UNIQUENESS OF A CLASS OF FUCHSIAN GROUPS[1]

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1. Let G be a fuchsian group of Moebius transformations acting on the upper half-plane $\mathbf{H}$, i.e., G is a discrete subgroup of $LF(2, \mathbf{R})$. As usual, we treat G as though it were a matrix group. Let G contain translations. We consider the parameter

$$c_0(G) \equiv c_0 = \min \{|c| \neq 0: (a, b: c, d) \in G\}. \tag{1.1}$$

It is well known that the minimum is attained and that $c_0 > 0$. Under certain circumstances the value of c_0 characterizes G up to conjugacy.

Since G contains translations, it will contain a smallest translation $z \to z + \lambda$, $\lambda > 0$. If $\lambda = 1$ we say G is *normalized.* Any group G can be normalized by conjugation with $\theta = (\lambda^{-1/2}, 0: 0, \lambda^{1/2})$ and we write

$$G^* = \theta G \theta^{-1} \tag{1.2}$$

for the normalized group. The notation K^* means that K is normalized. Obviously $c_0(G^*) = \lambda c_0(G)$.

Among the well-known groups in this class are the Hecke groups H_q. Here

$$H_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad 3 \leq q \leq \infty, \tag{1.3}$$

where

$$\lambda_q = 2 \cos \frac{\pi}{q}, \; 2 \leq q < \infty; \; \lambda_\infty = 2.$$

The Hecke groups are included in the more general class

$$H_{p,q} = \left\langle \begin{pmatrix} 1 & \lambda_p + \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -\lambda_p \end{pmatrix} \right\rangle, \quad 2 \leq p \leq q \leq \infty, p + q > 4; \tag{1.4}$$

in fact $H_q = H_{2,q}$. (There is no group $H_{2,2}$; see the lines following (2.6).) We shall see (Section 2) that

$$c_0(H_q) = c_0(H_{p,q}) = 1; \tag{1.5}$$

hence

$$c_0(H_q^*) = \lambda_q, \; c_0(H_{p,q}^*) = \lambda_p + \lambda_q. \tag{1.6}$$

It is known [3] that $H_{p,q}$ is the free product of a cyclic group of order p and one of order q when $p, q < \infty$.

In this paper all conjugacies will be over $SL(2, R)$.

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It should be noted that in Theorems 1–4 there is no *a priori* assumption that G^* is finitely generated; rather, this is a conclusion.

THEOREM 1. *If* $c_0(G^*) < 2$ *then* $c_0 = \lambda_q$ *for a* $q \geq 3$, $q < \infty$, *and* G^* *is conjugate to the Hecke group* H_q.

THEOREM 2. *Let* $2 < c_0(G^*) < 4$. *Then* G^* *has minimal elliptic elements (see Section 2). Let* $p \geq 2$ *be the lowest order of any such element. If*

$$c_0(G^*) < \lambda_p + 2, \tag{1.7}$$

then $c_0(G^*) = \lambda_p + \lambda_q$ *for a* $q \geq p$, $p + q > 4$, $q < \infty$, *and* G^* *is conjugate to* $H_{p,q}$.

THEOREM 3. *Let* $2 < c_0(G^*) < 4$ *and let*

$$c_0(G^*) = \lambda_p + 2, \quad 2 < p < \infty. \tag{1.8}$$

Then G^* *is conjugate to* $H_{p,\infty}$.

THEOREM 4. *Let* $c_0(G^*) = 2$. *Then* G^* *is conjugate either to* H_∞ *or to* $H_{3,3}$.

A group G is called *horocyclic* if every real number is a limit point of G; otherwise *nonhorocyclic*. The groups H_q, $H_{p,q}$ are horocyclic.

THEOREM 5. *Let* $2 < c_0(G^*) < 4$ *and let* (1.8) *be violated. Then* G^* *may be finitely generated (horocyclic or not) or it may be infinitely generated.*

In this case, then, there is no uniqueness.

I am greatly indebted to A. F. Beardon, who called my attention to this problem and kindly supplied a statement and proof (geometric) of Theorem 1.

2. Let G be a discrete subgroup of $SL(2, \mathbf{R})$. We can assume $-I = (-1, 0: 0 - 1) \in G$, for we can always adjoin $-I$ to G without affecting the transformation group $G/\{I, -I\}$.

An element $A \in G^*$ will be called *minimal* if

$$A = (a, b: c_0, d), \quad c_0 = c_0(G).$$

LEMMA 1. *If* E *is a minimal elliptic element of* G^* *of order* $p \geq 2$, *then*

$$\text{trace } E = \pm \frac{2 \cos \pi}{p}. \tag{2.1}$$

The point of the lemma is that in general we could assert only that $\text{tr } E = 2 \cos \pi k/p$. We may assume $\text{tr } E \geq 0$, otherwise replace E by $-E^{-1}$. Let $t = \text{tr } E$, $0 \leq t \leq 2$, and set

$$E^n = \alpha_n E + \beta_n I, \quad n \geq 0, \alpha_0 = \beta_1 = 0, \alpha_1 = \beta_0 = 1, \tag{2.2}$$

where $\alpha_n = \alpha_n(t)$, etc. Then

$$\alpha_{n+1} = t\alpha_n - \alpha_{n-1}, \quad n \geq 1; \qquad \alpha_n = \frac{\xi^n - \xi^{-n}}{\xi - \xi^{-1}}, \quad n \geq 0 \tag{2.3}$$

where ξ is either solution of

$$t = \xi + \xi^{-1}$$

It follows that E^n has third element $\alpha_n c_0$, so

$$|\alpha_n| \geq 1 \quad \text{or} \quad \alpha_n = 0, \quad n \geq 0. \tag{2.4}$$

Since E is of order p and $t \geq 0$, we may write

$$t = 2\cos\frac{\pi k}{p}, (k, p) = 1, \quad 1 \leq k \leq \frac{p}{2}; \qquad \xi = \exp\frac{\pi i k}{p}.$$

Obviously we may assume $p \geq 5$. Choose j so that $jk \equiv 1 \pmod{p}$, $1 \leq j < p$. Then

$$\alpha_j(t) = \frac{\sin(\pi jk/p)}{\sin(\pi k/p)} = \pm\frac{\sin(\pi/p)}{\sin(\pi k/p)}.$$

It follows that $\alpha_j(t) \neq 0$, hence $|\alpha_j(t)| \geq 1$ by (2.4). But if $1 < k \leq p/2$, $|\alpha_j(t)| < 1$. Hence $k = 1$ and the lemma is proved.

We say $K \subset SL(2, \mathbf{R})$ is *maximal* ([1]) if there is no discrete group L such that $K < L < SL(2, \mathbf{R})$. Here the inequality sign means "proper subgroup".

A finitely generated horocyclic fuchsian group containing translations (= H-group) has a known presentation:

$$G = \left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r, p_1, \ldots, p_t : x_1^{m_1} = \cdots = x_r^{m_r} = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{r} x_j \prod_{k=1}^{t} p_k = 1 \right\rangle, \tag{2.5}$$

$$m_j \geq 2, g \geq 0, r \geq 0, t > 0.$$

Such a group, then, is the free product of r cyclic groups of finite order and $2g + t - 1$ cyclic groups of infinite order. The x_j are elliptic, the p_k parabolic, the a_i, b_i hyperbolic, and g is called the genus of the group. Instead of (2.5) we also use the abbreviated symbol $\{g: m_1, \ldots, m_r, \infty, \ldots, \infty\}$ and this is called the signature of G; if $g = 0$ we write $\{m_1, \ldots, m_r, \infty, \ldots, \infty\}$. The m_i are called the periods of G.

The hyperbolic area of G, $\sigma(G)$ is given by the formula

$$\sigma(G) = g - 1 + \frac{1}{2}\left(t + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right), \tag{2.6}$$

and $\sigma(G) > 0$ if and only if G is a group of the above type. For example, there is no group $\{2, 2, \infty\}$. According to results of Siegel, the minimum area for a group with translations is 1/12, and the minimum is attained by the modular group, with signature $\{2, 3, \infty\}$.

We say a signature is maximal if every group having this signature is maximal.

LEMMA 2. *The signatures* $\{p, q, \infty\}$, $2 \leq p < q < \infty$, *are maximal. If* G *has signature* $\{p, p, \infty\}$, $p \geq 3$, G *is a subgroup of exactly one fuchsian group* G_0. *Moreover,* $[G_0: G] = 2$ *and* G_0 *has signature* $\{2, p, \infty\}$. *In particular,* $H_{p,p}$ *is contained only in* H_p.

This result can be deduced from results of Singerman [4]. The first statement appears on examination of Theorems 1 and 2 of [4]. (Note that if G has signature $\{p, q, \infty\}$ and $G < G_0$, then $0 < \sigma(G_0) < \sigma(G) < \infty$; hence $[G_0: G] = \sigma(G)/\sigma(G_0)$ is finite.) Now let G have signature $s = \{p, p, \infty\}$, $p \geq 3$ and let $G < G_0$. Then s is not maximal. According to [4], the only signature containing s is $s_0 = \{2, p, \infty\}$, hence G_0 has signature s_0.

We now make use of Proposition 4 of [4]. Let $A \subset A_0$, $[A_0: A] = N$, and let A_0 have signature $\{m_1, \ldots, m_r\}$, where now m_i can be ∞ (i.e., the corresponding generator is parabolic). Let $A_0 = \langle x_1, \ldots, x_r \rangle$, where $x_i^{m_i} = 1$. The *exponent* of x_i modulo A is the least positive integer n_i that $x_i^{n_i} \in A$; clearly $n_i < \infty$ and $n_i \mid m_i$ if m_i is finite. Proposition 4 states the following. If $n_i = m_i$, the period m_i does not appear among the periods of A. If $n_j < m_j$ it is easily seen that $m_j = n_j t_j$, $1 < t_j \leq \infty$. Then the period t_j appears N/n_j times among the periods of A and these constitute all the periods of A.

In the application G has signature s as above and presentation

$$\langle y_1, y_2, y_3 \colon y_1^p = y_2^p = y_1 y_2 y_3 = 1 \rangle,$$

while G_0 has signature s_0 and presentation

$$\langle x_1, x_2, x_3 \colon x_1^2 = x_2^p = x_1 x_2 x_3 = 1 \rangle.$$

Since the period p appears in s twice, $n_2 = 1$, and x_2 is conjugate to y_1 or y_2, say y_1. A generator can be replaced by a conjugate, so we may set $x_2 = y_1$. Also, $n_3 = 2$. Suppose $y_3 = (1, 2\lambda \colon 0, 1)$, then since $x_3^2 = y_3$, $x_3 = (1, \lambda; 0, 1)$. From $x_1 x_2 x_3 = 1$ we can now solve for $x_1 = x_3^{-1} x_2^{-1} = x_3^{-1} y_1^{-1}$. Hence G_0 is completely determined.

The last statement of the lemma is now easily checked.

LEMMA 3. *The signatures* $\{p, \infty, \infty\}$ *are maximal if* $p > 3$. *If* G *has signature* $\{2, \infty, \infty\}$ *or* $\{3, \infty, \infty\}$, *then* G *is contained in exactly one fuchsian group, which has signature* $\{2, 3, \infty\}$. *In particular* $H_{2,\infty}$ *is contained only in* H_3; *similarly* $H_{3,\infty}$ *is contained only in* H_3.

The first assertion follows from [4]. Now suppose G has signature $\{2, \infty, \infty\}$

and suppose there is a fuchsian group G_0 such that $G < G_0$; by the results of [4], G_0 has signature $\{2, 3, \infty\}$. Let

$$G_0 = \langle x_1, x_2, x_3 : x_1^2 = x_2^3 = x_1x_2x_3 = 1\rangle,$$
$$G = \langle y_1, y_2, y_3 : y_1^2 = y_1y_2y_3 = 1\rangle.$$

Since y_1 is conjugate to a power of x_1, y_1 must be conjugate to x_1, and we may take $x_1 = y_1$. Secondly, $x_2 \notin G$ since G has no elements of order 3. Suppose $x_3 \in G$, then $x_3^{-1} = x_1x_2 = y_1x_2 \in G$ and so $x_2 \in G$. Hence $x_3 \notin G$, which implies $x_3^2 \in G$. Since we may assume x_3 and y_3 have the same fixed point, x_3 is determined and thus so is $x_2 = x_1^{-1}x_3^{-1}$. Therefore G_0 is unique. It is clear there is a group G_0, since the group $\langle y_1, x_3 : x_3^2 = y_3\rangle$ satisfies the requirement.

The proof for the case $\{3, \infty, \infty\}$ is similar. This completes the proof of the lemma.

We shall now compute the c_0 of some groups. Let G be a group with minimum translation λ. The Ford fundamental region for G is the region contained in $\{|x| < \lambda/2, y > 0\}$ and lying outside all isometric circles of G. It is clear that $c_0(G)$ is the reciprocal of the radius of the largest isometric circle.

The well-known fundamental region of H_q is bounded below by the unit circle, hence $c_0(H_q) = 1$. Here $3 \le q \le \infty$. Next, let $p > 2$, $q \ge p$. Consider the region within $|x| < (\lambda_p + \lambda_q)/2$ and outside the circles $|z \mp \lambda_p/2| = 1$. The circles are the isometric circles of

$$E = \begin{pmatrix} \lambda_p/2 & \cdot \\ -1 & \lambda_p/2 \end{pmatrix}, \ E^{-1}$$

and the translation conjugating the vertical sides is $S = (1, \lambda_p + \lambda_q : 0, 1)$. Thus $E_1 = SE$ has trace $-\lambda_q$ and it fixes the point of intersection of the side $x = (\lambda_p + \lambda_q)/2$ and the isometric circle. According to Poincaré's theorem the above region is a fundamental region for the group G_1 generated by S and E. Clearly $c_0(G_1) = 1$. Now conjugate $G_1 \to G_2$ by $(1, \lambda/2 : 0, 1)$; S is unaffected and $E \to E_2 = (0, 1 : -1, \lambda_p)$. Furthermore $(a, b : c, d) \to (a', b' : c, d')$, so $c_0(G_1) = c_0(G_2)$. But $G_2 = H_{p,q}$ by (1.4). Hence (1.5).

3. Theorems 1–4 are consequences of

THEOREM 6. *Let G^* have a minimal elliptic element of smallest period $p \ge 2$. Assume*

$$c_0(G^*) < \lambda_p + 2. \tag{3.1}$$

Then G^* is conjugate to $H_{p,q}$ for a $q \ge p$. Moreover,

$$c_0(G^*) = \lambda_p + \lambda_q. \tag{3.2}$$

Proof. As usual we assume the minimal elliptic element has nonnegative trace; let it be $E = (a, b\colon c_0, d)$, $a + d \geq 0$. By Lemma 1, $a + d = \lambda_p$. Since

$$\begin{pmatrix} a & b \\ c_0 & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \cdot \\ c_0 & d - c_0 \end{pmatrix} = F$$

is in G^* with trace $\lambda_p - c_0$, and by (3.1)

$$-2 < \lambda_p - c_0 < \lambda_p < 2, \tag{3.3}$$

F is elliptic and minimal and has period $q \geq p$. By Lemma 1, $\lambda_p - c_0 = \pm\lambda_q$. But $q \geq p$ means $\lambda_q \geq \lambda_p$, and $\lambda_p - c_0 = \lambda_q$ would imply $\lambda_q < \lambda_p$ by (3.3). Hence (3.2).

Write $S = (1, 1\colon 0, 1)$. Conjugate G^* by

$$M = (-c_0, a\colon 0, 1), \quad c_0 = c_0(G^*). \tag{3.4}$$

The elements S, E, and F go into

$$S_1 = (1, -c_0\colon 0, 1),\ E_1 = (0, 1\colon -1, \lambda_p), \quad \text{and} \quad E_2 = (0, 1\colon -1, -\lambda_q),$$

in view of (3.2). The transformed group $G_1 = MG^*M^{-1}$ (no longer normalized) contains $-E_1$ and S_1^{-1}, hence contains $H_{p,q} = \langle S_1^{-1}, -E_1\rangle$. Note that the smallest translation in G_1 is $c_0(G^*) = \lambda_p + \lambda_q$.

Now if $q > p$, $H_{p,q}$ is maximal (Lemma 2); hence $G_1 = H_{p,q}$, and G^* is conjugate to G_1.

If $q = p$, $G_1 \supset H_{p,p}$. Here $p \geq 3$, for there is no fuchsian group $H_{2,2}$. Hence, again by Lemma 2, $G_1 = H_{p,p}$ or $G_1 = H_p$. The smallest translation in H_p is λ_p—see (1.3)—whereas the smallest translation in G_1 is $2\lambda_p$, as remarked above. But $2\lambda_p > \lambda_p$ since $p > 2$. It follows that $G_1 = H_{p,p}$.

4. We now turn to the proofs of Theorem 1–5. Observe that when $c_0 < 4$ there is a minimal elliptic element. For let $E = (a, b\colon c_0, d)$ be a minimal element of G^*; then

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c_0 & d \end{pmatrix} = \begin{pmatrix} a + uc_0 & \cdot \\ c_0 & d \end{pmatrix} = E_1, \quad u \in \mathbf{Z}$$

belongs to G^*, is minimal, and, for the proper choice of u,

$$-2 < -\frac{c_0}{2} \leq \operatorname{tr} E_1 = a + d + uc_0 < \frac{c_0}{2} < 2. \tag{4.1}$$

Hence E_1 is elliptic, as asserted. Let p be the smallest order of any minimal elliptic element in G^* and let E be a minimal element of order p with non-negative trace.

Suppose now $c_0 < 2$. By Lemma 1, $\operatorname{tr} E = 2\cos \pi/p$ and by (4.1), $-1 < 2\cos \pi/p < 1$. Hence $p = 2$, $\lambda_p = 0$. The hypotheses of Theorem 6 are satisfied and we conclude that G^* is conjugate to $H_{2,q} = H_q$. Necessarily $q > 2$.

This completes the proof of Theorem 1. In view of (1.6) we may restate the result:

THEOREM 1′. *A fuchsian group G is conjugate to the Hecke group H_p, $3 \leq p < \infty$, if and only if $c_0(G^*) = \lambda_p$.*

For the proof of Theorem 2 assume $2 < c_0 < 4$; as we have seen there is a minimal elliptic element E of lowest period $p \geq 2$. Because of the hypothesis (1.7) we can again apply Theorem 6, which produces the desired conclusion.

Proof of Theorem 3. Conjugate G^* with the M of (3.4), obtaining $G_1 = MG^*M^{-1}$, which contains $S_1 = (1, -c_0 \colon 0, 1)$, $E_1 = (0, 1 \colon -1, \lambda_p)$, and

$$P = S_1^{-1}E_1 = \begin{pmatrix} -c_0 & \cdot \\ -1 & \lambda_p \end{pmatrix}.$$

P is parabolic because of (1.8). Hence G_1 contains $H_{p,\infty} = \langle S_1^{-1}, -E_1 \rangle$. Now $H_{p,\infty}$ is maximal when $p > 3$ (Lemma 3); therefore $G_1 = H_{p,\infty}$. And when $p \leq 3$, $H_{2,\infty} \subset H_3$, $H_{3,\infty} \subset H_3$, so $H_{p,\infty} \subset G_1 \subset H_3$, $p = 2, 3$. But $c_0(H_3) = 1$ whereas $c_0(G_1) = c_0 = \lambda_p + 2 > 1$. Hence $G_1 \neq H_3$, Q.E.D.

Theorem 4 follows from previous results. Let G^* have $c_0 = 2$; then G^* has a minimal elliptic element of lowest order $p \geq 2$. If $p = 2$, so that $\lambda_p = 0$, we have $c_0 = 2 = \lambda_p + 2$ and we can use Theorem 3; then G^* is conjugate to $H_{2,\infty} = H_\infty$. If $p \geq 3$, $\lambda_p \geq 1$, $2 < \lambda_p + 2$ and Theorem 6 applies: G^* is conjugate to $H_{p,q}$, and also $c_0(G^*) = 2 = \lambda_p + \lambda_q$. Since $\lambda_p \geq 1$, $\lambda_q \leq 1$. Since also $q \geq p$, the only solution is $p = q = 3$.

To prove Theorem 5 we shall construct certain groups by the method of free products [2, pp. 118–120]. Fix an integer $p \geq 2$ and a real number $c_0 > \lambda_p + 2$ Let

$$E = \left(\frac{\lambda_p}{2}, \cdot \colon c_0, \frac{\lambda_p}{2}\right).$$

The isometric circles of E, E^{-1} are

$$I\colon \left|c_0 z + \frac{\lambda_p}{2}\right| = 1 \quad \text{and} \quad I'\colon \left|c_0 z - \frac{\lambda_p}{2}\right| = 1.$$

The extreme endpoints of I, I' are $x_1 = (\lambda_p/2 + 1)/c_0$ and $-x_1$. By hypothesis $-\frac{1}{2} < -x_1$, $x_1 < \frac{1}{2}$. Thus $I \cup I'$ lies in the strip $|x| < \frac{1}{2}$.

We shall construct 3 types of groups:

(1) Place a finite number of mutually tangent circles with centers in $(x_1, 1)$ so that the first is tangent to I and the last to the line $x = \frac{1}{2}$. The radii of the circles shall be less than $1/c_0$. Place symmetrical circles in the interval $(-\frac{1}{2}, -x_1)$.

(2) Same as in (1) except that the circles are not tangent; the first and last, however, are tangent as before to I and $x = \frac{1}{2}$.

(3) Place an infinite number of circles (tangent or not) with centers in $(x_1, 1)$ and radii less than $1/c_0$ so that the first is tangent to I and the centers of the circles $\to \frac{1}{2}$; place symmetrical circles in $(-\frac{1}{2}, -x_1)$.

In all cases the region bounded by the circles and by the half-lines

$$\{x = \pm\tfrac{1}{2}, y > 0\}$$

is a fundamental region for a fuchsian group G^*. Since $c_0(G^*)$ is the reciprocal of the radius of the largest bounding circle, we have $c_0(G^*) = c_0$. In case (1), G^* is horocyclic and finitely generated; in case (2), it is nonhorocyclic and finitely generated; in case (3) it is infinitely generated.

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