# ON DOLEANS-FÖLLMER'S MEASURE FOR QUASI-MARTINGALES

BY

MICHEL METIVIER[1] AND JEAN PELLAUMAIL

Let $(\Omega, (\mathscr{F}_t)_{t\in\mathbf{R}^+}, P)$ be the usual setting for studying stochastic processes. The idea of associating with every adapted process $(X_t)_{t\in\mathbf{R}^+}$ a set function $\mu_X$, defined on the boolean ring of subsets of $\mathbf{R}^+ \times \Omega$ generated by the family $\{]s, t] \times F; 0 < s \leq t, F \in \mathscr{F}_s\}$, through the formula

$$\mu_X(]s, t] \times F) = E[1_F \cdot (X_t - X_s)]$$

seems to have been used by C. Doleans in [2] for the first time, in the case of supermartingales. She proved that, if $X$ is a supermartingale of local class $D$, then $\mu_X$ is $\sigma$-additive. An extensive use has been made of $\mu_X$ in the case of quasi-martingales by J. Pellaumail [12]: he proved that the mere knowledge of $\mu_X$ allows building the natural process of $X$ in an easy way.

Recently Föllmer [5] proved, under particular conditions on $(\mathscr{F}_t)$ (which forbid the usual assumption of completeness on the $\mathscr{F}_t$'s and are of topological character), that $\mu_X$ is always $\sigma$-additive as soon as $X$ is a $L^1$-bounded quasi-martingale, and that the property for $X$ to be of class $D$ is equivalent to every evanescent predictable subset of $\mathbf{R}^+ \times \Omega$ being of $\mu_X$ measure zero. Moreover, it has been noticed that the previous decomposition theorem of quasi-martingales ($F$-processes in the work of Orey [11]) as gotten by Orey, Fisk, and Rao can be received as mere immediate consequences of known decomposition theorems for $\sigma$-additive measures [12], [5].

In this paper we intend to take over Föllmer's treatment without assuming topological properties for the $\sigma$-algebras $\mathscr{F}_t$'s, and with the usual assumptions of completeness. The results are slightly different: the measure $\mu_X$ is only simply-additive, and the property of $\sigma$-additivity is in this case equivalent to the property of being of class $D$ for $X$.

Sections 1–3 study the one-to-one correspondence $X \to \mu_X$ between quasi-martingales and a class of finitely additive-measures with bounded variation, wh ch is an isomorphism of the order structures defined by the positive cone of negiative submartingales and the positive cone of positive measures respectively. This part of the paper consists mainly of a synthesis of some results from [12].

Sections 4 and 5 study the $\sigma$-additive or purely finite additivity of $\mu$ in terms of the process $X$; in Section 6 the corresponding decomposition theorem are stated.

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Let us finally remark that, taking advantage of the simplicity of the method here used, we deal at the same time with Banach valued processes, stating the theorem for the real valued processes separately only when needed.

## 1. Notations and definitions

$(\mathscr{F}_t)_{t\in\mathbf{R}^+}$ is an increasing family of sub-algebras of a $\sigma$-algebra $\mathscr{F}$ of subsets of $\Omega$.

$(\Omega, \mathscr{F}, P)$ is a complete probability space. We set $\mathscr{F}_\infty = \bigvee_{t\in\mathbf{R}^+} \mathscr{F}_t$ ($\sigma$-algebra generated by $\bigcup_{t\in\mathbf{R}^+} \mathscr{F}_t$) and $\mathscr{N} = \{F\colon F \in \mathscr{F}_\infty, P(F) = 0\}$.

*Assumption.* $\mathscr{F}_t \supset \mathscr{N}$ for any $t$, and $(\mathscr{F}_t)_{t\in\mathbf{R}^+}$ is right-continuous.

We define the following systems of subsets of $\overline{\mathbf{R}} \times \Omega$ (where $\overline{\mathbf{R}}^+ = [0, \infty]$).

A *predictable rectangle* is a subset $]s, t] \times F$ of $\overline{\mathbf{R}}^+ \times \Omega$ such that $s < t$ and $F \in \mathscr{F}_s$. Let $\alpha \in [0, +\infty]$. We call $\mathscr{R}_\alpha$ the set of predictable rectangles in $]0, \alpha[ \times \Omega$ and $\bar{R}_\alpha$ the set of those included in $]0, \alpha]$.

$\mathscr{U}_\alpha$ is the boolean ring of subsets of $]0, \alpha[ \times \Omega$ which are finite union of predictable rectangles.

$\overline{\mathscr{U}}_\alpha$ is the boolean algebra of subsets of $]0, \alpha] \times \Omega$ which are finite union of predictable rectangles.

$\mathscr{P}_\alpha$ is the $\sigma$-ring generated by $\mathscr{U}_\alpha$.

$\overline{\mathscr{P}}_\alpha$ is the $\sigma$-ring generated by $\overline{\mathscr{U}}_\alpha$.

The elements of $\mathscr{P}_\alpha$ (resp. $\overline{\mathscr{P}}_\alpha$) are called the *predictable subsets* of $]0, \alpha[ \times \Omega$ (resp. $[0, \alpha] \times \Omega$).

The subsets of $\overline{\mathbf{R}}^+ \times \Omega$ included in some $[0, \alpha] \times \Omega$ with $\alpha < \infty$, will be said *bounded*.

For *all* the processes $X = (X_t)_{t\in\mathbf{R}^+}$ which will be considered we will define $X_\infty = 0$ ($X_\infty$ is to be distinguished from $X_\infty^- = \lim_{t\to\infty} X_t$ p.s. if such a limit exists).

$\overline{\mathscr{U}}$ will be the algebra generated by $\overline{\mathscr{U}}_\infty$ and the sets

$$\left\{\{\infty\} \times F; F \in \bigcup_{t\in\mathbf{R}^+} \mathscr{F}_t\right\}.$$

We recall that $\mathscr{U}_\alpha$ consists of finite unions of so-called "stochastic intervals" $]\sigma, \tau] = \{(u, w)\colon \sigma(w) < u \leq \tau(w)\}$ where $\sigma$ and $\tau$ are two finitely valued stopping times.

A function $f$ on $\overline{\mathbf{R}}^+ \times \Omega$ is said to be *evanescent* if

$$P(\{w\colon f(t, w) = 0 \quad \text{for all } t \in \overline{\mathbf{R}}^+\}) = 1.$$

A subset $G$ of $\overline{\mathbf{R}}^+ \times \Omega$ is called evanescent if its indicator function $1_A$ is evanescent.

Two processes $X$ and $Y$ are said indistinguishable if $X - Y$ is evanescent.

As to the variation of a finitely additive measure $\mu$, defined on $\mathscr{B}$ with values in a Banach space $\mathscr{F}$ with norm $\|\ \|$ we recall the definition:

$$|\mu|(B) = \sup \{\textstyle\sum \|\mu(B_i)\|\colon \{B_i\}\ \mathscr{B}\text{-partition of } B\}.$$

## 2. Simply additive measures associated with quasi-martingales

2.1. DEFINITION. An adapted Banach valued process $X$ is said[2] to be an $F$-process (Orey's definition in the real case) or a quasi-martingale on a compact interval $[0, \alpha]$ if

$$K_\alpha = \sup_{0 \le t_1 < \cdots < t_k \le \alpha} \sum_{i=0}^{k-1} E\|X_{t_i} - E(X_{t_{i+1}} \mid \mathscr{F}_{t_i})\| < +\infty$$

where the sup is to be taken on all the increasing finite sequences $t_1 < \cdots < t_k$ in $[0, \alpha]$.

A quasi-martingale on $[0, +\infty]$ (recall that $X_\infty = 0$), will be called shortly a quasi-martingale.

*Remark.* Such a process is clearly bounded in $L^1_{\mathbf{F}}$ on $[0, \alpha]$.

2.2. *Measures associated with a general adapted process.* We define the following functions $m^\alpha_X$ and $\mu^\alpha_X$ (resp. $\bar{m}^\alpha_X$ and $\bar{\mu}^\alpha_X$) on $\mathscr{R}_\alpha$ (resp. $\bar{\mathscr{R}}_\alpha$), for every adapted process $X$ with value in the Banach space $\mathbf{F}$ such that for all $t$, $X_t \in L^1_{\mathbf{F}}(\Omega, \mathscr{F}_t, P)$

(2.2.1) $\quad m^\alpha_X(]s, t] \times F) = 1_F \cdot (X_t - X_s) \in L^1_{\mathbf{F}}$ (resp. $\bar{m}^\alpha_X \cdots$)

(2.2.2) $\quad \mu^\alpha_X(]s, t] \times F) = E[1_F \cdot (X_t - X_s)] \in \mathbf{F}$ (resp. $\bar{\mu}^\alpha_X \cdots$).

It is quite immediate that this function can be extended into simply additive measures on the algebra $\mathscr{U}_\alpha$ (resp. $\bar{\mathscr{U}}_\alpha$). We will still denote the extensions by the same symbols $m_X$, $\bar{m}_X$, etc.

The following properties are immediate:

$-X$ is a martingale on $[0, \alpha[$ (resp. $[0, \alpha]$) if and only if $\mu^\alpha_X$ (resp. $\bar{\mu}^\alpha_X$) is identically zero;

$-X$ is a real supermartingale on $[0, \alpha[$ (resp. $[0, \alpha]$) if and only if $\mu^\alpha_X$ (resp. $\bar{\mu}^\alpha_X$) is negative or zero.

We then have the following.

PROPOSITION 1. *If $\sigma$ and $\tau$ are two finitely valued stopping times with values in $[0, \alpha]$, with $\sigma \le \tau$, then*

$$\bar{m}^\alpha_X(]\sigma, \tau]) = X_\tau - X_\sigma \quad \text{and} \quad \bar{\mu}^\alpha_X(]\sigma, \tau]) = E(X_\tau - X_\sigma).$$

*Proof.* The two stopping times can in fact be written ($\{t_1 \cdots t_n\}$ being the set of their values, with $t_1 < \cdots < t_n$),

$$\sigma = \sum_{i=0}^{n-1} (t_{i+1} - t_i)1_{F_i}, \quad F_0 \subset F_1 \cdots \subset F_{n-1}, F_i \in \mathscr{F}_{t_i},$$

$$\tau = \sum_{i=0}^{n-1} (t_{i+1} - t_i)1_{G_i}, \quad G_0 \subset G_1 \cdots \subset G_{n-1}, G_i \in \mathscr{F}_{t_i}.$$

[2] When we speak of a Banach valued process $X$, measurability of $X_t$ is always to be understood as strong measurability (i.e., $X_t$ is almost surely separably valued and weakly measurable).

The assumption $\sigma \leq \tau$ is equivalent to $F_i \subset G_i$ for all $i \leq n$, and the formulas of the proposition follow immediately from the definition of $\overline{m}_X$, $\bar{\mu}_X$ and the fact that

$$X_\tau = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i}) 1_{G_i}.$$

*Remark.* It should be emphasized that in our situation (all $P$-null sets are in $\mathscr{F}_t$ for all $t$), every evanescent predictable set is in $\mathscr{U}_\alpha$, with measure $\mu_X^\alpha$ zero, which makes basic difference with the Föllmer's situation.

2.3. *More on the correspondence* $X \to \bar{\mu}_X^\infty$. From the assumption $X_\infty = 0$, and the relation

$$\bar{\mu}_X^\infty(]t, \infty] \times F) = -E(1_F \cdot X_t)$$

it is clear that $\bar{\mu}_X^\infty \leftrightarrow X$ is a one-to-one correspondence between real finitely additive measures $\mu$ on $\overline{\mathscr{U}}_\infty$ such that for every $t$, $F \to \mu(]t, \infty] \times F)$ is a $P$-absolutely continuous measure on $\mathscr{F}_t$ with bounded variation, and real processes $X$ such that $X_t \in L^1$ for all $t$ (defined up to a modification).

The same is true for processes $X$ taking their values in a Banach space for which the usual Radon-Nikodym Theorem holds, in particular, separable duals of a Banach space. We will call such a space a R-N Banach space.

2.3. THEOREM 1. *$\bar{\mu}_X^\alpha$ of bounded variation on $\overline{\mathscr{U}}_\alpha \Leftrightarrow X$ is a Banach valued F-process on $[0, \alpha]$. In this case $|\bar{\mu}_X^\alpha|(]0, \alpha] \times \Omega = K_\alpha$ where $|\mu|$ denotes the variation of $\mu$.*

*Proof.* See [12, p. 47 and p. 96].

From the inequality

$$\begin{aligned} \|X_s\| &= \|E(X_s - X_t \mid \mathscr{F}_s) + E(X_t \mid \mathscr{F}_s)\| \\ &\leq \|E(X_s - X_t \mid \mathscr{F}_s)\| + E(\|X_t\| \mid \mathscr{F}_s) \end{aligned}$$

we see that $(\bar{\mu}_{\|X\|}^\alpha)^-$ is bounded by $|\mu_X|$. As $\mu_{\|X\|}(]0, \alpha] \times \Omega)$ is finite, this implies $(\bar{\mu}_{\|X\|}^\alpha)^+$ bounded.

COROLLARY. *If $X$ is a Banach valued quasi-martingale on $[0, \alpha]$, $\|X\|$ is a positive quasi-martingale on $[0, \alpha]$.*

## 3. Bounded variation of $\mu$ and regularity of trajectories of $X$

We recall the following.

THEOREM 2 (Orey). *Let $X$ be a separable real quasi-martingale on $[0, \alpha]$. Almost surely the trajectories have left and right limits.*

*Proof.* See [11], but we mention that, according to the method used in [12, p. 13] we may give the following proof which goes as the traditional proof for martingales due to Doob. Let $a$ and $b$ be two real numbers $a < b$. Let $S = \{s_1 < s_2 < \cdots < s_{2n}\} \subset [0, \alpha]$. We define the times of up crossings and down crossings over $[a, b]$, as follows:

$$\sigma_1 = s_1,$$

$$\sigma_{2k} = \begin{cases} \inf \{s: s \in S, s > \sigma_{2k-1}, X(s) \leq a\} & \text{if } \{\ \} \neq \emptyset \\ \alpha & \text{if } \{\ \} = \emptyset, \end{cases}$$

$$\sigma_{2k+1} = \begin{cases} \inf \{s: s \in S, s > \sigma_{2k}, X(s) \geq b\} & \text{if } \{\ \} \neq \emptyset \\ \alpha & \text{if } \{\ \} = \emptyset. \end{cases}$$

The condition of bounded variation on $\mu_X^\alpha$ implies

$$K_\alpha = |\mu_X^\alpha|(]0, \alpha] \times \Omega) \geq \sum_{k=1}^{n-1} |E(X_{\sigma_{2k+1}} - X_{\sigma_{2k}})|.$$

Because of the positivity of $X_{\sigma_{2k+1}}(\omega) - X_{\sigma_{2k}}(\omega)$, except, maybe, for one $k$,

$$K'_\alpha = E|X_\alpha - a| + K_\alpha \geq \sum_{k=1}^{n-1} E|X_{\sigma_{2k+1}} - X_{\sigma_{2k}}| \geq \sum_{k=1}^{n-1} j \cdot (b - a) P(F_{S,j}^{(a,b)})$$

where

$$F_{S,j}^{(a,b)} = \{w: j \text{ among the } X_{\sigma_{2k+1}} - X_{\sigma_{2k}} \text{ are } > 0\}.$$

We may then consider a dense denumerable set $S$ in $[0, \alpha]$, and an increasing sequence $(S_n)$ of finite subsets of $S$ such that $S = \bigcup_n S_n$, and the corresponding sets $F_{S_n,j}^{(a,b)}$. From

$$P(F_{S_n,j}^{(a,b)}) \leq \frac{K'_\alpha}{j \cdot (b - a)}$$

we deduce that the set $\Omega_\infty$ of trajectories having infinitely many crossings over $[a, b]$, on the set $S$, has probability 0.

The property of the theorem is deduced from there, by the usual argument.

## 4. Decomposition theorems

We recall that a real additive function $\mu$ on an algebra $\mathcal{U}$ of sets is the difference of two positive additive functions $\mu^+$ and $\mu^-$ if and only if $\mu$ is of bounded variation on many set $A$ of $\mathcal{U}$, i.e., if, for all $A \in \mathcal{U}$,

$$|\mu|(A) = \sup \left\{ \sum_i \mu(A_i): (A_i) \text{ being any finite partition of } A, A_i \in \mathcal{U} \right\} < \infty,$$

one has $|\mu|(A) = \mu^+(A) + \mu^-(A)$. One may view this as a Riesz decomposition in the ordered space (completely reticulated: see Bourbaki Integration I Section 1) of relatively bounded linear form on the space of step functions on $\mathcal{U}$. Every simply additive function $\mu$, with bounded variation, is isomorphically (linearly and for the order) associated with a linear form $\tilde{\mu}$ by

$$\left\langle \tilde{\mu}, \sum_i 1_{A_i} \right\rangle = \sum_i \alpha_i \mu(A_i).$$

We recall too, that the $\sigma$-additive-functions on $\mathscr{U}$ are easily seen to constitute a Riesz Band (cf. Bourbaki, reference above).

The band of the simply additive functions, which are orthogonal ("étrangères" to all $\sigma$-additive-functions consists of all the so-called "purely finitely additive functions," which may be characterized in the following way:

$\mu$ is purely finitely additive, i.e., if $0 < \nu < |\mu|$ and $\nu$ is $\sigma$-additive then $\nu = 0$. Every finitely additive measures with bounded variation is the sum $\mu_\sigma + \mu_s$ of a $\sigma$-additive measure and a purely finitely additive one. The decomposition is unique.

These decomposition theorems give us immediately the following theorem.

4.1. THEOREM 3. *Every real quasi-martingale $X$ on $[0, \alpha]$ is the difference of two positive $L^1$-bounded supermartingales $X^+$ and $X^-$: $X_t = X_t^- - X_t^+$. The decomposition is unique if we assume $X_\alpha = 0$ and impose $X^+(\alpha) = X^-(\alpha) = 0$ and for every $\varepsilon > 0$ there exists a sequence $\tau_1 < \cdots < \tau_n$ of finitely valued stopping times with values in $[0, \alpha]$ and a partition $I \cup J$* of $\{1, \ldots, n\}$ *such that*

$$\sum_{i \in I} E(X_{\tau_i}^+ - X_{\tau_{i+1}}^+) + \sum_{j \in J} E(X_{\tau_j}^- - X_{\tau_{j+1}}^-) \leq \varepsilon. \tag{4.1.2}$$

*Proof.* Decompose $\mu_X^\alpha = \mu_X^{\alpha+} - \mu_X^{\alpha-}$, and take Radon-Nikodym derivatives

$$X_t^+ = \left(\frac{d_{\nu_t^+}}{dP}\right)_{\mathscr{F}_t}, \quad X_t^- = \left(\frac{d_{\nu_t^-}}{dP}\right)_{\mathscr{F}_t}$$

of the measures $\nu_t^+$ and $\nu_t^-$ defined on $\mathscr{F}_t$ by

$$\nu_t^+(F) = \mu_X^{\alpha,+}(]t, \alpha] \times F) \quad \text{and} \quad \nu_t^-(F) = \mu_X^{\alpha,-}(]t, \alpha] \times F).$$

The decomposition $X_t = X_t^- - X_t^+$ follows from ($X_\alpha$ being zero)

$$\mu_X^\alpha(]t, \alpha] \times F) = -E(1_F \cdot X_t) = E(1_F \cdot X_t^+) + E(1_F \cdot X_t^-);$$

as for the uniqueness condition of the theorem, it only says that

$$\inf(\mu_X^{\alpha,+}, \mu_X^{\alpha,-}) = 0.$$

4.2. *Extension of $\bar{\mu}_X^\infty$.* Let us suppose that $X$ is a Banach valued $F$-process on $[0, \infty]$ (with the convention here always made that $X_\infty = 0$). It follows immediately from $|\bar{\mu}_X^\infty|([0, \infty] \times \Omega) < \infty$ that for all

$$F \in \bigcup_{t \in \mathbf{R}^+} \mathscr{F}_t, \quad (\bar{\mu}_X^\infty(]t, \infty] \times F))_{t \in \mathbf{R}^+}$$

is Cauchy when $t \to \infty$, and then $\lim_{t \to \infty} \bar{\mu}_X^\infty(]t, \infty] \times F) = -\lim_{t \to \infty} E(1_F \cdot X_t)$ exists.

It is then clear that if we set

$$\bar{\mu}_X(\{\infty\} \times F) = \lim_{t\to\infty} \bar{\mu}_X^\infty(]t, \infty] \times F)$$

and

$$\bar{\mu}_X(]s, t] \times F) = \bar{\mu}_X^\infty(]s, t] \times F) \quad \text{whenever } s < t \in [0, +\infty],$$

we define an additive extension $\bar{\mu}_X$ of $\bar{\mu}_X^\infty$ to the algebra called $\overline{\mathscr{U}}$ in Section 1, and those additive measures have same total variation.

It is evident that when $X$ is real, $\bar{\mu}_X$ is the difference of the extensions $\bar{\mu}_X^+$ and $\bar{\mu}_X^-$ of $\bar{\mu}_X^{\infty,+}$ and $\bar{\mu}_X^{\infty,-}$. As those extensions are such that $\inf(\bar{\mu}_X^+, \bar{\mu}_X^-) = 0$, they are respectively the positive part and negative part of $\bar{\mu}_X$.

From these definitions we immediately have the following:

PROPOSITION 2. *$(X_t)_{t\in\mathbf{R}^+}$ is a martingale if and only if $|\mu_X|(]0, \infty[ \times \Omega) = 0$. $(X_t)_{t\in\mathbf{R}^+}$ is a potential (i.e., a positive supermartingale such that $\lim_{t\to\infty} E(X_t) = 0$) if and only if $\bar{\mu}_X \leq 0$ and $\bar{\mu}_X(\{\infty\} \times \Omega) = 0$.*

*Every real quasi-martingale $X$ can be written uniquely as*

$$X = M + V^- - V^+$$

*where $V^-$ and $V^+$ are potentials verifying condition (4.1.2) ($X^+$ and $X^-$ being replaced by $V^+$ and $V^-$ in the statement of this condition), and $M$ is a martingale.*

The decomposition part of Proposition 2 cannot be stated in the same way when $X$ is Banach valued. We can only decompose $\bar{\mu}_X$ as the sum of a measure (finitely additive) which gives a mass zero to every set in $\overline{\mathscr{U}}$ which are included in $\mathbf{R}^+ \times \Omega$, and a measure which gives a mass zero to every set $\{\infty\} \times F$ where $F \in \bigcup_t \mathscr{F}_t$. This decomposition is clearly unique. If the Banach space has the Radon-Nikodym property, then a martingale $M$ on $\mathbf{R}^+$ is uniquely associated with the first measure, while the second one generates a unique quasi-martingale $V$ with the property

$$\lim_{t\to\infty} E(1_F \cdot V_t) = 0 \quad \text{for all } F \in \bigcup_{t\in\mathbf{R}^+} \mathscr{F}_t. \tag{4.2.1}$$

This leads to the following:

DEFINITION. A Banach valued process $V$ on $\mathbf{R}^+$, which is a quasi-martingale and for which (4.2.1) holds will be called a quasi-potential.

From there, we can now immediately state the following:

PROPOSITION 2′. *Let $X$ be a Radon-Nikodym Banach valued process on $\mathbf{R}^+$ which is a quasi-martingale on $[0, \infty]$. Then there exists a decomposition $X = M + V$ of $X$ where $M$ is a martingale and $V$ a quasi-potential.*

*The decomposition is unique in the following sense: if $X = M' + V'$, then for all $t \in \mathbf{R}^+$, $M_t' = M_t$ a.s. and $V_t = V_t'$ a.s.*

4.3. *Asymptotic behavior of Banach valued quasi-potentials.* For a real (positive) potential we have more than (4.2.1). In fact the convergence holds uniformly in $F$, or if we prefer $\lim_{t\to\infty} E\|V_t\| = 0$. This clearly does not follow immediately from the definition of a general quasi-potential. In the particular case of a real quasi-potential $V$, this follows from the decomposition in Proposition 2: if $\lim_{t\to\infty} E(1_F \cdot X_t) = 0$, the martingale $M$ in this decomposition is zero and

$$E|X_t| \leq E|V_t^-| + E|V_t^+|.$$

In this case it is also clear that $\lim_{t\to\infty} |\bar{\mu}_V^\infty|(]t, \infty] \times \Omega) = 0$. In the general case we have the following:

THEOREM 4. *If $V$ is a R. N. Banach valued quasi-potential, taking its values in* **B**, *then*

(a) $\lim_{t\to\infty} E(\|V_t\|) = 0$,
(b) $\lim_{t\to\infty} V_t = 0$ a.s.,
(c) $\lim_{t\to\infty} |\bar{\mu}_V^\infty|(]t, \infty] \times \Omega) = 0$.

*Proof.* As $V$ is a quasi-martingale, (b) follows immediately from (a). Let us prove (a).

The function $t \to |\bar{\mu}_V^\infty|(]t, \infty] \times \Omega)$ being decreasing, for every $\varepsilon$, there exists a $t_\varepsilon$ such that for all $s > t > t_\varepsilon$,

$$|\bar{\mu}_V^\infty|(]t, s] \times \Omega) < \varepsilon.$$

For every $t > t_\varepsilon$ and partition $\{F_i\}$ of $\Omega$ with sets in $\mathscr{F}_t$, we then have

$$\sum_i \|E(1_{F_i} \cdot V_t)\| = \lim_{s\to\infty} \sum_i \|E 1_{F_i} \cdot (V_s - V_t)\| < \varepsilon.$$

Let $x_i'$ be any continuous linear form on **B**, with norm $\leq 1$. Then

$$\sum_i |E\langle x_i', 1_{F_i} V_t\rangle| < \varepsilon.$$

This proves that $E\|V_t\| \leq \varepsilon$, for all $t \geq t_\varepsilon$.

We now prove (c). For every $\varepsilon$, take the same $t_\varepsilon$ as previously. Let

$$(]s_i, t_i] \times F_i)$$

be a partition of $]t, \infty] \times \Omega$, $t \geq t_\varepsilon$ such that

$$|\bar{\mu}_V^\infty|(]t, \infty] \times \Omega) \leq \sum \|\bar{\mu}_V^\infty(]s_i, t_i] \times F_i)\| + \varepsilon.$$

If $s = \max_i s_i$, by refining the partition if necessary, we see that

$$|\bar{\mu}_V^\infty|(]t, \infty] \times \Omega) \leq |\mu_V^\infty|(]t, s] \times \Omega) + \sum_{i: t_i = +\infty} \|E(1_{F_i} \cdot V_s)\| + \varepsilon \leq E\|V_s\| + 2\varepsilon \leq 3\varepsilon.$$

This proves (c).

In the sequel we will need the following:

4.4. LEMMA (Orey). *Let* $(\mathscr{F}_n)$ *be a decreasing sequence of* $\sigma$*-algebras with* $\mathscr{F} = \bigcap_n \mathscr{F}_n$.

*If the variables* $X_n$ *satisfy*

$$\sum_n E\|E(X_n - X_{n+1} \mid \mathscr{F}_{n+1})\| < \infty,$$

*then they are uniformly integrable.*

*Proof.* When $X$ is real, we refer to [11] for the proof, or the preceding theorem may be applied, and we can then use uniform integrability properties of supermartingales. If $X$ is Banach valued, we remark as in the corollary to Theorem 1, that the condition of the theorem implies

$$\sum_n E|E(\|X_n\| - \|X_{n+1}\| \mid \mathscr{F}_{n+1})| < \infty;$$

then the uniform integrability of $(\|X_n\|)$ follows from what precedes.

## 5. Characterization of $\sigma$-additive and purely finitely additive parts

5.1. *$\sigma$-additivity on* $\mathscr{P}_\infty$. We consider here the case where $X$ being a quasi-martingale on every bounded interval $[0, \alpha]$, $\bar{\mu}_X^\infty$ is of bounded variation only on the ring $\mathscr{P}_\infty$ generated by bounded predictable rectangles. So we take only its restriction $\mu_X^\infty$ to $\mathscr{P}_\infty$ into consideration.

DEFINITION. We recall that a process $X$ on $[0, \infty[$ is said to be *of class D* if the set $\{X_T\colon T \text{ any finite-stopping time}\}$ is uniformly integrable. It is said to be *locally of class D* if for every $\alpha < \infty$, the set $\{X_T\colon T \text{ any stopping time} \le \alpha\}$ is uniformly integrable.

PROPOSITION 3. *If* $\bar{\mu}_X^\alpha$ *is* $\sigma$*-additive on* $\overline{\mathscr{U}}_\alpha$, $\alpha \in \overline{\mathbf{R}}^+$, *and if* $X$ *is almost surely right continuous, then for every stochastic interval* $]T, \alpha]$, *its* $\sigma$*-additive extension to* $\overline{\mathscr{P}}_\alpha$ *has the property*

$$\bar{\mu}_X^\alpha(]T, \alpha]) = E(X_\alpha) - E(X_T).$$

*Proof.* This proposition is true for finitely valued stopping time $T$ according to Proposition 1. Using for a stopping time $T$, an upper decreasing approximating sequence $(T_n)$, of finitely valued stopping time $T$, we have then, for the $\sigma$-additivity,

$$\bar{\mu}_X^\alpha(]T, \alpha]) = \lim_n [E(X_\alpha) - E(X_{T_n})].$$

But as $\lim_n X_{T_n} = X_T$ a.s., applying Lemma 4.4 to the variable $X_{T_n}$ and $\sigma$-algebras $\mathscr{F}_{T_n}$ we get the convergence of $X_{T_n}$ toward $X_T$ in $L^1$, and from there Proposition 3.

The necessary part of the following theorem can be deduced from the Doob-Meyer decomposition theorem, as proved for Banach valued processes in [12], at least when $\mathbf{B}$ is a R. N. Banach space. We give here a simple direct proof.

THEOREM 5. *Let $X$ be a Banach valued process, right continuous, which is a quasi-martingale on every bounded interval $[0, \alpha]$. Then $\mu_X^\infty$ is $\sigma$-additive if and only if $X$ is locally of class $D$.*

*Proof. Necessity.* Using the corollary to Theorem 1 we may assume that $X$ is real. Let $\alpha < \infty$ and let $\bar{\mu}_X^\alpha$ be the restriction of $\mu_X^\infty$ to

$$\bar{\mathscr{P}}_\alpha = \mathscr{P}_\infty \cap [0, \alpha] \times \Omega.$$

If $\bar{\mu}_X^\alpha$ is $\sigma$-additive, its positive and negative parts are $\sigma$-additive too. Let us consider the positive part associated with the positive supermartingale $X^-$. From the $\sigma$-additivity of $\mu_X^+$, $\lim_{t \downarrow s} E(X_t^- - X_s^-) = 0$. Then there exists a right-continuous version of $X^-$.

We define the stopping times $R_n = \inf \{t: X_t^- > n\}$. For $u < \alpha$,

$$P\left[\bigcap_n ]R_n \wedge u, u] \neq \emptyset\right] = P\left(\bigcap_n [R_n < u]\right) = 0.$$

From the $\sigma$-additivity of $\mu_{X^-}$ and as $\bigcap_n ]R_n \wedge u, u]$ has measure $\mu_{X^-}$ zero, being evanescent, $\lim_n E(X_u - X_{R_n \wedge u}) = 0$. Using the same argument as in Meyer [10, p. 138] we will prove that this implies the uniform integrability of $\{X_T: T \leq u\}$.

Let us define

$$T'(w) = \begin{cases} T(w) & \text{if } X_{T(w)} > n \\ u & \text{if } X_{T(w)} \leq n. \end{cases}$$

One has $R_n \wedge u \leq T'$ and then

$$E(X_{R_n \wedge u}^-) \geq \int X_{T'}^- \, dP \geq \int_{[X_T^- \geq n]} X_T^- \, dP + \int_{[X_T^- < n]} X_u^- \, dP.$$

Then

$$\int_{[u < R_n]} X_u^- \, dP + \int_{[u \geq R_n]} X_{R_n}^- \, dP - \int_{[X_T^- < n]} X_u^- \, dP \geq \int_{[X_T^- \geq n]} X_T^- \, dP.$$

As $[u < R_n] \subset [X_T^- \leq n]$ the positivity of $X^-$ implies

$$\int_{[u \geq R_n]} X_{R_n}^- \, dP \geq \int_{[X_T^- \geq n]} X_T^- \, dP$$

which proves the uniform integrability property. We do the same reasoning for $X_T^-$.

*Sufficiency.* See [12, p. 50].

5.2. *$\sigma$-additivity on $\bar{\mathscr{P}}_\infty$.* The following theorems are mere corollaries of Theorem 5.

THEOREM 5′. *Let $X$ be a right continuous quasi-martingale on $[0, \infty]$. Then $\bar{\mu}_X$ is $\sigma$-additive if and only if $X$ is of class $D$.*

THEOREM 5″. *Let $X$ be a right continuous process which is a quasi-martingale on every $[0, \alpha]$, $\alpha < \infty$. Let $T$ be a finitely valued stopping time such that $\{X_\sigma: \sigma < T, \sigma \text{ stopping time}\}$ is uniformly integrable. Then $\bar{\mu}_X$ restricted to $[0, T] \cap \bar{\mathscr{P}}_\alpha$ is $\sigma$-additive.*

*Example.* The following example, suggested by the referee, illustrates what is happening when $X$ is not of class $D$.

Take $\Omega = ]0, 1]$, $P$ the Lebesgue measure, $(\mathscr{F}_t)$ the right continuous "interpolation," with *nullsets thrown in*, of $(\mathscr{F}_n^\circ)$, where $\mathscr{F}_n^\circ$ is generated by the intervals $]k \cdot 2^{-n}, (k+1) \cdot 2^{-n}]$. Now consider the right-continuous martingale $(X_t)$ defined by $X_n = 2^n$ on $]1 - 2^{-n}, 1]$ and 0 elsewhere. The martingale $X$ is clearly not of class $(D)$ on $[0, +\infty]$.

Let us see that $\mu_X$ has no $\sigma$-additive extension. In fact $\{(\infty, 1)\} \in \bar{\mathscr{U}}_\infty$ with $\mu_X\{(\infty, 1)\} = 0$ as an evanescent set, while

$$\{(\infty, 1)\} = \bigcap_n (]n, \infty] \times ]1 - 2^n, 1])$$

with

$$\mu_X(]n, \infty] \times ]1 - 2^{-n}, 1]) = 1 \quad \text{for all } n.$$

5.3. *Pure simple additivity of $\bar{\mu}_X$.* Before characterizing in term of $X$ the property for $\bar{\mu}_X$ to be purely finitely additive, we will extend the notion of purely finite additivity to some class of vector valued measures. This will make possible for us to deal at the same time with the real and the Banach valued processes.

PROPOSITION 4. *Let $E$ be a Banach space, $\mathscr{A}$ an algebra of subsets of a set $A$ and $m$ a finitely additive function on $\mathscr{A}$ taking its values in $E$, and with bounded variation $|m|$. Then there exist two uniquely defined functions $m_s$ and $m_\sigma$ on $\mathscr{A}$, with values in $E$, such that $m = m_\sigma + m_s$ and with the following properties:*

(a) *$m_\sigma$ is $\sigma$-additive and with $\sigma$-additive finite variation $|m_\sigma|$.*

(b) *$m_s$ is finitely additive and with purely finitely additive variation $|m_s|$.*

*Moreover $|m| = |m_\sigma| + |m_s|$.*

*Proof.* Let $\mu$ be the $\sigma$-additive part and $\nu$ the purely finitely additive part of $|m|$.

Let $(A_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{A}$ such that $\mu(A_n) + \nu(A - A_n) \leq 1/n$. Because $|m[B \cap (A_n \triangle A_{n+p})]| \leq 2/n$ for all $B \in \mathscr{A}$ we can define

$$m_\sigma(B) = \lim_{n \to \infty} m(B \cap A_n).$$

The following limit exists as a consequence:

$$m_s(B) = \lim_{n \to \infty} m(B \cap A_n).$$

We check immediately that $|m_\sigma| = \mu$ and $|m_s| = \nu$. It follows that $m_\sigma$ is $\sigma$-additive.

For every $x'$ in the dual of $E$, $\langle m_\sigma, x' \rangle$ is clearly the $\sigma$-additive part of the real measure $\langle m, x'\rangle$, while $\langle m_s, x'\rangle$ is the purely finitely additive part. The unicity of the decomposition follows.

DEFINITION. The additive Banach valued function $m$ on $\mathscr{A}$ is said to be purely finitely additive if in the decomposition of Proposition 4, $m_\sigma = 0$.

THEOREM 6. *Let $X$ be a right continuous process, taking its values in a R. N. Banach space $E$, which is a quasi-martingale on $[0, +\infty]$. (We recall the convention made throughout the paper: for every process $X$, $X_\infty = 0$, to be distinguished from $X_\infty^- = \lim_{t\to\infty} X_t$ if this limit exists almost surely.)*

*Then the following conditions are equivalent:*

(i) *$\bar{\mu}_X$ is purely finitely additive.*

(ii) *There exists a sequence $\sigma(n)$ of stopping times such that*

$$\lim_{n\to\infty} P[\sigma(n) < \infty] = 0,$$

*and such that for every $n$, $(X_{t\wedge\sigma(n)})_{t\in[0,+\infty]}$ is a martingale on the compact half line $[0, +\infty]$.*

(iii) *$X$ is a local martingale on $[0, \infty[$, (i.e., there exists an increasing sequence* $(\sigma(n))_{n\in^+\mathbf{N}}$ of stopping times such that $\lim_{n\to\infty} \sigma(n) = +\infty$ a.s., and $(X_{t\wedge\sigma(n)})_{t\in\mathbf{R}^+}$ *is a martingale on $\mathbf{R}^+$ for every $n$), and moreover*

$$\lim_{t\to\infty} X_t = 0 \quad \text{a.s.} \tag{6.3.1}$$

*Proof.* To prove (i) implies (ii) let us define $R_n = \inf\{t: |X_t| > n\}$ (with the convention $\inf\emptyset = +\infty$) and $Y_t^n = X_{t\wedge R_n}$. Were $R_n$ a finitely or denumerably valued stopping time, we would have immediately the inequality

$$|\bar{\mu}_{Y^n}^\infty| \leq |\bar{\mu}_X^\infty|. \tag{6.3.2}$$

But, if we take an upper approximation of $R_n$ by denumerably valued stopping times and go to the limit, we can approximate the values of the variation of $\bar{\mu}_{Y^n}^\infty$ and $\bar{\mu}_X^\infty$ in such a way that we can deduce that (6.3.2) actually is true. But as $Y^n$ is trivially a quasi-martingale of class $D$ on $[0, \infty]$, $\bar{\mu}_{Y^n}^\infty$ is at the same time purely finitely additive and $\sigma$-additive. It is then null, and $Y^n$ is a martingale on $[0, \infty]$.

To prove (ii) implies (iii) we define

$$A_n = [\sigma_n = +\infty] \quad \text{and} \quad U = \lim_{t\to\infty, t\in R^+} X_{t\wedge\sigma(n)} \quad \text{a.s.}$$

Since $(X_{t\wedge\sigma(n)})$ is a martingale on $[0, \infty]$, $0 = E(X_{\infty\wedge\sigma(n)} - U \mid \mathscr{F}_\infty^-)$. Therefore

$$E(U\cdot 1_{A_n} \mid \mathscr{F}_\infty^-) = 0$$

And as $U\cdot 1_{A_n}$ is $\mathscr{F}_\infty^-$-measurable, $\lim_{t\to\infty, t\in\mathbf{R}^+} X_t = 0$ a.s. on $A_n$. As $\lim_n P(A_n) = 0$, (6.3.1) is thus proved.

Let us prove now that (iii) implies (i). We use the decomposition $X = M + V$

of Proposition 2′. It is known (and easy to check) that the $\sigma$-additive part of $\bar{\mu}_M$ is $\bar{\mu}_{M^\infty}$ where $(M_t^\infty)_{t\in\mathbf{R}^+}$ is the uniformly integrable martingale

$$M_t^\infty = E\left(\lim_{t\to\infty,\, t\in\mathbf{R}^+} M_t \mid \mathscr{F}_t\right).$$

Theorem 4 and condition (6.3.1) prove that $M^\infty = 0$, and the purely finite additivity of $\bar{\mu}_M$. We have now only to prove that a quasi potential $V$, which is a local martingale, is such that $\bar{\mu}_V$ is purely finitely additive, that is every $\sigma$-additive positive measure $\nu$ such that $\nu \leq |\mu_V|$ is zero.

Let $Y$ be a positive supermartingale associated with $\nu$. Since $|\bar{\mu}_Y| \leq |\bar{\mu}_V|$, $Y$ has to be a potential and a local martingale. Then, as $|\bar{\mu}_Y|$ is $\sigma$-additive, $Y$ is, according to Theorem 5, a local martingale of class D, then a martingale. As $Y$ is at the same time a uniformly integrable martingale and a potential it is zero.

*Example.* In the example given at the end of 5.2, $\mu_X$ is actually purely finitely additive, while $\lim_{t\to\infty} X_t = 0$ almost surely, which is a typical example of what is stated in Theorem 6.

## 6. Final decomposition theorem

We may now summarize the previous results in the following.

THEOREM 7 (1st decomposition). (1) *Let $X$ be a right continuous process, taking its values in a R. N. Banach space, which is a quasi-martingale on* $[0, \infty]$. *Then there is a decomposition of $X$,*

$$X = M^\sigma + M^f + V^\sigma + V^f,$$

*where*

(a) *$M^\sigma$ is a uniformly integrable martingale on* $\mathbf{R}^+$,

(b) *$M^f$ is a martingale such that* $\lim_{t\to\infty} M_t^f = 0$ a.s.,

(c) *$V^\sigma$ is a quasi-potential of class D,*

(d) *$V^f$ is a quasi-potential which is at the same time a local martingale.*

*The decomposition is unique up to a modification (i.e., if $M'^\sigma + M'^f + V'^\sigma + V'^f$ is another decomposition, for all t,* $M_t'^\sigma = M_t^\sigma$ a.s. $M_t'^f = M_t^f$ a.s. *etc.*).

(2) *In case $X$ is a real quasi-martingale on* $[0, \infty]$, *we moreover can write*

$$V^\sigma = V^{\sigma+} - V^{\sigma-}, \qquad V^f = V^{f+} - V^{f-},$$

*where $V^{\sigma+}$, $V^{\sigma-}$ are potential of class D, while $V^{f+}$ and $V^{f-}$ are potentials which are local martingales.* (*As to the unicity of such a decomposition, see Theorem* 3.)

*Proof.* According to the equivalence previously proven between properties of processes and properties of measures, the theorem is an immediate consequence of the decomposition of $\mu_X$ (unique) into a finitely additive measure with bounded variation on $\{\infty\} \times \Omega$, a finitely additive measure with bounded variation on $\mathbf{R}^+ \times \Omega$, and next the decomposition of those two additive measures into their purely finitely additive and $\sigma$-additive part.

THEOREM 8 (2nd decomposition). *Let $X$ be a right continuous process, taking its values in a R. N. Banach space, which is a quasi-martingale on* $[0, \infty]$. *Then there is a decomposition of $X$,*

$$X = M + A + M^f + V^f,$$

*where*

(a) *$M$ is a uniformly integrable martingale on* $\mathbf{R}^+$,

(b) *$A$ is a predictable process, the paths of which have bounded variation, and such that* $A_0 = 0$,

(c) *$M^f$ is a martingale such that* $\lim_{t\to\infty} M_t^f = 0$,

(d) *$V^f$ is a quasi-potential, which is at the same time a local martingale.*

*The decomposition is unique up to a modification.*

*If moreover $X$ is real, $A$ can be written as the difference of two predictable increasing processes.*

*Proof.* We take the $\sigma$-additive part of $\bar{\mu}_X$, that is the part associated with the process $M^\sigma + V^\sigma$ which is of class $D$, and apply the decomposition theorem of Doob-Meyer type for vector valued quasi-martingales, as due to Pellaumail (cf. [12]).

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UNIVERSITY OF RENNES
RENNES, FRANCE