# ON THE HIERARCHY OF W. KRIEGER

BY

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In his paper "On ergodic flows and the isomorphism of factors" W. Krieger introduces a hierarchy $\Delta(n)$, $n \in \mathbf{N}$, labelling different weak equivalence classes of ergodic transformations of type $\mathrm{III}_0$. The aim of the present paper is to answer a question of W. Krieger, namely to prove the existence of a weak equivalence class of type $\mathrm{III}_0$ not in the above hierarchy. There is a close link between this hierarchy and the discrete decomposition $M = W^*(\theta, N)$ of factors of type $\mathrm{III}_0$ [2, part V]. In fact in such a decomposition the restriction of $\theta$ to the center of $N$ is unique, up to an induction on a non-zero projection in the sense of Kakutani [2, Theorem 5.4.2]. In particular the weak equivalence class of this restriction is uniquely associated to $M$. Starting from a weak equivalence class $\tau$ we get a factor $M$ by the group measure space construction, hence if $\tau$ is of type $\mathrm{III}_0$ we can associate to it the derived weak equivalence class $\tau'$ corresponding to discrete decompositions of $M$. A weak equivalence class $\tau$ belongs to the hierarchy if and only if $\tau^{(n)}$ fails to be of type $\mathrm{III}_0$ for some $n$.

We compute the discrete decomposition of a large class of infinite tensor product of type I factors. In fact we show that any of the automorphisms $T_p$ of W. Krieger [9, p. 87] which are strictly ergodic, appear as $\theta$/Center of $N$ in the discrete decomposition of some infinite tensor product of type I factors. Also we produce a weak equivalence class $\tau$ of transformation $T_p$ of type $\mathrm{III}_0$ such that $\tau' = \tau$ and hence not belonging to the above hierarchy.

We shall need some standard notations:

(1) Let $(k_i)_{i=1,2\ldots}$ be a sequence of integers, $X_i = \{n, 1 \leq n \leq k_i\}$ a totally ordered set with $k_i$ elements for each $i \in \mathbf{N}$, and $p = (p_i)_{i \in \mathbf{N}}$ a sequence of probability measures, $p_i$ on $X_i$ for each $i \in \mathbf{N}$. Then, as in [9, p. 87] we define an automorphism $T_p$ of the measure space $X = \prod_{i=1}^{\infty} (X_i, p_i)$ by setting, for $x = (x_i)_{i \in \mathbf{N}} \in X$,

$$\begin{aligned} I(x) &= \min \{i \in \mathbf{N}, x_i < k_i\}, \\ (T_p(x))_i &= 1 && \text{if } i < I(x) \\ &= x_i + 1 && \text{if } i = I(x) \\ &= x_i && \text{if } i > I(x). \end{aligned}$$

(2) Let $\{\lambda_{\nu,j}\}_{j=1,\ldots,n_\nu, \nu \in \mathbf{N}}$, be an eigenvalue list, i.e., for each $\nu$, $\lambda_\nu$ is a probability measure on a set $E_\nu$ with $n_\nu$ elements. Then for each $\nu$ we let $M_\nu$ be

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the algebra of $n_\nu \times n_\nu$ matrices over $\mathbf{C}$, with its canonical system of matrix units $(e^\nu_{i,j})_{i,j \in E_\nu}$ and the state $\phi_\nu = \mathrm{Tr}((\sum \lambda_{\nu,j} e_{jj})\cdot)$. For any finite subset $I$ of $\mathbf{N}$ we put $E(I) = \prod_{\nu \in I} E_\nu$, $\lambda_I = \prod_{\nu \in I} \lambda_\nu$ and we let $(e^I_{r,s})_{r,s \in E(I)}$ be the canonical system of matrix units in $M(I) = \otimes_{\nu \in I} M_\nu$. Finally $r(I)$ is the ratio set

$$r(I) = \left\{\frac{\lambda_{I,p}}{\lambda_{I,q}},\ p,\ q \in E(I)\right\}$$

and $|r(I)|$ the largest element of $r(I)$.

THEOREM 1. *Let $\{\lambda_{\nu,j}\}_{j=1,\dots,n_\nu,\nu \in \mathbf{N}}$ be an eigenvalue list such that for each $\nu \in \mathbf{N}$ the ratio set $r(\{\nu\})$ intersects the interval $[|r(\{1,\dots,\nu-1\})|^{-2}, |r(\{1,\dots,\nu-1\})|^2]$ in the point 1 only. Let $M = \otimes_\nu (M_\nu, \phi_\nu)$ be the infinite tensor product corresponding to $\lambda$. For each $\nu$ let $X_\nu$ be the totally ordered set of values of $\lambda_\nu$, and $p_\nu$ be the image on $X_\nu$ of the measure $\lambda_\nu$.*

*Then if $X = \prod_{\nu=1}^\infty (X_\nu, p_\nu)$ is a Lebesgue measure space, $M$ is a factor of type* $\mathrm{III}_0$ *which admits a discrete decomposition $M = W^*(\theta, N)$ in which the restriction of $\theta$ to the center of $N$ is equal to $T_p$ acting on $L^\infty(X)$.*

COROLLARY 2. *Let $(X_\nu, p_\nu)_{\nu \in \mathbf{N}}$ be a sequence of finite totally ordered probability spaces such that $(X, p) = \prod_{\nu=1}^\infty (X_\nu, p_\nu)_{\nu \in \mathbf{N}}$ is a Lebesgue measure space. Then $T_p$ acting on $L^\infty(X, p)$ is the restriction of $\theta$ to the center of $N$ in a discrete decomposition $M = W^*(\theta, N)$ of an infinite tensor product $M$ of type* I *factors,* (*$M$ of type* $\mathrm{III}_0$).

*Proof.* One has to produce probability spaces $E_\nu$, $\lambda_\nu$ satisfying the condition of Theorem 1, and such that $X_\nu, p_\nu$ is the range of $\lambda_\nu$. Replace each point, say $i$, of $X_\nu$, with measure $p_\nu(i)$ by sufficiently many points $i_1, \dots, i_{l_i}$ with $\lambda_\nu(i) = (1/l_i)p_\nu(i)$. Clearly if $l_i$ increases sufficiently fast when $i$ decreases, the image of $\lambda_\nu$ is isomorphic to $X_\nu$, $p_\nu$ as an ordered probability space, and the smallest ratio $> 1$ in $r(\{\nu\})$ is as large as desired.

COROLLARY 3. *There exists a weak equivalence class $\tau$ of ergodic transformations, which is of type* $\mathrm{III}_0$ *and satisfies $\tau' = \tau$.*

*Proof.* We just have to construct an eigenvalue list $\{\lambda_{\nu,j}\}_{j=1,\dots,n_\nu}$ such that the condition of Theorem 1 is fulfilled and the derived list $\{p_{\nu,l}\}_{l=1,\dots,k_\nu}$ gives a transformation $T_p$ weakly equivalent to $T_\lambda$ and not of type I. Those conditions will be fulfilled if we require that $E_\nu$, $\lambda_\nu$ is the same probability space as the range $X_{\nu+1}$, $p_{\nu+1}$ of $\lambda_{\nu+1}$ and that the largest element in the range of $\lambda_\nu$ is smaller than 1/2, for all $\nu$. (See [1, p. 61]). Construct $E_\nu$, $\lambda_\nu$ by induction, $E_{\nu+1}$, $\lambda_{\nu+1}$ being obtained by replacing each point, say $i$, of $E_\nu$ by $l_i$ points $i_{l'}$, $1 \le l' \le l_i$, $\lambda_{\nu+1}(i_{l'}) = (1/l_i)\lambda_\nu(i)$.

COROLLARY 4. *There exists a weak equivalence class $\tau$ of ergodic transformation of type* $\mathrm{III}_0$ *which does not belong to the hierarchy $\bigcup_{n \in \mathbf{N}} \Delta(n)$* [5, part 7].

*Proof.* By [3, part 2], for an arbitrary factor of type $\mathrm{III}_0$, $M$, the flow arising as the restriction to the center of $M_0$ of the one parameter group of automorphisms $(\theta_t^0)_{t\in\mathbf{R}}$ of $M_0$ in an arbitrary continuous decomposition [6] of $M$ is one of the flows built on the restriction to the center of $N$ of the automorphism $\theta^{-1}$, in an arbitrary discrete decomposition $M = W^*(\theta, N)$ of $M$. With the notations of [5] this means that for each ergodic transformation of type $\mathrm{III}_0$ the flow $W(T)$ is built on an ergodic transformation belonging to the weak equivalence class $\tau'$ derived from the weak equivalence class $\tau$ of $T$. Hence the conclusion follows Corollary 3. Q.E.D.

We now begin to prove Theorem 1. We keep the above notations.

LEMMA 5. *Let $\phi = \otimes_\nu \phi_\nu$ be the canonical product state on $M$.*

(a) *$\phi$ is an almost periodic state, more precisely the $e_{ij}^I$, $I$ finite subset of* $\mathbf{N}$, *$i, j \in E(I)$ are a total family of eigenvectors for $\sigma^\phi$ ($e_{ij}^I \in M(\sigma^\phi, \lambda_{I,j}/\lambda_{I,i})$ $i, j \in E(I)$).*

(b) *1 is an isolated point in the spectrum of $\Delta_\phi$, which is the closure of $r(\mathbf{N}) = \bigcup_{\nu=1}^\infty r(\{1, \ldots, \nu\})$.*

*Proof.* (a) is immediate, using $\sigma_t^\phi = \otimes_{\nu=1}^\infty \sigma_t^{\phi_\nu}$, $t \in \mathbf{R}$.

The formula $Sp\Delta_\phi = \bar{r}(N)$ follows from (a) and the hypothesis on the eigenvalue list $\{\lambda_{\nu,j}\}_{j=1,\ldots,n_\nu}$ gives (b). Q.E.D.

Now let $\nu \in \mathbf{N}$ and $\alpha_1^\nu < \cdots < \alpha_{k_\nu}^\nu$ the various values of $\lambda_\nu$. Put

$$a_j^\nu = \sum_{\lambda_{\nu,i}=\alpha_j^\nu} e_{ii}^\nu.$$

It is easy to check that $a_j^\nu$ is an atom in $C_\nu$ = Center of $M_{\nu,\phi_\nu}$ and is the central support in $M_{\phi_\nu}$ of $e_{ii}^\nu$ if $\lambda_{\nu,i} = \alpha_j^\nu$. Let $P_\nu$ be the restriction of $\phi_\nu$ to $C_\nu$, $(P_\nu(a_j^\nu) = p_\nu(\{j\})$.

LEMMA 6. *Let $C$ be the Center of $M_\phi$; then $C = \otimes_{\nu=1}^\infty (C_\nu, P_\nu)$.*

*Proof.* Let $f \in L^1(\mathbf{R})$ satisfy $\hat{f}(1) = 1$, support $\hat{f} \cap Sp\Delta_\phi = \{1\}$ where $\hat{f}(\lambda) = \int f(t)\lambda^{-it}\,dt$, $\lambda \in \mathbf{R}_+^*$. Then it is easy to check that $\sigma^\phi(f)$ [2, p. 170] restricted to $M_\phi$ is identity. By hypothesis, for $\nu \in \mathbf{N}$, $r_j \in r(\{j\})$, $j = 1, \ldots, \nu$ we have that $\prod_{j=1}^\nu r_j = 1$ implies $r_j = 1$ for all $j = 1, \ldots, \nu$. Writing any $x \in M_\phi$ as weak limit of finite linear combinations of the $\sigma^\phi(f)(e_{ij}^I)$, $i, j \in E(I)$, $I = \{1, \ldots, \nu\}$ we see that $M_\phi = \otimes_{\nu=1}^\infty M_{\phi_\nu}$ hence that Lemma 6 holds. Q.E.D.

LEMMA 7. *Let $\nu$ be an integer, $p_j \in \{1, \ldots, k_j\}$, $j = 1, \ldots, \nu$ with $\mu \in \{1, \ldots, \nu\}$ such that $p_1 = k_1, \ldots, p_{\mu-1} = k_{\mu-1}$, $p_\mu < k_\mu$. Put*

$$a = a_{p_1}^1 \otimes \cdots \otimes a_{p_\nu}^\nu \otimes 1,^1 \quad \textit{and} \quad b = a_{q_1}^1 \otimes \cdots \otimes a_{q_\nu}^\nu \otimes 1$$

[1] 1 stands, for short, for the unit of $\otimes_{\nu'>\nu}(M_{\nu'}, \phi_{\nu'})$.

*where* $q_j = 1$ *for* $1 \le j \le \mu - 1$, $q_\mu = p_\mu + 1$ *and* $q_j = p_j$ *for* $j > \mu$. *Then there exists a partial isometry* $u \in M$, *and a* $\lambda > 1$ *with*:

(1) $u \in M(\sigma^\phi, \{\lambda\})$.

(2) *Central support of* $uu^*$ (*resp.* $u^*u$) *in* $M_\phi$ *equal to* $a$ (*resp.* $b$).

(3) $x \in M(\sigma^\phi, ]1, \infty[)$ *implies* $ax \in M(\sigma^\phi, [\lambda, \infty[)$.

*Proof.* Let $I = \{1, \ldots, v\}$. Choose

$$i = (i_1, \ldots, i_v) \in E(I) \quad \text{and} \quad j = (j_1, \ldots, j_v) \in E(I)$$

such that for each $n$, $\lambda_{n,i_n} = \alpha^n_{p_n}$, $\lambda_{n,j_n} = \alpha^n_{q_n}$. Put $u = e^I_{ij}$,

$$\lambda = \prod_{n=1}^{v} \frac{\lambda_{n,j_n}}{\lambda_{n,i_n}} = \prod_{n=1}^{\mu} \frac{\lambda_{n,j_n}}{\lambda_{n,i_n}}.$$

Now (1) and (2) are easy to check. To prove (3) first observe that the $e^J_{k,l}$ belonging to $M(\sigma^\phi, ]1, \infty[)$ are total in $M(\sigma^\phi, ]1, \infty[)$. Then take $x = e^J_{k,l}$, $I \subset J$. If $ax \neq 0$ it follows that $\lambda_{n,k_n} = \alpha^n_{p_n}$, $n \in \{1, \ldots, v\}$. In particular, for $n \in \{1, \ldots, \mu\}$, $\lambda_{n,k_n}$ is the largest value of $\lambda_n$. Put $r_n = \lambda_{n,l_n}/\lambda_{n,k_n}$; then if $r_n \neq 1$ for some $n > 1$, the condition of Theorem 1 and the hypothesis $\prod_{n \in J} r_n > 1$, show that

$$\prod_{n \in J} r_n > |r\{1, \ldots, \mu\}| \geq \lambda.$$

One then easily checks that all the ratios $\prod_{n=1}^{\mu} \lambda_{n,l_n}/\lambda_{n,k_n}$, with $\lambda_{n,k_n} = \alpha^n_{p_n}$ which are $> 1$ are larger than $\prod_{n=1}^{\mu} \lambda_{n,j_n}/\lambda_{n,i_n} = \lambda$.

*Proof of Theorem* 1. Let $F_\infty$ be a factor of type $I_\infty$, put $P = M \otimes F_\infty$, $\psi = \phi \otimes$ trace. Our hypothesis says that the center of the centraliser of $\phi$ on $M$ is non-atomic; moreover 1 is an isolated point in $Sp\Delta_\phi$ so it follows that $M$ is of type $\mathrm{III}_0$ and that $\psi$ satisfies conditions of Lemma 5.3.2 of [2] on the factor $P$ isomorphic to $M$. We choose as in [2, proof of 5.3.1 p. 238] a unitary $U \in P(\sigma^\psi, ]1, \infty[)$ such that $P^\psi$ and $U$ generate $P$ and $UP^\psi U^* = P^\psi$. Let $v \in \mathbf{N}$, $p_j \in \{1, \ldots, k_j\}$, $j = 1, \ldots, v$. Take $a$ and $b$ as in Lemma 7, as well as $u$ and $\lambda$. We then have:

(1)′ $u \otimes 1 \in P(\sigma^\psi, \{\lambda\})$.

(2)′ Central support of $uu^* \otimes 1$ (resp. $u^*u \otimes 1$) in $P_\psi$ is $a \otimes 1$ (resp. $b \otimes 1$).

(3)′ $x \in P(\sigma^\psi, ]1, \infty[)$ implies $(a \otimes 1)x \in P(\sigma^\psi, [\lambda, \infty[)$.

To see this note that $P_\psi = M_\phi \otimes F_\infty$ has center $C \otimes 1$. By Lemma 5.3.3 of [2] we get a partial isometry $v \in P(\sigma^\psi, \{\lambda\})$ with initial support $b \otimes 1$, final support $a \otimes 1$. Condition (3)′ implies that $v$ belongs to the set $\mathscr{E}_1$ associated in [2, p. 235] to the action $\sigma^\psi$ of $\mathbf{R}$ on $P$. It hence follows that $Uv^* \in P_\psi$ using [2, p. 238, end of the proof of 5.3.4]. Hence the final support $U(b \otimes 1)U^*$ of $Uv^*$ is equal to its initial support $a \otimes 1$. As the restriction of $AdU$ to $P_\psi$ is the automorphism $\theta$ of the discrete decomposition of $P$, and as the center of $P_\psi$ is $C \otimes 1$, which is generated by the $a \otimes 1$, for $a$ as above, we have shown that the restriction of $\theta$ to the center is isomorphic to $T_p$ acting on $L^\infty(X)$. Q.E.D.

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