} = \sum_{H \supset H'} f(H). \tag{14}$$

The relation $H \supset H'$ must now be analyzed. If it holds, then $Q(\rho) = H \cap Q \supset H' \cap Q = Q(\rho')$, so $\rho \subset \rho'$. Next, $\pi'\tau^{\delta'} \in Q(\rho) \times H_\pi^\delta$, so

$$\pi'\tau^{\delta'} = \theta(\pi\tau^\delta)^m,$$

which implies that $\delta' = \delta m$ and $\pi' = \theta\pi^m$. These necessary conditions are easily seen to be sufficient, so $H \supset H'$ is equivalent to

$$(15)\qquad \begin{aligned} &\text{(i)}\quad \rho \subset \rho', \\ &\text{(ii)}\quad \pi'\pi^{-m} \in Q(\rho) \quad \text{for some } m \geqq 1, \\ &\text{(iii)}\quad \delta' = \delta m. \end{aligned}$$

If π' and π are related by (ii), we write $\pi \to \pi'(\rho)$. Given ρ, δ', π, and π', with $\pi \to \pi'(\rho)$, there is a minimum m for which (ii) holds, say μ, and every $m \equiv \mu \pmod{d(\pi)}$ for which $m \mid \delta'$ is also acceptable. Thus (14) may be written

$$(16)\qquad r'^{\delta'} = \sum_{\rho} \sum_{\rho'} \sum_{\pi \to \pi'(\rho)} \sum_{\substack{m \mid \delta' \\ m \equiv \mu \pmod{d(\pi)}}} f_{\rho,\pi}(\delta'/m).$$

We introduce the formal Dirichlet series

$$(17)\qquad K_r(s) = \sum_{\delta=1}^{\infty} \frac{r^{\delta}}{\delta^s} \qquad (r \neq q-1).$$

Then, multiplying (16) by δ'^{-s} and summing over all $\delta' \geqq 1$, we get

$$\begin{aligned} K_{r'}(s) &= \sum_{\rho \subset \rho'} \sum_{\pi \to \pi'(\rho)} \sum_{\delta'=1}^{\infty} \sum_{\substack{m \mid \delta' \\ m \equiv \mu \pmod{d(\pi)}}} f_{\rho,\pi}(\delta'/m)\delta'^{-s} \\ &= \sum_{\rho \subset \rho'} \sum_{\pi \to \pi'(\rho)} \sum_{m \equiv \mu \pmod{d(\pi)}} m^{-s} \sum_{\delta=1}^{\infty} f_{\rho,\pi}(\delta)\delta^{-s} \\ &= \sum_{\rho \subset \rho'} \sum_{\pi \to \pi'(\rho)} Z_{\rho,\pi}(s) \sum_{m \equiv \mu \pmod{d(\pi)}} m^{-s}. \end{aligned}$$

Define

$$(18)\qquad H_{\mu,d}(s) = \sum_{m \equiv \mu \pmod{d}} m^{-s}.$$

Then we have

$$(19)\qquad K_{r'}(s) = \sum_{\rho \subset \rho'} \sum_{\pi \to \pi'(\rho)} H_{\mu,d(\pi)}(s) Z_{\rho,\pi}(s).$$

This holds for all ρ' with $r(\rho') = r'$, all $\pi' \in Q(\rho'^*)$, and it is understood that $\pi \in Q(\rho^*)$.

We may normalize $\rho' = \{1, \cdots, r'\}$, so that $\pi' \in S_{r'}$. If $\rho, \bar{\rho} \subset \rho'$, $r(\rho) = r(\bar{\rho})$, there is a permutation which carries ρ into $\bar{\rho}$, ρ' into itself, and π' into $\bar{\pi}'$. For each $\pi \in Q(\rho^*)$, the image $\bar{\pi} \in Q(\bar{\rho}^*)$, and if $\pi \to \pi'(\rho)$, then $\bar{\pi} \to \bar{\pi}'(\bar{\rho})$. Furthermore $\mu(\pi, \pi', \rho) = \mu(\bar{\pi}, \bar{\pi}', \bar{\rho})$, $d(\bar{\pi}) = d(\pi)$, $H_{\bar{\mu},d(\bar{\pi})}(s) = H_{\mu,d(\pi)}(s)$, $Z_{\bar{\rho},\bar{\pi}}(s) = Z_{\rho,\pi}(s)$. Thus the inner sum in (19) depends only on $r(\rho)$. For each $r \leqq r'$, there are $\binom{r'}{r}$ sets $\rho \subset \rho'$ for which $r(\rho) = r$. Hence

$$(20)\qquad K_{r'}(s) = \sum_{r \leqq r'} \binom{r'}{r} \sum_{\pi \to \pi'(r)} H_{\mu,d(\pi)}(s) Z_{r,\pi}(s).$$

It is understood that $\pi \in S_r$, $\pi' \in S_{r'}$, and that $r' = 1, 2, \cdots, q-2, q$. If we want (20) to hold for $q - 1$, we may complete the definition (17) by putting $K_{q-1}(s) = 0$.

5. Solution of the equations

A straightforward solution of (20) for $Z_{r,\pi}$, followed by substitution in (13), solves our problem in a sense, but we can make further progress towards a more explicit result. We observe that we do not need to determine the individual $Z_{r,\pi}$ but only the particular combination appearing in (13). This turns out to be possible.

Let us multiply (20) by an unspecified function $Y(r', \pi')$ and sum over all r' and π'. The right side will yield a linear combination of the $Z_{r,\pi}$, with coefficients depending on Y. If we equate these coefficients with those appearing in (13) we obtain $\mathfrak{F}$ as our sum. Our problem is then transformed to finding Y. It is unimportant whether Y is unique, although this is probably true. We can make any a priori assumptions about Y that we please, the end justifying the means. For example, we shall assume that Y depends only on r' and $d(\pi')$.

Carrying out the above-mentioned summation, we obtain

$$\sum_{r',\pi'} Y(r', \pi')K_{r'} = \sum_{r',\pi'} \sum_{r \leq r'} \sum_{\pi \to \pi'(r)} \binom{r'}{r} H_{\mu,d(\pi)} Z_{r,\pi} Y(r', \pi')$$

$$= \sum_{r,\pi} Z_{r,\pi} \sum_{r' \geq r} \sum_{\substack{\pi' \in S_{r'} \\ \pi \to \pi'(r)}} \binom{r'}{r} H_{\mu,d(\pi)} Y(r', \pi').$$

Referring to (13), we shall require that

$$\frac{1}{r!} d(\pi)^{1-s} = \sum_{r' \geq r} \binom{r'}{r} \sum_{\substack{\pi' \in S_{r'} \\ \pi \to \pi'(r)}} H_{\mu,d(\pi)} Y(r', \pi'). \tag{21}$$

If (21) is satisfied, then

$$\mathfrak{F}(s) = \sum_{r',\pi'} Y(r', \pi)K_{r'}. \tag{22}$$

Now, for fixed r, r', and $\pi \in S_r$, the relations $\pi \to \pi'(r)$, $\pi' \in S_{r'}$ are equivalent to $\pi' = \pi^{\mu}\theta$, where $\theta \in Q(r) = Q(\{1, 2, \cdots, r\})$. But $\theta = \pi'\pi^{-\mu}$ belongs to $S_{r'}$, since $S_r \subset S_{r'}$. Hence θ belongs to the symmetric group $\bar{S}$ on $\{r + 1, \cdots, r'\}$. Conversely, if $\theta \in \bar{S}$, then for every m, $\pi^m\theta$ belongs to the group generated by S_r and $\bar{S}$, which is contained in $S_{r'}$. The minimum positive m (which we have denoted by μ) will be one of the integers $1, \cdots, d(\pi)$. Hence (21) may be written

$$\frac{1}{r!} d(\pi)^{1-s} = \sum_{r' \geq r} \binom{r'}{r} \sum_{\theta \in \bar{S}} \sum_{\mu=1}^{d(\pi)} H_{\mu,d(\pi)} Y(r', \pi^{\mu}\theta). \tag{23}$$

Now π^{μ} and θ operate on the disjoint sets of symbols $\{1, \cdots, r\}$ and $\{r + 1, \cdots, r'\}$, so they commute, and $d(\pi^{\mu}\theta)$ is equal to the least common multiple of $d(\pi^{\mu})$ and $d(\theta)$, denoted by $[d(\pi^{\mu}), d(\theta)]$. Also, $d(\pi^{\mu}) = d(\pi^{v}) = d(\pi)/v$, where v is the greatest common divisor $(\mu, d(\pi))$. Recalling our assumption on Y, the inner sum is expressible as

$$(24)\quad \sum_{v\mid d(\pi)} \sum_{\substack{\mu=1\\(\mu,d(\pi))=v}}^{d(\pi)} H_{\mu,d(\pi)}\, Y(r', \pi^v\theta) = \sum_{v\mid d(\pi)} Y(r', \pi^v\theta) \sum_{\substack{\mu=1\\(\mu,d(\pi))=v}}^{d(\pi)} H_{\mu,d(\pi)}\,.$$

Write $\mu = kv$, $k = 1, 2, \cdots, d(\pi)/v$, k prime to $d(\pi)/v$. Then

$$\sum_{\substack{\mu=1\\(\mu,d)=v}}^{d} H_{\mu,d} = \sum_{\substack{k=1\\(k,d/v)=1}}^{d/v} H_{kv,(d/v)\cdot v}\,.$$

Now

$$H_{kv,lv}(s) = \sum_{m\equiv kv(\mathrm{mod}\ lv)} m^{-s} = \sum_{n\equiv k(\mathrm{mod}\ l)} n^{-s}v^{-s} = v^{-s}H_{k,l}(s).$$

Our inner sum (24) becomes

$$(25)\quad \sum_{v\mid d(\pi)} Y(r', \pi^v\theta)v^{-s} \sum_{\substack{k=1\\(k,d(\pi)/v)=1}}^{d(\pi)/v} H_{k,d(\pi)/v} = \sum_{v\mid d(\pi)} Y(r', \pi^v\theta)v^{-s}M\left(\frac{d(\pi)}{v}\right),$$

where

$$M(\delta) = \sum_{\substack{k=1\\(k,\delta)=1}}^{\delta} H_{k,\delta} = \sum_{\substack{k=1\\(k,\delta)=1}}^{\infty} k^{-s}.$$

But it is easy to see, by expanding each factor in a geometric series, that

$$\prod_{p\nmid\delta}\frac{1}{1-p^{-s}} = \sum_{\substack{k=1\\(k,\delta)=1}}^{\infty} k^{-s},$$

the product being over all primes p which do not divide δ. Hence

$$(26)\quad M(\delta) = \prod_{p\nmid\delta}\frac{1}{1-p^{-s}} = \prod_{p\mid\delta}(1-p^{-s})\cdot\prod_{p}\frac{1}{1-p^{-s}} = \zeta(s)\prod_{p\mid\delta}(1-p^{-s}).$$

Returning to (25), we replace v by its complementary divisor $\delta = d(\pi)/v$ to get

$$\sum_{\delta\mid d(\pi)} Y(r', \pi^{d(\pi)/\delta}\theta)\left(\frac{d(\pi)}{\delta}\right)^{-s} M(\delta) = d(\pi)^{-s}\sum_{\delta\mid d(\pi)} Y(r', \pi^{d(\pi)/\delta}\theta)\delta^{s}M(\delta).$$

Putting this back in (23), simplifying, and changing the order of summation, we get

$$\frac{1}{r!}\,d(\pi) = \sum_{\delta\mid d(\pi)} \delta^s M(\delta)\sum_{r'\geqq r}\binom{r'}{r}\sum_{\theta\in\bar{S}} Y(r', \pi^{d(\pi)/\delta}\theta).$$

Observe that the order of $\pi^{d(\pi)/\delta}\theta$ is $[\delta, d(\theta)]$, so the sum

$$(27)\quad V(r,\delta) = \sum_{r'\geqq r}\binom{r'}{r}\sum_{\theta\in\bar{S}} Y(r', \pi^{d(\pi)/\delta}\theta)$$

depends only on r and δ. The equation

$$\frac{1}{r!}\,d(\pi) = \sum_{\delta\mid d(\pi)} \delta^s M(\delta)V(r,\delta)$$

has an obvious solution

$$\delta^s M(\delta) V(r, \delta) = (1/r!)\varphi(\delta),$$

where φ denotes the Euler function

$$\varphi(\delta) = \delta \prod_{p \mid \delta} (1 - p^{-1}),$$

since it is well known that

$$d = \sum_{\delta \mid d} \varphi(\delta).$$

Thus we have reduced our problem to

$$V(r, \delta) = \varphi(\delta)\delta^{-s}/r!\, M(\delta),$$

where V is given by (27). This is equivalent to

$$g(\delta) = \sum_{\substack{r'=r \\ r' \neq q-1}}^{q} \frac{r'!}{(r'-r)!} \sum_{\theta \epsilon \bar{S}} Y(r', \pi^{d(\pi)/\delta}\theta), \tag{28}$$

where we have defined

$$g(\delta) = \varphi(\delta)\delta^{-s}/M(\delta). \tag{29}$$

Observe that the condition $r' \neq q - 1$ in (28) can be removed by requiring that $Y(q - 1, \pi') = 0$.

A few trial computations suggest that Y, which has depended implicitly on q, does so only as a function of $q - r'$. Therefore we make the following change of notation:

$$Y(r', \pi') = \frac{y(q - r', d(\pi'))}{r'!\,(q - r')!}. \tag{30}$$

It will be seen later that the extra factors lead to a convenient symmetry. (28) becomes

$$g(\delta) = \sum_{r'=r}^{q} \frac{1}{(r'-r)!\,(q-r')!} \sum_{\theta \epsilon \bar{S}} y(q - r', [\delta, d(\theta)]). \tag{31}$$

Now we introduce $j = q - r'$, $m = r' - r$, $b = q - r$, so that j and m are nonnegative integers whose sum is b. Hence

$$g(\delta) = \sum_{j+m=b} \frac{1}{j!\,m!} \sum_{\theta \epsilon \bar{S}} y(j, [\delta, d(\theta)]). \tag{32}$$

Recall that $\bar{S}$ is the symmetric group on the $r' - r = m$ symbols $\{r + 1, \cdots, r'\}$. The particular set is unimportant now, since only the order of θ is involved, so we may write

$$g(\delta) = \sum_{j+m=b} \frac{1}{j!\,m!} \sum_{\theta \epsilon S_m} y(j, [\delta, d(\theta)]). \tag{33}$$

Again, a few trials indicate that we should try a solution in the form

$$y(j,k) = \sum_{\pi \in S_j} A_j(\pi) g([k, d(\pi)]). \tag{34}$$

Putting this into (33) and observing that $[[\delta, d(\theta)], d(\pi)] = [\delta, d(\theta), d(\pi)]$, we get

$$g(\delta) = \sum_{j+m=b} \frac{1}{j!\, m!} \sum_{\theta \in S_m} \sum_{\pi \in S_j} A_j(\pi) g([\delta, d(\theta), d(\pi)]). \tag{35}$$

This should be satisfied identically in g, so we collect the coefficient of $g(k\delta)$ on the right, and require that it be equal to 1 if $k = 1$ and to 0 if $k > 1$. This coefficient is

$$C_\delta(k) = \sum_{j+m=b} \frac{1}{j!\, m!} \sum_{\theta,\pi} A_j(\pi), \tag{36}$$

where the inner sum is over all $\theta \in S_m$, $\pi \in S_j$ for which $[\delta, d(\theta), d(\pi)] = k\delta$. If $k = 1$, this means that $d(\theta) \mid \delta$ and $d(\pi) \mid \delta$, that is, $\theta^\delta = e$ and $\pi^\delta = e$. Thus we require that

$$1 = \sum_{j+m=b} \frac{1}{j!\, m!} \sum_{\substack{\pi \in S_j \\ \pi^\delta = e}} A_j(\pi) \sum_{\substack{\theta \in S_m \\ \theta^\delta = e}} 1. \tag{37}$$

Although (37) is needed only for certain values of b and δ, we shall insist on it for all $b \geqq 0$ and $\delta \geqq 1$. It will then follow that $C_\delta(k) = 0$ for $k > 1$, as required. For if we sum $C_\delta(k)$ over all $k \mid a$, the inner sum in (36) is then over all θ, π for which $[\delta, d(\theta), d(\pi)]$ divides $a\delta$. But this is equivalent to $[a\delta, d(\theta), d(\pi)] = a\delta$, so

$$\sum_{k \mid a} C_\delta(k) = C_{a\delta}(1) = 1,$$

by assumption. The result then follows from the Möbius inversion formula.

To solve (37), we define

$$B_j(\delta) = \frac{1}{j!} \sum_{\substack{\pi \in S_j \\ d(\pi) \mid \delta}} A_j(\pi), \qquad L_m(\delta) = \frac{1}{m!} \sum_{\substack{\theta \in S_m \\ d(\theta) \mid \delta}} 1,$$

$$\beta(\delta) = \beta(\delta, x) = \sum_{j \geqq 0} B_j(\delta) x^j, \qquad \lambda(\delta) = \lambda(\delta, x) = \sum_{m \geqq 0} L_m(\delta) x^m. \tag{38}$$

Then (37) is equivalent to

$$\beta(\delta)\lambda(\delta) = \sum_{j \geqq 0} B_j(\delta) x^j \sum_{m \geqq 0} L_m(\delta) x^m$$

$$= \sum_{b \geqq 0} x^b \sum_{j+m=b} B_j(\delta) L_m(\delta) = \sum_{b \geqq 0} x^b = \frac{1}{1-x}.$$

That is,

$$\beta(\delta) = \frac{1}{(1-x)\lambda(\delta)}. \tag{39}$$

It is known[4] that

$$(40) \qquad \lambda(\delta) = \exp \sum_{d \mid \delta} \frac{x^d}{d}.$$

Hence

$$(41) \qquad \begin{aligned} \beta(\delta) &= \exp(-\log(1-x)) \cdot \exp\left(-\sum_{d \mid \delta} \frac{x^d}{d}\right) \\ &= \exp\left(\sum_{d=1}^{\infty} \frac{x^d}{d} - \sum_{d \mid \delta} \frac{x^d}{d}\right) = \exp \sum_{d \nmid \delta} \frac{x^d}{d}. \end{aligned}$$

Either (41) or the equivalent

$$(42) \qquad \beta(\delta) = \frac{1}{1-x} \exp\left(-\sum_{d \mid \delta} \frac{x^d}{d}\right)$$

may be used, whichever is more convenient. We may therefore regard $B_j(\delta)$ as known.

To determine $A_j(\pi)$, we introduce the function

$$(43) \qquad X_j(k) = \frac{1}{j!} \sum_{\substack{\pi \in S_j \\ d(\pi) = k}} A_j(\pi).$$

Clearly

$$\sum_{k \mid \delta} X_j(k) = B_j(\delta).$$

By the Möbius inversion formula,

$$(44) \qquad X_j(k) = \sum_{\delta \mid k} \mu\left(\frac{k}{\delta}\right) B_j(\delta).$$

With X_j determined by (44), we can satisfy (43) in many ways, for example by making $A_j(\pi)$ depend only on $d(\pi)$. This does not affect the value of $y(j, k)$. For, by (34),

$$\begin{aligned} y(j, k) &= \sum_{d \geqq 1} \sum_{\substack{\pi \in S_j \\ d(\pi) = d}} A_j(\pi) g([k, d(\pi)]) \\ &= \sum_{d \geqq 1} g([k, d]) \sum_{\substack{\pi \in S_j \\ d(\pi) = d}} A_j(\pi) = j! \sum_{d \geqq 1} X_j(d) g([k, d]), \end{aligned}$$

the last step following from (43).

Now, from (44), we have

$$(45) \qquad \frac{y(j, k)}{j!} = \sum_{d \geqq 1} \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) B_j(\delta) g([k, d]) = \sum_{\delta \geqq 1} B_j(\delta) \sum_{c \geqq 1} \mu(c) g([k, c\delta]).$$

We may write the inner sum in the form

$$\sum_{a \geqq 1} g(a[k, \delta]) \sum_{[k, c\delta] = a[k, \delta]} \mu(c).$$

[4] S. Chowla, I. N. Herstein, and W. R. Scott, *The solutions of $x^d = 1$ in symmetric groups*, Norske Vid. Selsk. Forh., Trondheim, vol. 25 (1952), pp. 29–31.

For fixed k and δ, define

$$V(a) = \sum_{[k,c\delta]=a[k,\delta]} \mu(c).$$

Then for arbitrary d,

$$(46) \qquad \sum_{a|d} V(a) = \sum_{[k,c\delta]|d[k,\delta]} \mu(c) = \sum_{c\delta|d[k,\delta]} \mu(c) = \sum_{c|(d[k,\delta]/\delta)} \mu(c).$$

But $\sum_{c|t} \mu(c) = 1$ if $t = 1$, $= 0$ if $t > 1$. Thus the sum in (46) is 0 unless $d[k, \delta] = \delta$, that is, unless $d = 1$ and $k \mid \delta$, in which case it is equal to 1. Defining $\chi_k(\delta) = 1$ if $k \mid \delta$, $= 0$ if $k \nmid \delta$, we have

$$\sum_{a|d} V(a) = \sum_{a|d} \chi_k(\delta)\mu(a).$$

By the Möbius inversion formula the summands must be equal, so

$$V(a) = \chi_k(\delta)\mu(a).$$

Therefore the inner sum in (45) is

$$\sum_{a\geqq 1} g(a[k, \delta])\chi_k(\delta)\mu(a) = \sum_{a\geqq 1} \mu(a)g(a\delta)\chi_k(\delta),$$

and (45) becomes, on putting $\delta = vk$,

$$(47) \qquad \frac{y(j, k)}{j!} = \sum_{v\geqq 1} B_j(vk) \sum_{a\geqq 1} \mu(a)g(avk).$$

We can now reduce the inner sum in (47). Writing $\delta = vk$, define $h(a) = g(a\delta)/g(\delta)$. It is easily verified that $h(a)$ is a multiplicative[5] function of a, and so is $\mu(a)$. Hence

$$\sum_{a\geqq 1} \mu(a)h(a) = \prod_p (1 + \mu(p)h(p) + \mu(p^2)h(p^2) + \cdots) = \prod_p (1 - h(p)).$$

Now if $p \mid \delta$,

$$h(p) = \frac{g(p\delta)}{g(\delta)} = \frac{\varphi(p\delta)(p\delta)^{-s}}{\prod_{p'|p\delta} (1 - p'^{-s})} \cdot \frac{\prod_{p'|\delta} (1 - p'^{-s})}{\varphi(\delta)\delta^{-s}} = p^{1-s}.$$

If $p \nmid \delta$,

$$h(p) = \frac{(p - 1)p^{-s}}{1 - p^{-s}}.$$

Therefore

$$\prod_p (1 - h(p)) = \prod_{p|\delta} (1 - p^{1-s}) \prod_{p\nmid\delta} \frac{(1 - p^{1-s})}{(1 - p^{-s})}$$

$$= \prod_p (1 - p^{1-s}) \cdot \frac{\prod_{p|\delta} (1 - p^{-s})}{\prod_p (1 - p^{-s})} = \frac{\zeta(s)}{\zeta(s - 1)} \prod_{p|\delta} (1 - p^{-s}).$$

[5] That is, $(a, b) = 1$ implies that $h(ab) = h(a)h(b)$.

Hence

$$\sum_{a \geqq 1} \mu(a)g(a\delta) = g(\delta) \sum_{a \geqq 1} \mu(a)h(a)$$

$$= \frac{\varphi(\delta)\delta^{-s}}{\zeta(s) \prod_{p|\delta} (1 - p^{-s})} \cdot \frac{\zeta(s) \prod_{p|\delta} (1 - p^{-s})}{\zeta(s - 1)} = \frac{\varphi(\delta)\delta^{-s}}{\zeta(s - 1)}.$$

Putting this back in (47), we get

$$\frac{y(j, k)}{j!} = \sum_{v \geqq 1} \frac{1}{\zeta(s - 1)} B_j(vk)(vk)^{-s}\varphi(vk). \tag{48}$$

We return now to the evaluation of $\mathfrak{F}(s)$. From (22),

$$\mathfrak{F}(s) = \sum_{r=1}^{q} K_r \sum_{\pi \epsilon S_r} Y(r, \pi) = \sum_{r=1}^{q} T(r)K_r,$$

say. By (30),

$$T(r) = \sum_{\pi \epsilon S_r} \frac{y(j, d(\pi))}{r!j!},$$

where we put $j = q - r$ for brevity. Continuing,

$$T(r) = \sum_k \sum_{\substack{\pi \epsilon S_r \\ d(\pi)=k}} \frac{y(j, d(\pi))}{r!j!} = \sum_k \frac{y(j, k)}{j!} \cdot \frac{1}{r!} \sum_{\substack{\pi \epsilon S_r \\ d(\pi)=k}} 1 = \sum_k \frac{y(j, k)}{j!} D_r(k),$$

say. Recalling the definition of $L_r(k)$ in (38), we see that

$$L_r(k) = \sum_{d|k} D_r(d).$$

By the Möbius inversion formula,

$$D_r(k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) L_r(d).$$

Hence

$$T(r) = \sum_k \sum_{d|k} \mu\left(\frac{k}{d}\right) L_r(d) \frac{y(j, k)}{j!}. \tag{49}$$

Now equation (48) may be written

$$\zeta(s - 1) \frac{y(j, k)}{j!} = \sum_{a \geqq 1} \chi_k(a)B_j(a)\varphi(a)a^{-s}.$$

Using this in (49), we have

$$\zeta(s - 1)T(r) = \sum_{k \geqq 1} \sum_{d|k} \sum_{a \geqq 1} \chi_k(a)B_j(a)a^{-s}\varphi(a)\mu\left(\frac{k}{d}\right) L_r(d)$$

$$= \sum_{a \geqq 1} B_j(a)\varphi(a)a^{-s} \sum_{d \geqq 1} L_r(d) \sum_{\substack{d|k \\ k \geqq 1}} \chi_k(a)\mu\left(\frac{k}{d}\right).$$

In the inner sum, we write $k = md$, to obtain

$$\sum_{m \geq 1} \chi_{md}(a)\mu(m) = \sum_{m \mid (a/d)} \mu(m) = \begin{cases} 1 & \text{if } d = a, \\ 0 & \text{if } d \neq a. \end{cases}$$

Hence

$$\zeta(s-1)T(r) = \sum_{a \geq 1} B_j(a)L_r(a)\varphi(a)a^{-s}. \tag{50}$$

Therefore

$$\begin{aligned} \zeta(s-1)\mathfrak{F}(s) &= \sum_{r=1}^{q} (\zeta(s-1)T(r))K_r(s) \\ &= \sum_{r=1}^{q} \sum_{a \geq 1} B_j(a)L_r(a)\varphi(a)a^{-s}K_r(s). \end{aligned} \tag{51}$$

Now

$$\zeta(s-1) = \sum_{m \geq 1} \frac{1}{m^{s-1}}, \qquad \mathfrak{F}(s) = \sum_{d \geq 1} \frac{F(d)}{d^{s-1}},$$

so

$$\zeta(s-1)\mathfrak{F}(s) = \sum_{n \geq 1} \frac{1}{n^{s-1}} \sum_{d \mid n} F(d) = \sum_{n \geq 1} \frac{nF^*(n)}{n^s}.$$

Using Dirichlet multiplication on the right of (51), we get

$$\sum_{n \geq 1} \frac{1}{n^s} \sum_{r=1}^{q} \sum_{a \mid n} B_j(a)L_r(a)\varphi(a)r^{n/a},$$

and equating coefficients of n^{-s},

$$nF^*(n) = \sum_{r=1}^{q} \sum_{a \mid n} B_j(a)L_r(a)\varphi(a)r^{n/a}. \tag{52}$$

We recall that $F^*(n)$ is the total number of classes with period n, not necessarily primitive, and that $F(n)$ may be recovered from it by Möbius inversion.

Equation (52) is not quite satisfactory, since it apparently requires a knowledge of the coefficients $B_j(a)$ and $L_r(a)$ for all values of a. It turns out, however, that these coefficients are periodic in a. In fact, let σ be the least common multiple of the integers $1, 2, \cdots, q$, and suppose that $a \equiv b \pmod{\sigma}$. From (38) we see that the values of $B_j(a)$ and $L_r(a)$ depend only on the set of π in S_j and S_r for which $d(\pi)$ divides a. But $j \leqq q$ and $r \leqq q$, so $d(\pi)$ always divides σ, and $d(\pi) \mid a$ is equivalent to $d(\pi) \mid b$. Therefore the sets for a and b are identical, and $B_j(a) = B_j(b)$, $L_r(a) = L_r(b)$. By the same token, if every divisor of a and σ is also a divisor of b and σ, and conversely, then the coefficients have the same values. That is, if $(a, \sigma) = (b, \sigma)$, then $B_j(a) = B_j(b)$, $L_r(a) = L_r(b)$. We may therefore write

$$nF_q^*(n) = \sum_{r=1}^{q} \sum_{d \mid n} r^{n/d}\varphi(d)h_r^*(d), \tag{53}$$

where

$$h_r^*(d) = L_r((d, \sigma))B_{q-r}((d, \sigma)). \tag{54}$$

Once we have computed $F^*(n)$, we have immediately

$$F_q(n) = \sum_{m \mid n} \mu\left(\frac{n}{m}\right) F_q^*(m). \tag{55}$$

Another expression for $F_q(n)$ may be derived as follows. We have already proved that

$$T(r) = \sum_k \frac{y(j, k)}{j!} D_r(k)$$

and

$$\frac{y(j, k)}{j!} = \sum_d X_j(d)g([k, d]).$$

Combining these, we get

$$T(r) = \sum_b W_r(b)g(b), \tag{56}$$

where

$$W_r(b) = \sum_{[k,d]=b} X_j(d)D_r(k). \tag{57}$$

Recalling the definitions of X_j and D_r, we see that to each pair d, k contributing to the sum (57), there corresponds a pair $\theta \in S_j$ and $\pi \in S_r$, with $d(\theta) = d$ and $d(\pi) = k$. Since $j = q - r$, it follows that $[d, k]$ divides σ (we may regard θ as acting on $\{r + 1, \cdots q\}$, so that $\theta\pi \in S_q$ and $[d, k] = d(\theta\pi) \mid \sigma$). Hence $W_r(b) = 0$ unless $b \mid \sigma$, and the sum in (56) extends over such b. We can evaluate $W_r(b)$ in terms of L_r and B_j. In fact, we have

$$\sum_{b \mid c} W_r(b) = \sum_{[k,d] \mid c} X_j(d)D_r(k) = \sum_{k \mid c} D_r(k) \sum_{d \mid c} X_j(d) = L_r(c)B_j(c).$$

By the Möbius inversion formula,

$$W_r(b) = \sum_{c \mid b} \mu\left(\frac{b}{c}\right) L_r(c)B_{q-r}(c). \tag{58}$$

Now define

$$\begin{aligned} \mu_b(m) &= \mu(m) \qquad \text{if} \quad (m, b) = 1, \\ &= 0 \qquad\quad\ \text{if} \quad (m, b) > 1. \end{aligned} \tag{59}$$

We shall also adopt the convention that $\mu_b(y) = 0$ if y is not an integer. Then

$$g(b) = \frac{\varphi(b)b^{-s}}{\zeta(s) \prod_{p \mid b} (1 - p^{-s})} = \varphi(b)b^{-s} \sum_{m \geq 1} \mu_b(m)m^{-s} = \varphi(b) \sum_{m \geq 1} \mu_b\left(\frac{m}{b}\right) m^{-s}.$$

Putting this back in (56), multiplying by K_r, and picking out the coefficient of n^{-s}, we find

$$[T(r)K_r]_n = \sum_{b|\sigma} W_r(b)\varphi(b) \sum_{d|n} \mu_b\left(\frac{d}{b}\right) r^{n/d}.$$

Summation over r yields

$$nF_q(n) = \sum_{r=1}^{q} \sum_{d|n} r^{n/d} h_r(d), \tag{60}$$

where

$$h_r(d) = \sum_{b|\sigma} W_r(b)\varphi(b)\mu_b\left(\frac{d}{b}\right), \tag{61}$$

and $W_r(b)$ is given by (58).

Equations (53) and (60) constitute the solution to our problem. In any numerical case they may be checked against each other by means of (55) or (1).

6. Some special results

Suppose that $(n, \sigma) = 1$. Then, in (53), the only admissible value of a is 1, so

$$nF_q^*(n) = \sum_{r=1}^{q} L_r(1)B_{q-r}(1) \sum_{d|n} \varphi(d)r^{n/d}.$$

But

$$L_r(1) = \frac{1}{r!}, \qquad B_{q-r}(1) = \sum_{t=0}^{q-r} \frac{(-1)^t}{t!}.$$

Therefore

$$nF_q^*(n) = \sum_{d|n} \varphi\left(\frac{n}{d}\right) \sum_{r=0}^{q} \sum_{t=0}^{q-r} \frac{(-1)^t}{r!\,t!} r^d = \sum_{d|n} \varphi\left(\frac{n}{d}\right) \sum_{s=1}^{q} \frac{1}{s!} \sum_{r+t=s} (-1)^t \binom{s}{r} E^r 0^d,$$

$$nF_q^*(n) = \sum_{d|n} \varphi\left(\frac{n}{d}\right) \sum_{s=1}^{q} \frac{\Delta^s 0^d}{s!} \qquad ((n, \sigma) = 1). \tag{62}$$

Similarly, if $(n, \sigma) = 1$, (60) becomes

$$nF_q(n) = \sum_{r=1}^{q} L_r(1)B_{q-r}(1) \sum_{d|n} \mu(d)r^{n/d},$$

which reduces in exactly the same way to

$$nF_q(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{s=1}^{q} \frac{\Delta^s 0^d}{s!} \qquad ((n, \sigma) = 1). \tag{63}$$

It may be worthwhile to work out the general case for $q = 2$ and $q = 3$. For this purpose we list the values of $L_r(a)$ and $B_j(a)$ for $r, j = 0, 1, 2, 3$ and $a = 1, 2, 3, 4, 5, 6$.

For $q = 2$, $\sigma = 2$, and (53) yields

$$2nF_2^*(n) = \sum_{\substack{d|n \\ d \text{ odd}}} \varphi(d)2^{n/d} + 2 \sum_{\substack{d|n \\ d \text{ even}}} \varphi(d)2^{n/d}. \tag{64}$$

TABLE 1

$L_r(a)$

a	r = 0	1	2	3
1	1	1	$\frac{1}{2}$	$\frac{1}{6}$
2	1	1	1	$\frac{2}{3}$
3	1	1	$\frac{1}{2}$	$\frac{1}{2}$
4	1	1	1	$\frac{2}{3}$
5	1	1	$\frac{1}{2}$	$\frac{1}{6}$
6	1	1	1	1

TABLE 2

$B_j(a)$

a	j = 0	1	2	3
1	1	0	$\frac{1}{2}$	$\frac{1}{3}$
2	1	0	0	$\frac{1}{3}$
3	1	0	$\frac{1}{2}$	0
4	1	0	0	$\frac{1}{3}$
5	1	0	$\frac{1}{2}$	$\frac{1}{3}$
6	1	0	0	0

Similarly, by computing $W_2(1) = \frac{1}{2}$, $W_2(2) = \frac{1}{2}$ from (58), we find that

$$\begin{aligned} h_2(d) &= \tfrac{1}{2}(\mu(d) + \mu_2(d/2)) \\ &= \tfrac{1}{2}\mu(d) \qquad (d \text{ odd}), \\ &= 0 \qquad\quad (d \text{ even}). \end{aligned}$$

Therefore

$$2nF_2(n) = \sum_{\substack{d \mid n \\ d \text{ odd}}} \mu(d) 2^{n/d}. \tag{65}$$

For $q = 3$, $\sigma = 6$, we compute $h_1^*(d)$ and $h_3^*(d)$ (in general, $h_{q-1}^*(d) = h_{q-1}(d) = 0$). We find that $h_1^*(d) = \frac{1}{2}$ if d is odd, $= 0$ if d is even, and that

$$\begin{aligned} h_3^*(d) &= \tfrac{1}{6} && d \equiv 1, 5 \pmod 6, \\ &= \tfrac{2}{3} && d \equiv 2, 4 \pmod 6, \\ &= \tfrac{1}{2} && d \equiv 3 \pmod 6, \\ &= 1 && d \equiv 0 \pmod 6. \end{aligned} \tag{66}$$

Now if $\bar{n}$ is the largest odd divisor of n, we have

$$\sum_{\substack{d \mid n \\ d \text{ odd}}} \varphi(d) = \sum_{d \mid \bar{n}} \varphi(d) = \bar{n}.$$

Hence

$$nF_3^*(n) = (1/2)\bar{n} + \sum_{d \mid n} \varphi(d) h_3^*(d) 3^{n/d}. \tag{67}$$

By referring to our tables and (58), we find that $W_1(1) = \frac{1}{2}$, $W_1(2) = -\frac{1}{2}$, $W_1(3) = W_1(6) = 0$, so

$$\begin{aligned} h_1(d) &= \tfrac{1}{2}\mu(d) && (d \text{ odd}), \\ &= \mu(d) && (d \text{ even}). \end{aligned}$$

A little computation shows that

$$\begin{aligned} h(n) = \sum_{d \mid n} h_1(d) &= \tfrac{1}{2} && (n = 1), \\ &= -\tfrac{1}{2} && (n = 2^\alpha, \quad \alpha > 0), \\ &= 0, && \text{otherwise}. \end{aligned} \tag{68}$$

Again, we find that $W_3(1) = \frac{1}{6}$, $W_3(2) = \frac{1}{2}$, $W_3(3) = \frac{1}{3}$, $W_3(6) = 0$, so

$$h_3(d) = \tfrac{1}{6}\mu(d) + \tfrac{1}{2}\mu_2\left(\frac{d}{2}\right) + \tfrac{2}{3}\mu_3\left(\frac{d}{3}\right) = \tfrac{1}{6}h'(d)\mu(d),$$

where

$$(69)\qquad \begin{aligned} h'(d) &= 1 && d \equiv 1, 5 \pmod 6,\\ &= -2 && d \equiv 2, 4 \pmod 6,\\ &= -3 && d \equiv 3 \pmod 6,\\ &= -6 && d \equiv 0 \pmod 6. \end{aligned}$$

Combining these results, we have

$$(70)\qquad nF_3(n) = h(n) + \tfrac{1}{6}\sum_{d\mid n}\mu(d)h'(d)3^{n/d}.$$

Following is a table of $F_2(n)$, $F_2^*(n)$, $F_3(n)$, and $F_3^*(n)$, for $n = 1, \cdots, 10$, computed from (65), (64), (70), and (67), respectively. The numerical work was checked by (1).

TABLE 3

n	1	2	3	4	5	6	7	8	9	10
$F_2(n)$	1	1	1	2	3	5	9	16	28	51
$F_2^*(n)$	1	2	2	4	4	8	10	20	30	56
$F_3(n)$	1	1	2	4	8	22	52	140	366	992
$F_3^*(n)$	1	2	3	6	9	26	53	146	369	1002

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