AVERAGING OPERATORS ON $C_\infty(X)$

BY

J. L. KELLEY[1]

G. Birkhoff [1] investigated a linear operator T which is supposed to be defined on a Banach algebra A, and satisfies, for all f and g in A, the identity: $T(fT(g)) = T(f)T(g)$. It turns out that a very interesting class of operators satisfy this weakened form of the condition that T associate with multiplication. Birkhoff showed that if A is the algebra of real valued continuous functions on a compact Hausdorff space Y, and if in addition to the above identity T is positive and idempotent, then (1) Y may be decomposed into slices, and on each slice $T(f)$ is an average of the values of f on this slice, and (2) if Y is a topological group, and, in addition to the above requirements, T commutes with right translation, then T is convolution on the left by Haar measure of a subgroup. (Definitions and more precise statements of these theorems occur in the text.) This last result suggests a connection with a result of Kawada and Itô [4], who showed that a positive, finite, idempotent (under convolution) measure on a compact topological group is necessarily Haar measure on some subgroup. The purpose of this note is to exhibit this connection and to extend the results mentioned above.

Let X be a locally compact Hausdorff space, let $C_\infty(X)$ be the algebra of continuous real valued functions on X which vanish at ∞, and let an operator T on $C_\infty(X)$ be called *averaging* if condition (1) of the preceding paragraph holds. The results of this note are: (1) T is averaging if and only if $T(fT(g)) = T(f)T(g)$. (2) If T is positive and idempotent, then T is averaging if and only if the range of T is a subalgebra of $C_\infty(X)$. (3) If X is a topological group, then T commutes with right translation if and only if T is convolution on the left by a finite signed measure m. (4) Given the hypothesis of (3) and the fact that T is averaging, then m is $\pm$ Haar measure on a compact subgroup of X. (5) On a locally compact topological group X, a finite (nonnegative) measure which is idempotent under convolution is necessarily Haar measure on a compact subgroup (proved for X compact by Kawada and Itô [4]). Finally, we note in Section 4 that Halmos' form of a theorem of Dieudonné [3] is a consequence of the earlier results. The theorem in question, which states that a certain Radon-Nikodym differentiation is averaging, arises from the general probabilistic question as to when a conditional expectation has "nice" properties. The work of Moy [5], characterizing conditional expectation as a linear operator, contains this theorem and many other results in this direction, and the theorem in question has been vastly extended by Maharam [7].

As a matter of convenience, not necessity, the discussion is limited to alge-

Received March 25, 1957.

[1] Presented to the American Mathematical Society April 24, 1953.

bras of real functions. ((2) above would require the additional hypothesis that the subalgebra be self-adjoint.)

The techniques used are those of elementary measure theory, and the first section is devoted to a few lemmas on this subject. These are given in detail simply because there seems to be no reference for the results.

1. Preliminaries

Let X be a locally compact Hausdorff space, and let Y be its one-point compactification, obtained from X by adjoining a single point, ∞, and agreeing that the complement in Y of each compact subset of X is a neighborhood of ∞. The space $C_\infty(Y)$ of all continuous real valued functions on Y which vanish at the point ∞ is normed, as usual, by $\| f \| = \sup \{ | f(y) | : y \in Y \}$. $C_\infty(X)$ is defined to be the space obtained by restricting the domain of each member of $C_\infty(Y)$ to X; clearly $C_\infty(X)$ is isometric to $C_\infty(Y)$.

The relation between Baire measures on X and those on Y is of importance. (The terminology here, as in the other measure theoretic considerations, is that of Halmos [2].) Since the Baire σ-ring of X is contained in that of Y, each measure on Y corresponds (by restricting its domain of definition) to a measure on X. On the other hand:

1.1. LEMMA. *For each finite Baire measure m on X there is a unique Baire measure n on Y such that n is an extension of m and the outer n-measure of $\{ \infty \}$ is zero (i.e. for some Baire set E, $\infty \in E$ and $n(E) = 0$).*

Proof. First, E is a Baire subset of Y if and only if either E or $Y - E$ is a Baire subset of X. To see that this is the case, consider the family S of all subsets E of Y such that either E or $Y - E$ is a Baire set in X. Without difficulty, it can be seen that S is a σ-ring, and to show that S is the Baire σ-ring of Y it is only necessary to verify (in routine fashion) that the complement of a compact G_δ set containing ∞ is a Baire set in X. It follows that the intersection of a Baire set in Y with a Baire set in X is a Baire set in X. Given m, a finite Baire measure on X, there is a set E such that $m(E \cap F) = m(F)$ for all Baire sets F in X. For each Baire set G in Y set $n(G) = m(E \cap G)$. Then n is an extension of m, $\infty \in Y - E$, and $n(Y - E) = 0$. If p is another extension of m such that the outer p measure of $\{ \infty \}$ is zero, then both p and n assign measure zero to some Baire set G which contains ∞, and p and n agree on subsets of $Y - G$. It follows that the extension is unique.

A Baire measure on Y such that the outer measure of ∞ is zero will be said to vanish at ∞, and a signed measure which is the difference of two such measures will also be said to vanish at ∞. The set of all signed measures which vanish at ∞ is denoted $M_\infty(Y)$, and is normed by $\| m \| =$ *variation of* $m = \sup \{ | \int f \, dm | : f \in C_\infty(Y) \}$. (It is not hard to see, because each Baire measure is regular, that $\sup \{ | \int f \, dm | : f \in C_\infty(Y) \} = \sup \{ | \int f \, dm | : f$ *an arbitrary continuous function on* $Y \}$, whenever $m \in M_\infty(Y)$.) Clearly the space of all signed measures on X is in one to one norm preserving correspondence with $M_\infty(Y)$, and we use either measures on X, or measures vanishing at ∞ on the compactification Y of X, as convenience dictates.

1.2. LEMMA. *For each bounded linear functional F on $C_\infty(Y)$ there is a unique signed measure m in $M_\infty(Y)$ such that $F(f) = \int f\,dm$ for all f in $C_\infty(Y)$. Moreover, $\| F \| = \| m \|$.*

Proof. Using the Hahn-Banach theorem, extend F to a linear functional F' on the space of all continuous functions on X such that $\| F \| = \| F' \|$. By the Riesz-Kakutani theorem there is a unique signed Baire measure m on X such that $\int f\,dm = F'(f)$ for all continuous f on X, and $\| F' \| = \| m \|$. It must be shown that m vanishes at ∞. For $e > 0$, there is f in $C_\infty(Y)$, of norm at most one, such that $\int f\,dm \geqq \| m \| - e$. Because $f \in C_\infty(Y)$, there is a continuous function g on Y such that $\| g \| \leqq 1, f + g \leqq 1$, and $\int g\,dm + e$ is greater than or equal to the outer measure of $\{ \infty \}$. Then

$$\int f\,dm + \int g\,dm \leqq \| m \|$$

and the outer measure of $\{ \infty \}$ is less than or equal to $\| m \| - \int f\,dm + e < 2e$. Hence $m \in M_\infty(Y)$. The uniqueness of m follows from regularity of Baire measures.

The w^* topology for $M_\infty(Y)$ is the topology of elementwise convergence of the corresponding functionals on $C_\infty(Y)$. This is related to w^* convergence in the adjoint of the space of *all* continuous functions on Y as follows: The signed measures m_a converge to m relative to the latter topology if and only if m_a converges to m relative to the w^* topology for $M_\infty(Y)$ and $\int 1\,dm_a = m_a(Y)$ converges to $m(Y)$. (This is easily proved since the direct sum of the set of constant functions and $C_\infty(Y)$ is the space of all continuous functions on Y.)

The *carrier* of a signed Baire measure m is defined to be the set of all points y such that each neighborhood of y contains a set E with $m(E) \neq 0$. (If m is a measure, this is equivalent to requiring that the measure of each Baire neighborhood of x is not zero.) Clearly the carrier of a measure is a closed set; in general it is not a Baire set. The carrier of a signed measure on X and the carrier of the corresponding measure on the compactification Y are related in a simple fashion: the latter is the closure of the first. The following results will be stated for signed measures on Y, but clearly the corresponding propositions about signed measures on X are correct.

1.3. LEMMA. *If $f \in C_\infty(Y)$, $m \in M_\infty(Y)$, and $f = 0$ on the carrier of m, then $\int f\,dm = 0$. Consequently, if $f = g$ on the carrier of m, then $\int f\,dm = \int g\,dm$. If m is a measure and $f \geqq 0$, then $\int f\,dm > 0$ if and only if $f(x) > 0$ for some x in the carrier of m.*

This is an elementary consequence of the definition of integral—one only needs the fact that if f vanishes on the carrier of m then f vanishes on a Baire set containing the carrier of m.

1.4. LEMMA. *If $g \in C_\infty(Y)$, $m \in M_\infty(Y)$, and $\int fg\, dm = 0$ for all f in $C_\infty(Y)$, then g is zero on the carrier of m.*

Proof. If g is not zero on the carrier of m, there is a neighborhood U of a point x of the carrier of m on which g is nonzero and of constant sign—say $g(x) > 0$. We may assume that $g(x)/2 \leqq g(u) \leqq 3g(x)/2$ for all u in U. There is then a Baire set E contained in U such that $m(E) \neq 0$ and the measure of each Baire subset of E is zero or of the same sign as $m(E)$. Since m is regular, there is a compact Baire subset K of E of nonzero measure, and a Baire neighborhood V of K such that for each subset F of $V - K$, $m(F) < m(K)/6$. Then a simple calculation shows that if f is nonnegative, 1 on K and 0 outside V, then $\int fg\, dm \neq 0$.

1.5. LEMMA. *If m is a nonnegative member of $M_\infty(Y)$, f is continuous and $\int f\, dm = \| m \| \sup \{f(x)$: x in the carrier of $m\}$, then f is constant on the carrier of m.*

This is, again, an elementary consequence of the definitions of integral and of carrier.

Let F be a continuous map of the compact Hausdorff space Y into another compact Hausdorff space Z. Then for each signed Baire measure m on Y there is a unique signed measure, which we denote $F^*(m)$, such that

$$F^*(m)(E) = m(F^{-1}[E])$$

for each Baire set E in Z; equivalently, for f on Z, $\int f\, dm = \int f \circ F\, dF^*(m)$ where $f \circ F$ is the composition of the two functions. The map F^* is said to be *induced by* F.

If F is a continuous map of a locally compact Hausdorff space X into another such space Z, and if F is continuous at ∞ in the sense that the inverse under F of a compact set is compact, then F may be extended to a continuous map of the one-point compactification of X into the compactification of Z. The following discussion applies directly to this situation. (Actually, by using regular Borel measures instead of Baire, the condition of continuity at ∞ may be dispensed with.)

If Y is a closed subset of a compact Hausdorff space Z, the identity map induces a map of the Baire measures on Y into those on Z. The image, n, under this induced map of a signed measure m is called the *normal extension* of m. The carrier of n is surely a subset of Z.

1.6. LEMMA. *If Z is a closed subset of Y, n is a signed measure on Y, and if the carrier of n is contained in Z, then n is the normal extension of a unique signed Baire measure m on Z. If f is continuous on Z and g is an arbitrary continuous extension of f on Y, then $\int f\, dn = \int g\, dm$.*

Proof. First, each Baire set in Z is the intersection with Z of a Baire set in Y, because: each compact G_δ in Z is the set of zeros of a continuous function on Z, this function has a continuous extension to Y, the set of zeros of the extended function is a compact G_δ in Y, and a routine argument then extends the proposition to arbitrary Baire sets in Z. Since the carrier C of n is contained in Z, if E and F are Baire sets in Y such that $E \cap C = F \cap C$, then $n(E) = n(F)$. For a Baire set G in Z we then define $m(G)$ to be $n(F)$ where F is an arbitrary Baire set F in Y such that $F \cap C = G$. It is clear that m is a signed measure, that n is its normal extension, and that the equality on integrals is a special case of the corresponding formula for the image of a signed measure under an induced map.

2. Averaging operators

Throughout this section, Y and Z are compact Hausdorff spaces, each with a distinguished point, ∞. The following lemma is a mild variant of one of Birkhoff's. Recall that the w^* topology for the adjoint $M_\infty(Y)$ of $C_\infty(Y)$ is the topology of pointwise convergence on $C_\infty(Y)$.

2.1. LEMMA. *Let T be a bounded linear operator on $C_\infty(Y)$ to $C_\infty(Z)$, and for each $z \in Z$ let n_z be the signed measure such that $T(f)(z) = \int f(s)\, dn_z s$ for each $f \in C_\infty(Y)$. Then the function n on Z to $M_\infty(Y)$ is continuous relative to the w^* topology, $n_\infty = 0$, and $\| T \| = \sup \{\| n_z \| : z \in Z\}$.*

On the other hand, if n is continuous on Z to $M_\infty(Y)$ and $n_\infty = 0$, the operator T defined by $T(f)(Z) = \int f\, dn_z$, for $f \in C_\infty(Y)$, is a bounded linear operator carrying $C_\infty(Y)$ into $C_\infty(Z)$.

Proof. If T is a given linear operator, the fact that for each f in $C_\infty(Y)$ the function $T(f)$ is continuous on Z shows that n is continuous relative to the w^* topology. Clearly $n_\infty = 0$, and $\| T \| = \sup\{\| T(f) \| : \| f \| \leqq 1\} = \sup\{| T(f)(z) | : z \in Z \text{ and } \| f \| \leqq 1\} = \sup\{\| n_z \| : z \in Z\}$. On the other hand, if n is given, continuous on Z to $M_\infty(Y)$, the range of n is w^* compact and hence bounded, and there is no difficulty in showing that the corresponding T is a bounded linear operator.

An operator T on $C_\infty(Y)$ to $C_\infty(Y)$ is to be called *averaging* if Y can be broken up into "slices" such that for each function f the function $T(f)$ assumes on each slice an average of the values of f on this slice. This notion is made precise as follows: For each y in Y let $D_y = \{x : T(f)(x) = T(f)(y) \text{ for all } f \text{ in } C_\infty(Y)\}$, and let n_y be the signed measure such that $T(f)(y) = \int f\, dn_y$. Then T is averaging if and only if the carrier of n_y is a subset of D_y for each y in Y.

2.2. THEOREM. *A bounded linear operator T on $C_\infty(Y)$ to $C_\infty(Y)$ is averaging if and only if $T(fT(g)) = T(f)T(g)$ for all f and g in $C_\infty(Y)$.*

Proof. Suppose that $T(fT(g)) = T(f)T(g)$ and that $T(f)(x) = \int f(t)\, dn_x t$. Then for f and g in $C_\infty(Y)$ and y in Y it is true that

$$T(fT(g))(y) = \int [f(s) \int g(r)\, dn_s r]\, dn_y s = \int f(s)\, dn_y s \int g(t)\, dn_y t,$$

and hence $\int f(s)[\int g(t)\, d(n_s - n_y)t]\, dn_y\, s = 0$. By Lemma 1.4 it follows that $\int g(t)\, d(n_s - n_y)t$ vanishes for s belonging to the carrier of n_y. Consequently, since this is the case for all functions g, $n_s = n_y$ if $s \in$ *carrier* n_y, hence $T(f)(y) = T(f)(s)$ for such s, and it is proved that T is averaging. Conversely, if T is averaging, then $n_s - n_y = 0$ when $s \in$ *carrier* n_y, and

$$\int f(s)[\int g(t)\, d(n_s - n_y)t]\, dn_y\, s = 0,$$

because the function within square brackets vanishes on the carrier of n_y. Consequently $T(fT(g)) = T(f)T(g)$.

2.3. *Remark.* An averaging operator need be neither idempotent ($T^2 = T$) nor positive ($T(f) \geqq 0$ when $f \geqq 0$). Clearly an averaging operator T is idempotent if and only if for each x, $\int 1 \cdot dn_x$ is zero or one, and is positive if and only if for each x, n_x is a nonnegative measure.

2.4. *Remark.* Each bounded operator T on a space $C_\infty(Y)$ to another function space $C_\infty(Z)$ may be realized as the restriction of an averaging operator to a subalgebra, in the following simple way. For each function f which is continuous on the cartesian product $Y \times Z$, let $T^-(f)(y, z) = \int f(t, z)\, dn_z\, t$, where $T(g)(z) = \int g\, dn_z$ for $g \in C_\infty(Y)$. Clearly T^- is averaging, for $T^-(f)(y, z)$ is an average of the values of f on $Y \times \{z\}$, and $T^-(f)$ is constant on this set. The algebra $C_\infty(Y)$ is isomorphic with the subalgebra of $C(Y \times Z)$ consisting of functions which are constant on $\{y\} \times Z$ for each y in Y, and 0 on $\{\infty\} \times Z$, and the algebra $C_\infty(Z)$ is isomorphic with the subalgebra consisting of functions constant on each set of the form $Y \times \{z\}$, and 0 on $Y \times \{\infty\}$. Moreover, under these two isomorphisms, T corresponds exactly to T^-.

The range of an averaging operator is automatically a subalgebra of $C_\infty(Y)$, since $T(f)T(g) = T(fT(g))$. The structure of a subalgebra A of $C_\infty(Y)$ is a well known consequence of the Stone-Weierstrass theorem [6]. The subalgebra A divides Y together with the point ∞ into a family D of equivalence classes, two points x and y belonging to the same class if $f(x) = f(y)$ for all f in A, and A is dense in the algebra of all those continuous functions which are constant on each member of D and vanish on the class containing ∞. If A is the range of an operator T, then D is precisely the family of all sets D_y, where $D_y = \{x : T(f)(x) = T(f)(y) \text{ for all } f \text{ in } C_\infty(Y)\}$. If T is positive but not averaging, then there is y in Y such that the carrier of n_y is not a subset of D_y, and it is then possible to find f in A such that $f(y) = 0$ but $\int f\, dn_y \neq 0$, i.e. $T(f)(y) \neq 0$. But this cannot happen if T is idempotent, for $f = T(g)$ for some g, and $T(g)(y) = 0$ while $T \circ T(g)(y) \neq 0$. Hence:

2.5. THEOREM. *A positive idempotent operator on $C_\infty(Y)$ is averaging if and only if its range is a subalgebra.*

3. Operators commuting with group translation

Throughout this section it will be assumed that X is a locally compact topological group. Then for each $x \in X$, *right translation by x* is defined to be the operator R_x such that $R_x(f)(y) = f(yx^{-1})$. If m is a signed measure, then *convolution on the left* by m is defined by $m * f(x) = f(y^{-1}x)\, dmy$. If n is another signed measure, the *convolution of m and n, $m * n$,* is defined by $\int f\, dm * n = \int\int f(xy)\, dmx\, dny$. These definitions are arranged so that $m * (n * f) = (m * n) * f$.

The following simple theorem states the relationship between convolution on the left and operators commuting with right translation.

3.1. THEOREM. *A bounded linear operator T on $C_\infty(X)$ is convolution on the left by a signed measure if and only if T commutes with right translation by each group element.*

Proof. Since $m * R_x(f)(y) = \int R_x(f)(z^{-1}y)\, dmz = \int f(z^{-1}yx^{-1})\, dmz = m * f(yx^{-1}) = R_x(m * f)(y)$, convolution on the left commutes with right translation. On the other hand, suppose that a bounded linear operator T commutes with right translation, that e is the identity element of X, and that m is the signed measure such that for all $f \in C_\infty(X)$,

$$T(f)(e) = \int f(x^{-1})\, dmx.$$

Then for each $g \in C_\infty(X)$, $T(g)(y) = R_{y-1} \circ T(g)(e) = T \circ R_{y-1}(g)(e) = \int R_{y-1}(g)(x^{-1})\, dmx = m * g(y)$.

It is necessary to establish the connection between the measure m, which according to Theorem 3.1, completely describes an operator T which commutes with right translation, and the measures n_x, for x in X, which were used in the preceding section to describe T. Suppose then that $T(f) = m * f$, that for each $x \in X$, $T(f)(x) = \int f\, dn_x$, that $x^\cdot$ is the "point" measure defined by $\int f\, dx^\cdot = \int f(t)\, dx^\cdot t = f(x)$, and that m^- is the measure such that $\int f\, dm^- = \int f(y^{-1})\, dmy$. Then for each $f \in C_\infty(X)$ and each $x \in X$,

$$\int f\, dn_x = \int f(y^{-1}x)\, dmy = \int f(yx)\, dm^-y = \int f(yz)\, dm^-y\, dx^\cdot z = \int f\, dm^- * x^\cdot.$$

Consequently, $n_x = m^- * x^\cdot$. Either from this formula, or directly from the definition of T, it is not hard to see that the right translate by the carrier of the measure n_x is (*carrier* m^-)x (that is, the right translate by x of the carrier of m^-). For convenience, in what follows, the carrier of m^- will be denoted by C^-.

The operator T, where $T(f) = m * f$, is averaging if the carrier of $n_x = m^- * x^\cdot$ is a subset of $D_x = \{y\colon \textit{for all } f,\ T(f)(x) = T(f)(y)\}$. Rewritten, $D_x = \{y\colon n_x = n_y\} = \{y\colon m^- * x^\cdot = m^- * y^\cdot\} = \{y\colon m^- * (xy^{-1})^\cdot = m^-\}$. Define H to be $\{z\colon m^- * z^\cdot = m^-\}$; by a simple calculation,

$$H = \{z\colon \textit{for all } f,\ T(f)(z) = T(f)(e)\}.$$

Then H is a subgroup of X, and by the last equality above, it is clear that H is a closed subgroup. Moreover, D_x is precisely the right coset of x modulo H.

It follows that T is averaging if and only if for each $x \in X$, $C^{-}x \subset Hx$, which is the case if and only if $C^{-} \subset H$. Finally, since

$$(m^{-} * x^{\cdot})(A^{-1}) = (x^{-1})^{\cdot} * m(A)$$

for each Baire set A, the subgroup H is precisely the set of all x such that $x^{\cdot} * m = m$, and since the carrier C of m is $(C^{-})^{-1}$, T is averaging if and only if $C \subset H$. Since, from above, $H = \{z\colon \textit{for all } f,\ T(f)(z) = T(f)(e)\}$, an equivalent statement is: T is averaging if and only if for each $x \in C$, and each $f \in C_\infty(X)$, $T(f)(x) = T(f)(e)$. Hence:

3.2. LEMMA. *If T is convolution on the left by a Baire measure m, then T is averaging if and only if the carrier C of m is a subset of the group H of all x such that m is invariant under left translation by x (i.e. $x^{\cdot} * m = m$). Equivalently, T is averaging if and only if for each $x \in C$, $T(f)(x) = T(f)(e)$ for all $f \in C_\infty(x)$.*

It is almost obvious that if convolution by m is averaging, then m is, in some sense, the Haar measure of H. A few minor technical details remain. It may be that no Baire subset of H is a Baire set in X. However, (Lemma 1.6) there is a unique signed measure h on H such that for $f \in C_\infty(H)$, $f\,dh = g\,dm$, where g is any member of $C_\infty(X)$ which is an extension of f. The signed measure m is the normal extension of h. If g is an extension of f, then the translation of g by a member of H is an extension of the translation of f by the same element, and it follows that h is invariant under left translation by members of H. Finally, it must be shown that either h or $-h$ is nonnegative, from which it will follow that h or $-h$ is a left Haar measure for H, and since h is finite, H will be compact. Let J be the union of all Baire sets A in H such that each Baire subset of A has nonnegative h measure. Then J is invariant under left translation, and $HJ \subset J$. Hence either $J = H$, or J is void. In the first case h is nonnegative, and in the latter (see Halmos, loc. cit. p. 121) h is nonpositive. It is then proved:

3.3. THEOREM. *If X is a locally compact group and T is a bounded linear operator on $C_\infty(X)$ such that T is averaging and commutes with right translation, then T is convolution on the left by (the normal extension of) $\pm$ Haar measure on a compact subgroup of X.*

Using the preceding result, the following generalization of a theorem of Kawada and Itô [4] will be demonstrated.

3.4. THEOREM. *If X is a locally compact group and m is a nonnegative finite measure which is idempotent under convolution, then m is necessarily (the normal extension of) Haar measure on a compact subgroup of X.*

Proof. Suppose that m is nonnegative, $m \neq 0$, and $m * m = m$. It will be shown that for each y belonging to the carrier C of m, and each $f \in C_\infty(X)$, $m * f(y) = m * f(e)$, from which, using 3.2 and 3.3, the theorem will follow.

First, it must be shown that, if $m \neq 0$, then $\| m \| = 1$. Because $\| m \| =$

$\| m * m \| \leqq \| m \|^2$, $\| m \| \geqq 1$. On the other hand, if D is a compact Baire set and $f \in C_\infty(X)$, $0 \leqq f(x) \leqq 1$ for all $x \in X$ and $f = 1$ on $D^{-1}D^{-1}$, then $\| m \| \geqq \int f(y^{-1})\, dmy = m * f(e) = m * m * f(e) = \int\int f(y^{-1}z^{-1})\, dmy\, dmz$, and since, for a fixed z in D, the integrand is one on D, the double integral is greater than or equal to $(m(D))^2$. Hence $\| m \| \geqq \| m \|^2$, and $\| m \| = 1$.

Next, it will be shown that C is the closure of CC. Let f be a nonnegative member of $C_\infty(X)$ which is zero save on an open set U. Then $m * f(x) = \int f(y^{-1}x)\, dmy$ is positive if and only if $f(y^{-1}x) > 0$ for some y in C, which is true if and only if for some y in C, $y^{-1}x \in U$, or equivalently $x \in yU$. Hence $m * f$ is zero precisely on the complement of the set CU. Consequently, $m * m * f = m * f$ is zero except on CCU, and therefore $CU = CCU$. Taking the intersection of the sets CU for all neighborhoods U of e, we obtain, on the one hand, C, and on the other, the closure of CC, and the assertion is proved.

Finally, for an arbitrary nonnegative f in $C_\infty(X)$, let x be a fixed member of C^{-1} such that $m * f(x) = \sup\{m * f(z) : z \in C^{-1}\}$. Then

$$m * f(x) = \int m * f(z^{-1}x)\, dmz,$$

because m is idempotent. Now if $z \in C$, $z^{-1}x \in C^{-1}x \subset C^{-1}C^{-1}$, and by virtue of the preceding paragraph, $z^{-1}x \in C^{-1}$. Consequently, $m * f(x)$ is equal to the integral of a function, which on the carrier C of the measure m, is everywhere less than or equal to $m * f(x)$. By Lemma 1.5 it follows that for $z \in C$, $m * f(z^{-1}x) = m * f(x)$, and since $x^{-1} \in C$, $m * f(x) = m * f(e)$. Hence $m * f$ is constant on C, and since x is then an arbitrary point of C, for all $x \in C$, $m * f(x) = m * f(e)$. If $h = f - g$, where f and g are nonnegative, then $m * h(x) = m * f(x) - m * g(x) = m * f(e) - m * g(e) = m * h(e)$ and hence $m * h(x) = m * h(e)$ for every h in $C_\infty(X)$, and the theorem is proved.

4. A remark on measure theory

A theorem of Dieudonné has been reformulated by Halmos [3] in such a way that it states, in our terminology, that a certain operator is averaging. This theorem will be shown to be a consequence of 2.4 and 2.5. The argument is given, not as a simplification of Halmos' (which is really about as simple as one could hope), but because it seems to throw a little additional light on the theorem.

The problem is the following. We are given a set X, a σ-algebra of subsets $\mathcal{S}$, and a finite measure m on $\mathcal{S}$. We are also given a set Y, a σ-algebra $\mathcal{T}$ of subsets, and an $\mathcal{S}$-$\mathcal{T}$ measurable function P on X onto Y, and p is defined to be the measure on $\mathcal{T}$ induced by m (that is, $p(A) = m(P^{-1}[A])$). The question: under what conditions does there exist, for each y in Y, a measure n_y on $\mathcal{S}$ such that $\int f\, dm = \int [\int f(t)\, dn_y t]\, dpy$ for all f in $L_\infty(m)$? It is not true that such measures n_y always exist; the content of the Dieudonné-Halmos theorem is that they do exist, provided that X and Y are the "natural coordinate spaces" for the measures.

For convenience, let us assume that $X = Y$ and $\mathcal{T} \subset \mathcal{S}$, and that P is the identity mapping. Then p is simply $m \mid \mathfrak{J}$, the restriction of m to the domain $\mathfrak{J}$. For each member f of $L_\infty(m)$ let $f \cdot m$, the indefinite integral, be the signed measure such that $f \cdot m(A) = \int_A f\, dm$. Clearly $f \cdot m \mid \mathfrak{J}$ is absolutely continuous with respect to $m \mid \mathfrak{J}$, and in fact the Radon-Nikodym derivative belongs to $L_\infty(m \mid \mathfrak{J})$. Moreover, the map carrying f into its derivative $d(f \cdot m \mid \mathfrak{J})/d(m \mid \mathfrak{J})$ is a positive idempotent mapping of $L_\infty(m)$ onto $L_\infty(m \mid \mathfrak{J})$. But $L_\infty(m)$ is isomorphic to the space of all continuous functions on a compact Hausdorff space Z, where Z is the class of all multiplicative linear functionals on $L_\infty(m)$ with the weak* topology (equivalently, Z is the Stone space of the Boolean measure algebra of m). Since $L_\infty(m \mid \mathfrak{J})$ is a subalgebra of $L_\infty(m)$, Theorem 2.5 shows that the Radon-Nikodym differentiation is averaging. The desired equality then follows from application of 2.4 to the equation

$$\int f\, dm = \int 1\, d(f \cdot m) = \int 1[d(f \cdot m \mid \mathfrak{J})/d(m \mid \mathfrak{J})]\, d(m \mid \mathfrak{J}).$$

It is also possible to verify directly that the differentiation T satisfies the identity $T(fT(g)) = T(f)T(g)$.

REFERENCES

1. G. BIRKHOFF, *Moyennes de fonctions bornées*, Algèbre et Théorie des Nombres, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 143–153, Centre National de la Recherche Scientifique, Paris, 1950.
2. P. R. HALMOS, *Measure theory*, New York, 1950.
3. ———, *On a theorem of Dieudonné*, Proc. Nat. Acad. Sci. U. S. A., vol. 35 (1949), pp. 38–42.
4. Y. KAWADA AND K. ITÔ, *On the probability distribution on a compact group. I*, Proceedings of the Physico-Mathematical Society of Japan (3), vol. 22 (1940), pp. 977–998.
5. SHU-TEH CHEN MOY, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math., vol. 4 (1954), pp. 47–63.
6. M. H. STONE, *The generalized Weierstrass approximation theorem*, Math. Mag., vol. 21 (1948), pp. 167–184, 237–254.
7. D. MAHARAM, *Decompositions of measure algebras and spaces*, Trans. Amer. Math. Soc., vol. 69 (1950), pp. 142–160.

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA