L^1-ALGEBRAS ON SEMIGROUPS

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The purpose of the paper is to extend the theory of L^1-algebras on semigroups. We first consider a finite product of idempotent intervals, with Lebesgue measure. Then we extend the theory to products of intervals with a group. In the next stage, all homomorphic images of the aforementioned type semigroups are examined. Finally, in the last section, we consider some examples. One example which comes readily out of this is a generalization of Bergman's work [2] on algebraically irreducible semigroups S in which S' is an idempotent thread (see Example 4.2).

1. Products of idempotent intervals

Let $I = \{a, b\}$ denote an interval of real numbers from a to b, where a or b may be infinite and I may or may not contain either of the endpoints a and b. With multiplication on I given by $xy = \max(x, y)$, and the natural topology, the space I is a topological semigroup. For any $E \subseteq I$, and point $x \in I$, define the set Ex^{-1} by

$$Ex^{-1} = \{y \in I \mid xy \in E\}.$$

Then $M(I)$ is the space of all bounded, regular, Borel measures on I equipped with total variation norm and multiplication given by

$$\mu * \nu(E) = \int_I \int_I \chi_E(xy)\, d\mu(x)\, d\nu(y) = \int_I \mu(Ey^{-1})\, d\nu(y),$$

where χ_E is the characteristic function of E. If m is Lebesgue measure on I, define $L^1(I, m)$ to be the space of all measures in $M(I)$ which are absolutely continuous with respect to m.

Lardy, in [5], has shown that $L^1(I, m)$ is a closed subalgebra of $M(I)$. Define $(a, b]$ as follows: $(a, b]$ is the ordinary open-closed interval if $b \neq \infty$, and if $b = \infty$, $(a, b]$ is obtained by "compactifying the right half" of $(a, \infty]$. In other words, $(a, \infty] = (a, \infty) \cup \{\infty\}$, where a neighborhood base of ∞ is given by $\{(c, \infty) \cup \{\infty\} \mid c \in (a, \infty)\}$. With this definition, Lardy has identified the maximal ideal space of $L^1(I, m)$ with the interval $(a, b]$. In addition, he has shown that $L^1(I, m)$ is regular, possesses approximate identities, and satisfies a Herglotz-Bochner theorem.

All through this discussion, S will denote the semigroup $\prod_{n=1}^{N} I_n$ (which we will write as $\prod I_n$ when the indexing is understood), where $I_n = \{a_n, b_n\}$ and

Received November 25, 1974.

[1] Many of the following results are contained in the author's doctoral dissertation written under the direction of Professor Neal J. Rothman at the University of Illinois.

multiplication is given by $(x_n)(y_n) = (\max(x_n, y_n))$. Under the Cartesian product topology, S is a commutative topological semigroup. The measure algebra $M(S)$ has been studied by Baartz in [1] and many of the following results draw heavily on his work. If λ is product Lebesgue measure on S and convolution is defined exactly as before, we must first show that $L^1(S, \lambda)$ is a closed subalgebra of $M(S)$. From now on, we simply write $L^1(S)$ for $L^1(S, \lambda)$.

At this point, we introduce several definitions.

DEFINITION. Suppose that A is a nonempty, proper subset of $\{1, 2, \ldots, N\}$, $B = \{1, 2, \ldots, N\} \sim A$, and $x = (x_n) \in \prod_{n \in A} I_n$. Then for E a subset of S, $E_x = \{y = (y_n) \in \prod_{n \in B} I_n \mid$ there exists a point $z = (z_1, \ldots, z_N) \in E$ where $z_n = x_n$ for all $n \in A$ and $z_n = y_n$ for all $n \in B\}$. Let λ_A denote Lebesgue measure on $\prod_{n \in A} I_n$.

If $x \in \prod_{n \in A} I_n$ and $y \in \prod_{n \in B} I_n$ as in the above definition, consider the pair $(x, y) \in S$ in the natural way. In other words, identify (x, y) with the point $z = (z_1, \ldots, z_N) \in S$ having the property that $z_n = x_n$ for all $n \in A$ and $z_n = y_n$ for all $n \in B$. We can now state and prove the two key lemmas necessary to show that $L^1(S)$ is an algebra.

LEMMA 1.1. *Let E be a subset of S such that $\lambda(E) = 0$ and let $x \in J_A = \prod_{n \in A} I_n$ and $y \in J_B = \prod_{n \in B} I_n$, where A and B are nonempty subsets of $\{1, 2, \ldots, N\}$ with $A \cap B = \emptyset$ and $A \cup B = \{1, 2, \ldots, N\}$. Then $\lambda_A(E_y) = 0$ for almost all $(\lambda_B) y \in J_B$ and $\lambda_B(E_x) = 0$ for almost all $(\lambda_A) x \in J_A$.*

Proof. It suffices to prove one of the statements, for the other will then follow in exactly the same manner. We have

$$0 = \lambda(E) = \int_S \chi_E(z)\, d\lambda(z) = \int_{J_B} \int_{J_A} \chi_E(x, y)\, d\lambda_A(x)\, d\lambda_B(y)$$

$$= \int_{J_B} \lambda_A(E_y)\, d\lambda_B(y).$$

Since $\lambda_A(E_y)$ is a nonnegative function of y and λ_B is a positive measure, the fact that the integral is equal to zero implies that $\lambda_A(E_y) = 0$ for almost all (λ_B) $y \in J_B$.

LEMMA 1.2. *If $E \subseteq S$ and $\lambda(E) = 0$, then $\lambda(Ez^{-1}) = 0$ for almost all (λ) $z \in S$.*

Proof. Given a nonempty, proper subset A of $\{1, 2, \ldots, N\}$, let $B = \{1, 2, \ldots, N\} \sim A$ and define the set $D_A = \{x \in \prod_{n \in A} I_n \mid \lambda_B(E_x) > 0\}$. By Lemma 1.1, $\lambda_A(D_A) = 0$ for all such sets A. Suppose

$$D = \bigcup \left\{D_A \times \prod_{n \notin A} I_n\right\},$$

where the union runs through all nonempty, proper subsets of $\{1, 2, \ldots, N\}$, and let $K = E \cup D$. Then $\lambda(K) = 0$. For any such subset A of $\{1, 2, \ldots, N\}$, let π_A be the projection of S onto $\prod_{n \in A} I_n$. The claim is that for $z = (z_1, \ldots, z_N) \notin K$,

$$Ez^{-1} \subseteq \bigcup_A \left\{E_{\pi_A(z)} \times \prod_{n \in A} I_n\right\} \cup E = F_z.$$

Before proving the claim, it should be noted that $\lambda(F_z) = 0$, since for any set A, $(z_1, \ldots, z_N) \notin K$ implies that $(z_1, \ldots, z_N) \notin D_A \times \prod_{n \notin A} I_n$, which in turn implies that for $B = \{1, 2, \ldots, N\} \sim A$, $\lambda_B(E_{\pi_A(z)}) = 0$.

To prove the claim, suppose that $u = (u_1, \ldots, u_N) \in Ez^{-1}$ and consider the three cases:

Case I. If $u \geq z$, then $uz = u \in E \subseteq F_z$. (Recall that $u \geq z$ if $u_i \geq z_i$ for all $i = 1, \ldots, N$.)

Case II. If $u \leq z$, then $uz = z \in E$, which contradicts the assumption that $z \notin K$.

Case III. If $u \nleq z$ and $u \ngeq z$, then $u_n \leq z_n$ for $n \in A$, where A is a nonempty, proper subset of $\{1, 2, \ldots, N\}$. If

$$B = \{1, 2, \ldots, N\} \sim A = \{n \mid u_n > z_n\},$$

we conclude that $uz = w = (w_1, \ldots, w_N) \in E$, where $w_n = z_n$ for $n \in A$ and $w_n = u_n$ for $n \in B$. Consequently, $u \in E_{\pi_A(z)} \times \prod_{n \in A} I_n \subseteq F_z$ and

$$\lambda(Ez^{-1}) \leq \lambda(F_z) = 0.$$

The fact that $L^1(S)$ is closed under multiplication is an easy consequence of these two lemmas. Rothman [9] shows that Lemma 1.2 is a sufficient condition.

THEOREM 1.3. *The space $L^1(S)$ is a closed subalgebra of $M(S)$.*

DEFINITION. We denote the space of all nonzero, measurable semicharacters on S, identified almost everywhere with respect to λ, by $\hat{S}$.

Baartz has shown in [1] that all of the semicharacters on S are characteristic functions of sets A of the form $A = \prod \{a_n, c_n\}$, $c_n \leq b_n$, where the left-hand bracket is open or closed depending on whether it is open or closed in $\{a_n, b_n\}$, and the right-hand bracket is open or closed. The proof of the following theorem is a fairly standard application of the Riesz Representation Theorem and can be found in Lardy [5] and Rothman [9]. Crucial to this whole theory is the fact that elements of $L^1(S)$ can be considered as either absolutely continuous measures or absolutely integrable functions, which is guaranteed by the Radon-Nikodym Theorem. The symbol $\mathcal{M}(L^1(S))$ stands for the maximal ideal space of $L^1(S)$; that is, the space of all nonzero, multiplicative linear functionals.

THEOREM 1.4. *Each $\chi \in \hat{S}$ gives rise to an $h \in \mathscr{M}(L^1(S))$ by*

$$h(\mu) = \int_S \chi \, d\mu \quad \textit{for all } \mu \in L^1(S).$$

Conversely, each $h \in \mathscr{M}(L^1(S))$ defines a measurable semicharacter $\chi \in \hat{S}$ such that $h(\mu) = \int_S \chi \, d\mu$.

The sets $\prod \{a_n, c_n\}$, regardless of whether the right brackets are open or closed, determine the same member of $\mathscr{M}(L^1(S))$ since their characteristic functions are equal almost everywhere with respect to λ. Hence, we obtain the following theorem.

THEOREM 1.5. *The space $\hat{S}$, with the Gelfand topology, is homeomorphic to $\prod (a_n, b_n]$, with the interval topology. The elements of $\hat{S}$ form a semigroup under minimum multiplication.*

The idea of this proof is the following. We show that the set of functions $\{\hat{\mu} \mid \mu \in L^1(S)\}$ are continuous on $\prod_{n=1}^N (a_n, b_n]$, separate points, vanish at infinity, and do not all vanish at a particular point. This implies that the topologies are equivalent.

Proof. It should be mentioned that we are using the same interpretation of $(a_n, \infty]$ as was defined earlier. We can now identify $\prod (a_n, b_n]$ and $\hat{S}$ by associating to each element $(x_n) \in \prod (a_n, b_n]$ the semicharacter χ which is the characteristic function of $\prod (a_n, x_n]$; where now, if $x_n = \infty$, we interpret $(a_n, x_n]$ as $[a_n, \infty]$. In order to see that the topologies coincide, let $x = (x_n) \in \prod (a_n, b_n]$, $\mu \in L^1(S)$, and $\varepsilon > 0$ be given. Since μ is absolutely continuous, we can find an open neighborhood G of $\prod (a_n, x_n]$ of the type $G = \prod (a_n, d_n)$ (where $(a_n, d_n) = (a_n, b_n]$ if $x_n = b_n < \infty$) such that

$$|\mu|(\prod (a_n, d_n)) < |\mu|(\prod (a_n, x_n]) + \frac{\varepsilon}{2}.$$

Similarly, there exists a compact set $K \subseteq \prod (a_n, x_n)$, which we can assume without loss of generality is of the type $K = \prod [e_n, c_n]$, such that

$$|\mu|(K) + \frac{\varepsilon}{2} > |\mu|(\prod (a_n, x_n]).$$

In this manner we obtain an open neighborhood (in the interval topology) of (x_n), namely $\prod (c_n, d_n)$, with the property that for any $(y_n) \in \prod (c_n, d_n)$,

$$|\mu(\prod (a_n, y_n]) - \mu(\prod (a_n, x_n])| < \varepsilon.$$

Therefore, the Gelfand image $\hat{\mu}$ of μ is continuous on $\hat{S}$ with the interval topology which implies that the Gelfand topology is weaker than or equal to the interval topology.

Both $\hat{S}$, with the Gelfand topology, and $\prod (a_n, b_n]$, with the interval topology, are locally compact spaces. Choose $\mu \in L^1(S)$ and without loss of generality, $\mu \geq 0$. We want to show that $\hat{\mu}$ vanishes at the ∞ of the one-point compactification of $\prod (a_n, b_n]$. Define the measure $\mu_k \in L^1(a_k, b_k)$ as follows:

$$\mu_k(E) = \mu\left(\prod_{n=1}^{k-1} I_n \times E \times \prod_{n=k+1}^{N} I_n\right).$$

Given an $\varepsilon > 0$, choose a compact set $C_k \subseteq (a_k, b_k)$ such that

$$\mu_k((a_k, b_k) \sim C_k) < \varepsilon.$$

Suppose t_k is the minimal element of C_k and define $K = \prod [t_n, b_n]$. If $x = (x_n) \in \hat{S} \sim K$, then for some k, we must have that $x_k < t_k$. Hence,

$$\begin{aligned} \hat{\mu}(x) &= \mu(\prod (a_n, x_n]) \\ &\leq \mu\left(\prod_{n=1}^{k-1} I_n \times (a_k, x_k] \times \prod_{n=k+1}^{N} I_n\right) \\ &= \mu_k((a_k, x_k]) \\ &< \varepsilon, \end{aligned}$$

from which we conclude that $\hat{\mu}$ vanishes at ∞.

To see that not all of the functions in $\{\hat{\mu} \mid \mu \in L^1(S)\}$ vanish at any particular point of $\prod (a_n, b_n]$, let $x = (x_n) \in \prod (a_n, b_n]$. Choose $\varepsilon > 0$ with the property that $x_n - \varepsilon \in (a_n, b_n]$ for all n. If $A = \prod [x_n - \varepsilon, x_n]$, define $\mu \in L^1(S)$ by $\mu(E) = \lambda(E \cap A)$. Then $\hat{\mu}(x) = \mu(\prod (a_n, x_n]) = \varepsilon^N > 0$. Finally, notice that the set $\{\hat{\mu} \mid \mu \in L^1(S)\}$ separates points of $\prod (a_n, b_n]$ by the Gelfand theory. Hence, by a theorem in general topology [6], the two topologies coincide.

The following propositions are easy generalizations of the ones given by Lardy and hence, they will be included here without proof.

PROPOSITION 1.6. *Every proper closed ideal in $L^1(S)$ is contained in a regular, maximal ideal.*

PROPOSITION 1.7. *The algebra $L^1(S)$ is regular.*

PROPOSITION 1.8. *The algebra $L^1(S)$ is symmetric and semisimple.*

DEFINITION. A sequence $\{u_n\}$ in a Banach algebra A is called an *approximate identity* for A if $\lim_{n\to\infty} u_n x = x$ for all $x \in A$.

THEOREM 1.9. *The algebra $L^1(S)$ possesses approximate identities.*

Lardy points out in [5] that if I is a bounded interval, then $L^1(I)$ is singly generated. In fact, it is generated by the function which is identically one on I. In the case of $L^1(S)$, however, it is still unknown as to whether or not it is even finitely generated.

The next two ideas are essentially different from those presented in Lardy's paper and hence are given here in complete detail. Recall that the Choquet boundary of $L^1(S)$ is the set of points in $\hat{S}$ which possess unique representing measures.

PROPOSITION 1.10. *The Choquet boundary of $L^1(S)$ equals $\hat{S}$.*

Proof. Since $L^1(S)$ is symmetric, the Stone-Weierstrass theorem implies that the Gelfand image of $L^1(S)$ is dense in $C_0(\hat{S})$. Hence, suppose that $\chi \in \hat{S}$ and λ_1, λ_2 are representing measures for χ. In other words,

$$\int \hat{f}\, d\lambda_1 = \hat{f}(\chi) = \int \hat{f}\, d\lambda_2 \quad \text{for all } f \in L^1(S);$$

or equivalently, $\int \hat{f}\, d(\lambda_1 - \lambda_2) = 0$ for all $f \in L^1(S)$. The fact that $L^1(S)\hat{}$ is dense in $C_0(\hat{S})$ implies that $\lambda_1 - \lambda_2 = 0$, or $\lambda_1 = \lambda_2$.

Although $L^1(S)$ does not contain point masses, convolving a point mass δ_x with a particular measure in $L^1(S)$, depending on δ_x, yields a new measure in $L^1(S)$ for almost all (λ) points x in S. However, the proof of the following proposition will point out that convolving δ_x with the wrong measure in $L^1(S)$ will yield a measure which is not in $L^1(S)$, showing us that unlike in the group algebra case, $L^1(S)$ is not an ideal in $M(S)$. For certain topological semigroups R, Rothman, in [9], uses a result along the lines of the following proposition to obtain a concrete representation of the measurable semicharacter associated with a multiplicative homomorphism (see Theorem 1.4). In other words, if $h \in \mathcal{M}(L^1(R))$, the function t defined by $t(x) = h(\mu * \delta_x)/h(\mu)$ for some $\mu \in L^1(R)$ with $\mu * \delta_x \in L^1(R)$ and $h(\mu) \neq 0$ is a measurable semicharacter. Hence, the next proposition gives us an alternative proof to the second part of Theorem 1.4.

PROPOSITION 1.11. *If $x = (x_n) \in \prod \{a_n, b_n\}$, then there exists a measure $\mu \in L^1(S)$ such that $\mu * \delta_x \in L^1(S)$.*

Proof. Define the set A by $A = \{y = (y_n) \in S \mid x_n \leq y_n \leq \min(b_n, x_n + 1)\}$ and notice that $0 < \lambda(A) < \infty$. If μ is given by $\mu(E) = \lambda(E \cap A)$ then the support of μ is contained in A. For any $B \subseteq S$ with $\lambda(B) = 0$,

$$\begin{aligned} \mu * \delta_x(B) &= \int_S \int_S \chi_B(ty)\, d\mu(t)\, d\delta_x(y) \\ &= \int_S \chi_B(tx)\, d\mu(t) \\ &= \int_S \chi_{Bx^{-1}}(t)\, d\mu(t) \\ &= \mu(Bx^{-1}) \\ &= \lambda(Bx^{-1} \cap A). \end{aligned}$$

However, $Bx^{-1} \cap A = \{y \in A \mid yx \in B\} = \{y \in A \mid y \in B\} = A \cap B$. Therefore, $\mu * \delta_x(B) = \lambda(B \cap A) = 0$.

2. Products of intervals with a group

This section is inserted merely as a stepping stone to work in further sections. The ideas of the proofs are very straightforward and similar to those in Section 1 and for that reason, will be omitted.

As in the previous section, let $S = \prod \{a_n, b_n\}$ and suppose that G is a locally compact Abelian group. Throughout this section, T will denote the topological semigroup $S \times G$. If λ is Lebesgue measure on S and m is Haar measure on G, define $\mu = \lambda \times m$. Then μ is a measure on T and we can consider the space $L^1(T, \mu)$. This space turns out to be a closed subalgebra of $M(T)$ and we can summarize the major results in the following theorem.

THEOREM 2.1. (1) *Each $\chi \in \hat{T}$ gives rise to an $h \in \mathscr{M}(L^1(T))$ by*

$$h(v) = \int_T \chi \, dv \quad \text{for all } v \in L^1(T).$$

Conversely, each $h \in \mathscr{M}(L^1(T))$ defines a measurable semicharacter $\chi \in \hat{T}$ such that $h(v) = \int_T \chi \, dv$.

(2) *The space $\hat{T}$ is equal to $\hat{S} \times \hat{G}$.*

(3) *The Gelfand topology on $\hat{T} \cong (\prod (a_n, b_n]) \times \hat{G}$ coincides with the Cartesian product topology, where $\prod (a_n, b_n]$ has the interval topology and $\hat{G}$ has the compact-open topology.*

The results corresponding to semisimplicity, approximate identities, and existence of a measure $v \in L^1(T)$ such that $v * \delta_z \in L^1(T)$ are exactly the same as in Section 1.

3. Homomorphic images

Suppose U is a locally compact topological semigroup and T and μ are as in Section 2. Let ψ be a continuous homomorphism of T onto U and define by v by $v(E) = \mu(\psi^{-1}(E))$ for all Borel sets $E \subseteq U$.

In order to verify that v is indeed an element of $M(U)$, first notice the $v(\emptyset) = \mu(\psi^{-1}(\emptyset)) = \mu(\emptyset) = 0$. We see that v is bounded since v is positive and $v(U) = \mu(\psi^{-1}(U)) = \mu(T) < \infty$. If $\{E_n\}_{n=1}^{\infty}$ are disjoint sets in U,

$$v(\cup E_n) = \mu(\psi^{-1}(\cup E_n)) = \mu(\cup (\psi^{-1}(E_n)) = \sum \mu(\psi^{-1}(E_n)) = \sum v(E_n).$$

Finally, if $E \subseteq U$, $\varepsilon > 0$ is given, and $E_1 = \psi^{-1}(E)$, choose a compact set $K_1 \subseteq T$ such that $\mu(E_1 \sim K_1) < \varepsilon$. If $K = \psi(K_1)$, then

$$\begin{aligned} v(E \sim K) &= \mu(\psi^{-1}(E \sim K)) = \mu(\psi^{-1}(E) \sim \psi^{-1}(K)) \\ &= \mu(E_1 \sim \psi^{-1}(K)) \leq \mu(E_1 \sim K_1) < \varepsilon. \end{aligned}$$

Therefore, $v \in M(U)$.

Throughout this section we consider the space $L^1(U, \nu)$ where U is a homomorphic image of T, and ν is defined by $\nu(E) = \mu(\psi^{-1}(E))$. The following lemma is needed in order to show that $L^1(U, \nu)$ is a subalgebra of $M(U)$.

LEMMA 3.1. *If $\nu(E) = 0$, then $\nu(Ex^{-1}) = 0$ for almost all (ν) $x \in U$.*

Proof. Define $A = \{x \in U \mid \nu(Ex^{-1}) > 0\}$ and suppose $\nu(A) > 0$. If $E_0 = \psi^{-1}(E)$ and $A_0 = \psi^{-1}(A)$, then $\mu(E_0) = 0$ and $\mu(A_0) > 0$. For $x_0 \in A_0$ and $x = \psi(x_0)$, we assert $\psi^{-1}(Ex^{-1}) = E_0 x_0^{-1}$. This follows since $y_0 \in \psi^{-1}(Ex^{-1})$ if and only if $\psi(y_0) \in Ex^{-1}$ if and only if $\psi(y_0)x \in E$ if and only if $\psi(y_0)\psi(x_0) \in E$ if and only if $\psi(y_0 x_0) \in E$ if and only if $x_0 y_0 \in \psi^{-1}(E) = E_0$ if and only if $y_0 \in E_0 x_0^{-1}$. Therefore,

$$\mu(E_0 x_0^{-1}) = \mu(\psi^{-1}(Ex^{-1})) = \nu(Ex^{-1}) > 0 \quad \text{for all } x_0 \in A_0,$$

which is a contradiction arising from the analog of Lemma 1.2 for T.

From this lemma, the proof of the following theorem proceeds exactly as in 1.3.

THEOREM 3.2. *The space $L^1(U)$ is a closed subalgebra of $M(U)$.*

Similarly, the proof of the characterization of the maximal ideal space is exactly the same as in the case of a finite product of intervals.

THEOREM 3.3. *Each $\chi \in \hat{U}$ gives rise to an $h \in \mathscr{M}(L^1(U))$ by*

$$h(\mu) = \int_U \chi \, d\mu \quad \text{for all } \mu \in L^1(U).$$

Conversely, each $h \in \mathscr{M}(L^1(U))$ defines a measurable semicharacter $\chi \in \hat{U}$ such that $h(\mu) = \int_U \chi \, d\mu$.

The next project is to identify the Gelfand topology. In the following result and remark, it is noticed that $\hat{U}$ can be embedded in $\hat{T}$, telling us quite a bit about the structure of $\hat{U}$ since the space $\hat{T}$ was thoroughly described in 2.1. Then a precise description of the subset of $\hat{T}$ identified with $\hat{U}$ is given.

THEOREM 3.4. *If $\psi^*\colon \hat{U} \to \hat{T}$ is defined by $\psi^*(\xi) = \xi \circ \psi$, then ψ^* is one-to-one and continuous with respect to the Gelfand topologies on $\hat{U}$ and $\hat{T}$.*

Proof. To see that the range of ψ^* is actually $\hat{T}$, first note that $\psi^*(\xi)$ is clearly a semicharacter on T. If $\psi^*(\xi) = \chi$ were zero almost everywhere (μ), choose $E \subseteq T$ with the property that $\chi(t) = 0$ for all $t \in E$ and $\mu(T \sim E) = 0$. Letting $D = \psi(E)$, we see that $\xi(u) = 0$ for all $u \in D$ since if $u = \psi(t) \in D$ where $t \in E$, then $\xi(u) = \xi \circ \psi(t) = \chi(t) = 0$. Moreover, $\nu(U \sim D) = 0$, since

$$\begin{aligned} \nu(U \sim D) &= \mu(\psi^{-1}(U \sim D)) = \mu(\psi^{-1}(U) \sim \psi^{-1}(D)) \\ &= \mu(T \sim \psi^{-1}(D)) \leq \mu(T \sim E) = 0. \end{aligned}$$

This is a contradiction since $\xi \in \hat{U}$.

The map ψ^* is one-to-one, for suppose $\xi_1, \xi_2 \in U$ and $\xi_1 \neq \xi_2$ as elements of $\hat{U}$. Then for some $D \subseteq U$ with $\nu(D) > 0$, we have $\xi_1(u) \neq \xi_2(u)$ for all $u \in D$. If $K = \psi^{-1}(D)$, then $\mu(K) > 0$ and for $y \in K$,

$$\psi^*(\xi_1)(y) = \xi_1(\psi(y)) \neq \xi_2(\psi(y)) = \psi^*(\xi_1)(y)$$

since $\psi(y) \in D$. Therefore $\psi^*(\xi_1) \neq \psi^*(\xi_2)$ as elements of $\hat{T}$.

In order to establish the continuity of ψ^*, let $\{\xi_\alpha\}$, $\xi \in \hat{U}$ with $\xi_\alpha \to \xi$ in the Gelfand topology of $\hat{U}$. We must verify that if $\psi^*(\xi_\alpha) = \chi_\alpha$ and $\psi^*(\xi) = \chi$, then $\chi_\alpha \to \chi$ in the Gelfand topology of $\hat{T}$. In other words, it must be shown that $\int_T \chi_\alpha \, d\beta \to \int_T \chi \, d\beta$ for all $\beta \in L^1(T)$. Fix a measure $\beta \in L^1(T)$ and define the measure $\pi \in M(U)$ by

$$\pi(E) = \beta(\psi^{-1}(E)).$$

Then $\pi \in L^1(U)$, since for any set $B \subset U$ with $\nu(B) = 0$, we have $\mu(\psi^{-1}(B)) = 0$ which implies $\beta(\psi^{-1}(B)) = 0$, and thus, $\pi(B) = 0$. Since $\xi_\alpha \to \xi$, we have

$$\int_U \xi_\alpha \, d\pi \to \int_U \xi \, d\pi$$

Therefore, the proof will be complete if we can show that $\int_U \xi_\alpha \, d\pi = \int_T \chi_\alpha \, d\beta$ for all α and $\int_U \xi \, d\pi = \int_T \chi \, d\beta$. The image of ξ is contained in the closed unit disc Δ, and hence there is a decomposition of Δ, say $\{B_i\}_{i=1}^m$, such that if $A_i = \{u \in U \mid \xi(u) \in B_i\}$, and b_i is an arbitrary point of B_i, and $\varepsilon > 0$, then

$$\left| \int_U \xi \, d\pi - \sum_{i=1}^{m} b_i \pi(A_i) \right| < \varepsilon.$$

Suppose $C_i = \psi^{-1}(A_i)$. Then $C_i = \{t \in T \mid \chi(t) \in B_i\}$, since $t_i \in C_i$ implies $\psi(t) \in A_i$ implies $\chi(t) = \xi \circ \psi(t) \in B_i$, and $t_i \in \{t \in T \mid \chi(t) \in B_i\}$ implies $\chi(t) = \xi \circ \psi(t) \in B_i$ implies $\psi(t) \in A_i$ implies $t \in \psi^{-1}(A_i) = C_i$. Therefore, $\sum_{i=1}^m b_i\beta(C_i)$ approximates $\int_T \chi \, d\beta$ and all that remains to be seen is that

$$\sum_{i=1}^{m} b_i\beta(C_i) = \sum_{i=1}^{m} b_i\pi(A_i),$$

and this is because $\beta(C_i) = \beta(\psi^{-1}(A_i)) = \pi(A_i)$.

Remark. The one point compactifications of $\hat{U}$ and $\hat{T}$, written $\hat{U}_\infty$ and $\hat{T}_\infty$ respectively, are obtained by adjoining the zero semicharacter. The map $\psi^*\colon \hat{U}_\infty \to \hat{T}_\infty$, which extends ψ^* by sending the zero semicharacter on U to the zero semicharacter on T, is a one-to-one, continuous mapping of a compact space onto a compact space, namely $\psi^*_\infty(\hat{U})$. Therefore, ψ^*_∞ is a homeomorphism, from which we conclude that ψ^* is a homeomorphism.

DEFINITION. Suppose X and Y are topological spaces and $p\colon X \to Y$ is a continuous, onto mapping. The *fibre* over a point $y_0 \in Y$ is the set $p^{-1}(y_0) \subseteq X$.

As a corollary to the previous theorem, we identify the image of ψ^*.

COROLLARY 3.5. *The space $\hat{U}$ is homeomorphic to*

$$\{\chi \in \hat{T} \mid \chi \text{ is constant on fibers of } T\}.$$

Proof. If $A = \{\chi \in \hat{T} \mid \chi$ is constant on fibers of $T\}$, we must show that $\psi^*(\hat{U}) = A$. Suppose that $\xi \in \hat{U}$ and $\chi = \psi^*(\xi) = \xi \circ \psi$. Then $\chi \in A$ since if x and y are in the same fiber of T, then $\psi(x) = \psi(y)$, and hence, $\chi(x) = \xi \circ \psi(x) = \xi \circ \psi(y) = \chi(y)$.

On the other hand, suppose that $\chi \in A$ and define $\xi \in \hat{U}$ by

$$\xi(u) = \chi(\psi^{-1}(u)) \quad \text{for all } u \in U.$$

This is well defined since χ is constant on fibers of T. Moreover, ξ is not zero almost everywhere (ν) for if $B \subseteq T$ has the properties that $\mu(B) > 0$ and $\chi(t) \neq 0$ for all $t \in B$, then $\chi(t) \neq 0$ for all $t \in B^1 = \psi^{-1}(\psi(B)) \supseteq B$. Consequently, $\xi(u) \neq 0$ for all $u \in \psi(B)$, where

$$\nu(\psi(B)) = \mu(\psi^{-1}(\psi(B))) \geq \mu(B) > 0.$$

To see that ξ is a semicharacter, suppose $u_1, u_2 \in U$. Notice that

$$\psi^{-1}(u_1)\psi^{-1}(u_2) \subseteq \psi^{-1}(u_1u_2),$$

since $x \in \psi^{-1}(u_1)$, $y \in \psi^{-1}(u_2)$ implies $\psi(x) = u_1$, $\psi(y) = u_2$ implies $\psi(xy) = \psi(x)\psi(y) = u_1u_2$ implies $xy \in \psi^{-1}(u_1u_2)$. Therefore, if $x \in \psi^{-1}(u_1), y \in \psi^{-1}(u_2)$ we have

$$\begin{aligned}\xi(u_1u_2) &= \chi(\psi^{-1}(u_1u_2)) = \chi(xy) = \chi(x)\chi(y)\\ &= \chi(\psi^{-1}(u_1))\chi(\psi^{-1}(u_2)) = \xi(u_1)\xi(u_2).\end{aligned}$$

Finally, $\psi^*(\xi) = \chi$, since $\psi^*(\xi)(t) = \xi(\psi(t)) = \chi(\psi^{-1}(\psi(t))) = \chi(t)$.

The following three results are very straightfowrard and will complete our study of $L^1(U)$.

PROPOSITION 3.6. *The algebra $L^1(U)$ is semisimple.*

PROPOSITION 3.7. *The algebra $L^1(U)$ possesses approximate identities.*

PROPOSITION 3.8. *If $u \in \psi(\prod \{a_n, b_n) \times G)$, then there exists a measure $\pi \in L^1(U)$ such that $\pi * \delta_u \in L^1(U)$.*

4. Examples

In this section, two specific types of semigroups are considered. Both of these semigroups can be realized as continuous homomorphic images of semigroups of the form $\prod \{a_n, b_n\} \times G$, where G is a locally compact group, and hence the theory of Section 3 is applicable. In the first example, we deal with the quotient spaces as discussed in [9].

Through this section, the semigroup T and the measure μ are defined as in Section 2 of this paper.

Example 4.1. Suppose I is an ideal in T and $U = T/I$. Then the canonical map $\psi: T \to T/I = U$ is a continuous onto homomorphism. It is well known that the space $(T/I) \sim \{I\}$ is homeomorphic to $T \sim I$. Therefore, the measure $\nu \in L^1(U)$ defined by $\nu(E) = \mu(\psi^{-1}(E))$ can also be defined in the following manner:

$$\nu(E) = \mu(E) \text{ for } E \subseteq T/I \sim \{I\} \quad \text{and} \quad \nu(\{I\}) = \mu(I).$$

As in the preceding section, define $\psi^*: \hat{U} \to \hat{T}$ by $\psi^*(\xi) = \chi = \xi \circ \psi$. Then, by 3.5, we know that

$$\begin{aligned} \hat{U} &\cong \{\chi \in \hat{T} \mid \chi \text{ is constant on fibers of } T\} \\ &\cong \{\chi \in \hat{T} \mid \chi \text{ is constant on } I\}. \end{aligned}$$

In this particular example, it should be noted that this last fact could be easily verified on a straightforward manner rather than resorting to the previous, more general result.

The following example generalizes the algebras discussed by Bergman in [2]. In it, the idea of an algebraically irreducible semigroup is extended to some degree and the L^1-theory of such semigroups is discussed. The manner in which the first three sections were developed was geared precisely to the study of these new types of semigroups.

Example 4.2. Before we begin, some definitions and notation are needed.

DEFINITION. If U is a compact commutative semigroup with identity, and $x, y \in U$, then $x \equiv y(\mathscr{L})$ if $xU = yU$.

It is a known fact that the quotient space, $U/\mathscr{L}$, is again a compact semigroup which is denoted by U'. Let

$$H = \text{the maximal subgroup of } U \text{ containing the identity } 1,$$

and

$$K = \text{the kernel of } U = \text{the minimal ideal contained in } U.$$

It is known that H is a compact topological group. The semigroup U under consideration in this example is defined next.

DEFINITION. We call U a *quasi-irreducible* ($Q - I$) *semigroup* if:

(1) U is a compact, commutative semigroup with identity 1;

(2) $U' \cong \prod [a_n, b_n]$, where $[a_n, b_n]$ are idempotent intervals;

(3) if $q: U \to /\mathscr{L} = U'$ is the canonical map, then there exists a subsemigroup $P \subseteq U$ such that $q|_p$ is an isomorphism onto U';

(4) U is the union of the orbits of elements of H under action by P.

It should be pointed out that the third condition in this definition is implied by the first two. The proof of this fact, which will be given later, is very similar to a proof given by Hunter in [4].

Because both H and P are contained in U, we can define the map $\psi: H \times P \to U$ by $\psi((h, e)) = he$. By part (4) of this definition, ψ is onto, and ψ is clearly continuous since multiplication is continuous. To see that ψ is a homomorphism, choose $(h, e), (k, f) \in H \times P$. Then

$$\psi((h, e)(k, f)) = \psi(hk, ef) = (hk)(ef) = (he)(kf) = \psi(h, e)\psi(k, f).$$

Therefore, ψ satisfies all of the conditions of the previous section. Hence, if we define $\nu \in L^1(U)$ by $\nu(E) = \mu(\psi^{-1}(E))$, then $\nu \geq 0$, $\|\nu\| < \infty$, $\nu(D) > 0$ for D open in U, and $L^1(U, \nu)$ is a closed subalgebra of $M(U)$. Furthermore, the maximal ideal space of $L^1(U)$ can be identified with $\hat{U}$.

By part (4) of the definition, $U = \bigcup_{x \in P} Hx$. We now claim that this union is disjoint. Suppose $x, y \in P$, $x \neq y$, and M_x and M_y are the maximal groups in U containing x and y respectively. It is a standard fact in semigroup theory that $M_x \cap M_y = \emptyset$. We next show that Hx is a group in U containing x. If $h, k \in H$, then $(hx)(kx) = (hk)x^2 = (hk)x$, and thus, Hx is closed under multiplication. The element $x = 1x \in Hx$ is the identity for Hx since $x(hx) = hx^2 = hx$. Finally, if $hx \in Hx$, then $h^{-1}x \in Hx$ and $(h^{-1}x)(hx) = 1x^2 = x$. From this we conclude that $Hx \subseteq M_x$. Similarly, $Hy \subseteq M_y$ and the fact that $M_x \cap M_y = \emptyset$ implies that $Hx \cap Hy \subseteq M_x \cap M_y = \emptyset$, which proves the claim.

Suppose $L_x = \{y \in U \mid xU = yU\}$ = the equivalence class of x(modulo $\mathscr{L}$), and

$$T_x: H \to L_x \quad \text{by } T(h) = hx.$$

For $h \in H$, $hx \in L_x$ since

$$(hx)U = x(hU) \subseteq xU$$

and

$$xU = (xhh^{-1})U = (hx)(h^{-1}U) \subseteq (hx)U.$$

Observe that T_x is a homomorphism since

$$T_x(h_1h_2) = h_1h_2x = h_1h_2x^2 = (h_1x)(h_2x) = T_x(h_1)T_x(h_2).$$

Finally, T_x is onto, since if $z \in L_x$ and $z \notin T_x(H) = Hx$, then $z \in Hy$ for some $y \in P$, $y \neq x$. This implies that $z \in L_y$ (since $H_y \subseteq L_y$ via the map T_y), which is a contradiction since $L_x \cap L_y = \emptyset$. Therefore, $Hx = L_x$.

We can now analyze the structure of the maximal ideal space of $L^1(U)$. Partition $H \times P$ as follows:

$$\{T_x^{-1}(z) \times \{x\} \mid z \in L_x\}_{x \in P}. \tag{1}$$

Notice that this partition corresponds to the fibers in $H \times P$ with respect to ψ, for if $x, y \in P$, $x \neq y$, then $Hx \cap Hy = \emptyset$. Consequently, in each fiber, there is only a single element of P and $hx = kx$ if and only if $h, k \in T_x^{-1}(hx)$. Therefore, by 3.5, we have

$$\hat{U} \cong \{\chi \in \hat{H} \times \hat{P} \mid \chi \text{ is constant on sets of the form (1)}\}.$$

We conclude this section with the following proposition:

PROPOSITION 4.3. *Conditions* (1) *and* (2) *in the definition of a* $Q - I$ *semigroup imply condition* (3).

Proof. If E is the set of idempotents of U, then we assert that E is isomorphic to U'. Given $a \in U'$ and $x, y \in q^{-1}(a)$, we have $xU = yU$. To see that $xy \in q^{-1}(a)$, first note that $L_{xy} = (L_x)(L_y) = (L_x)(L_x) = L_x$, since U' is idempotent. Therefore, $x \equiv xy(\text{modulo } \mathscr{L})$, from which we conclude that $xy \in q^{-1}(a)$ and hence, $q^{-1}(a)$ is also compact. Thus, $q^{-1}(a)$ contains an idempotent element e, and we can write $q^{-1}(a) = L_e$. We have just seen that $L_e = He$, and consequently, L_e is a group with identity element e. Moreover, since L_e is a group, it can contain only one idempotent, namely e. Therefore, $q|_E$ is one-to-one and onto and hence, an isomorphism.

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